SECTION D

Answer **FIVE** of the six questions from this section.

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1. (a) Find the values of x for which $\dfrac{x(x-2)}{x+6} > 2$.

(b) Solve the equation $x^2 + 2x + \dfrac{12}{x^2+2x} = 7$.

(c) Solve the equations $2^{x+y} = 6^y$ and $3^x = 6(2^y)$ simultaneously, writing your solution(s) in terms of $\log_2(3)$.

2. (a) Show that the sum of the arithmetic series

$$a + (a+d) + (a+2d) + \cdots + (a + [n-1]d),$$

is given by

$$\frac{n}{2}(2a + [n-1]d).$$

Hence show that $1 + 2 + 3 + \cdots + n = \dfrac{n}{2}(n+1)$.

(b) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n}{6}(n+1)(2n+1)$.

[Hint: The identity $k^3 - (k-1)^3 = 3k^2 - 3k + 1$ may be useful.]

(c) Show that

$$\log(a) + \log(ar) + \log(ar^2) + \cdots + \log(ar^{n-1}) = \frac{n}{2}\log(a^2 r^{n-1}).$$

(d) Consider the arithmetic progression a, $a+2$, $a+4$, $\ldots$, $a+2(n-1)$. If s is the sum of these terms and S is the sum of the squares of these terms, show that

$$3(nS - s^2) = n^2(n^2 - 1).$$

3. Consider the function $f(x) = \dfrac{2x^2 + x - 6}{x^2 + x - 6}$ for values of x where it is defined.

(a) Find and classify any stationary points of $f(x)$.

(b) Find the horizontal and vertical asymptotes of the curve $y = f(x)$.

(c) Find the x and y-intercepts of the curve $y = f(x)$.

(d) Sketch the curve $y = f(x)$.

4. (a) Show that the derivative of $\tan x$ is $\sec^2 x$.

(b) For what values of a does

$$\int_1^a \left(x + \frac{1}{2}\right) \mathrm{d}x = 2 \int_0^{\pi/4} \sec^2 x \, \mathrm{d}x\,?$$

(c) Find substitutions to show that each of the integrals

$$\text{(i)} \quad \int_0^1 \frac{x^2}{(1+x^2)^2} \, \mathrm{d}x \quad \text{and} \quad \text{(ii)} \quad \int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}} \, \mathrm{d}x,$$

is equal to the integral

$$\int_0^{\pi/4} \sin^2 \theta \, \mathrm{d}\theta.$$

What is the value of this integral?

5. It costs a firm $C(q) = q^3 - 4q^2 + 5q + 18$ to produce q units of its product. Given that they sell all of this product at a fixed price p, then their profit is given by

$$\pi(q) = pq - C(q),$$

where p is a constant.

(a) Find an equation relating p and q that must hold if the profit is to have a stationary point. For what values of q will this stationary point be a local maximum? Find, in terms of q only, an expression for the profit, $\pi^*(q)$, of the firm when p and q are related by this equation.

(b) Using your expression for $\pi^*(q)$, find the quantity, $q^* > 0$, which makes the profit the same as the profit obtained when the firm produces nothing. That is, find the quantity, $q^* > 0$, that makes $\pi^*(q^*) = \pi^*(0)$. Also, use your equation relating p and q from (a) to find the price, p^*, that corresponds to this quantity.

(c) Taking p to be the vertical axis and q to be the horizontal axis, sketch the curve given by the equation relating p and q from (a).

(d) The firm decides that, if the price, p, is less than p^* then they will supply nothing whereas if the price, p, is greater than or equal to p^*, then they will supply the quantity q determined by the equation relating p and q from (a). Using this information, indicate on your sketch from (c) all the points (q, p) that determine what the firm will supply for $p \geq 0$.

6. A health-conscious lecturer wants to ensure that he gets enough vitamins each week by taking vitamin tablets. He wants to have at least 36 units of vitamin A, 60 units of vitamin C and 16 units of vitamin D. Two kinds of vitamin tablet are available: MultiVit and KeepVit. Each tablet of MultiVit contains 3 units of vitamin A, 3 units of vitamin C, and 1 unit of vitamin D. Each tablet of KeepVit contains 2 units of vitamin A, 5 units of vitamin C, and 1 unit of vitamin D.

(a) Explain, **very briefly**, why the inequalities:

$$3x + 2y \geq 36, \quad 3x + 5y \geq 60, \quad x + y \geq 16, \quad x \geq 0, \quad y \geq 0,$$

must hold if the lecturer is to meet his desired intake of vitamins by taking x MultiVit tablets and y KeepVit tablets.

(b) Indicate on a diagram the region, $\mathcal{R}$, of all points (x, y) in the plane that satisfy these inequalities.

(c) Each tablet of MultiVit costs \$3 and each tablet of KeepVit costs \$4. The lecturer wishes to spend as little as possible on vitamin tablets, while still meeting his desired intake of vitamins. By sketching some lines of the form

$$3x + 4y = \text{constant},$$

on your diagram from (b), find the values of x and y which will provide the lecturer with the vitamins he wants and will minimise his expenditure.

(d) Suppose that the price of each KeepVit tablet rises to \$6. Use a similar method to the one in (c) to find the values of x and y which will provide the lecturer with the vitamins he wants and will minimise his expenditure in this case.