Low independence number and Hamiltonicity implies pancyclicity

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Definitions

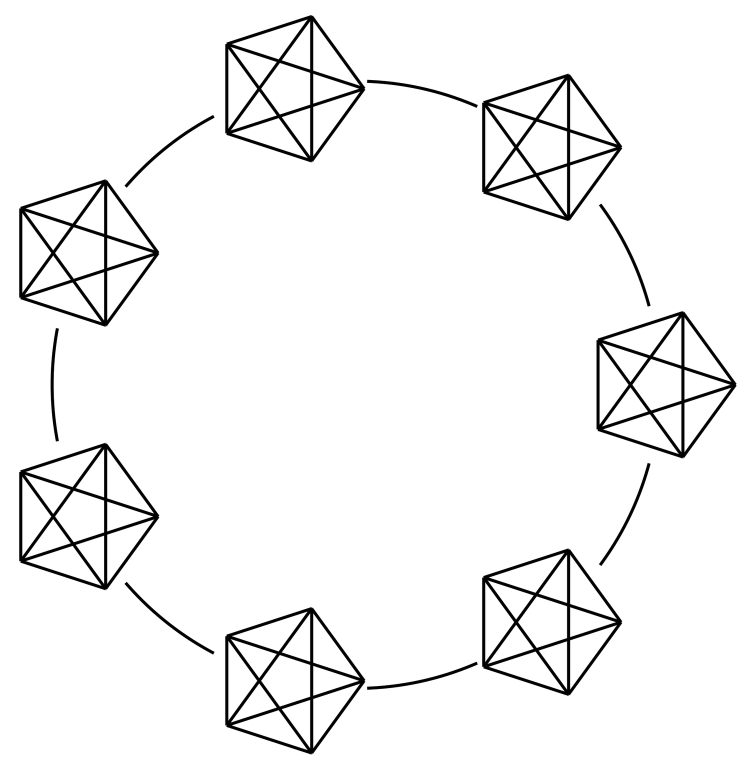
A graph is *pancyclic* if it contains a cycle of every length $3 \leq l \leq n$ where n denotes the number of vertices. The *independence number* of a graph G is the size of the largest independent set.

Given a Hamiltonian graph G with independence number at most k we are looking for the minimum number of vertices $f(k)$ that guarantees that G is pancyclic.

The problem of finding $f(k)$ was raised by Erdős who showed that $f(k) \leq 4k^4$ and conjectured that $f(k) = \Theta(k^2)$. Formerly the best known upper bound was $f(k) = O(k^{7/3})$ by Lee and Sudakov [1].

A lower bound construction

Hamiltonian graph with no cycle of length 6. 35 vertices, independence number 7.



Main result

$f(k) = O(k^{11/5})$.

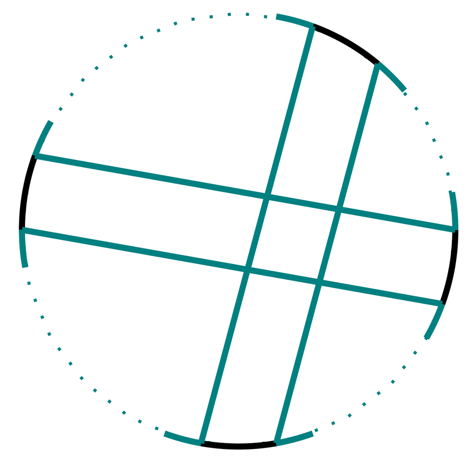
The idea of the proof

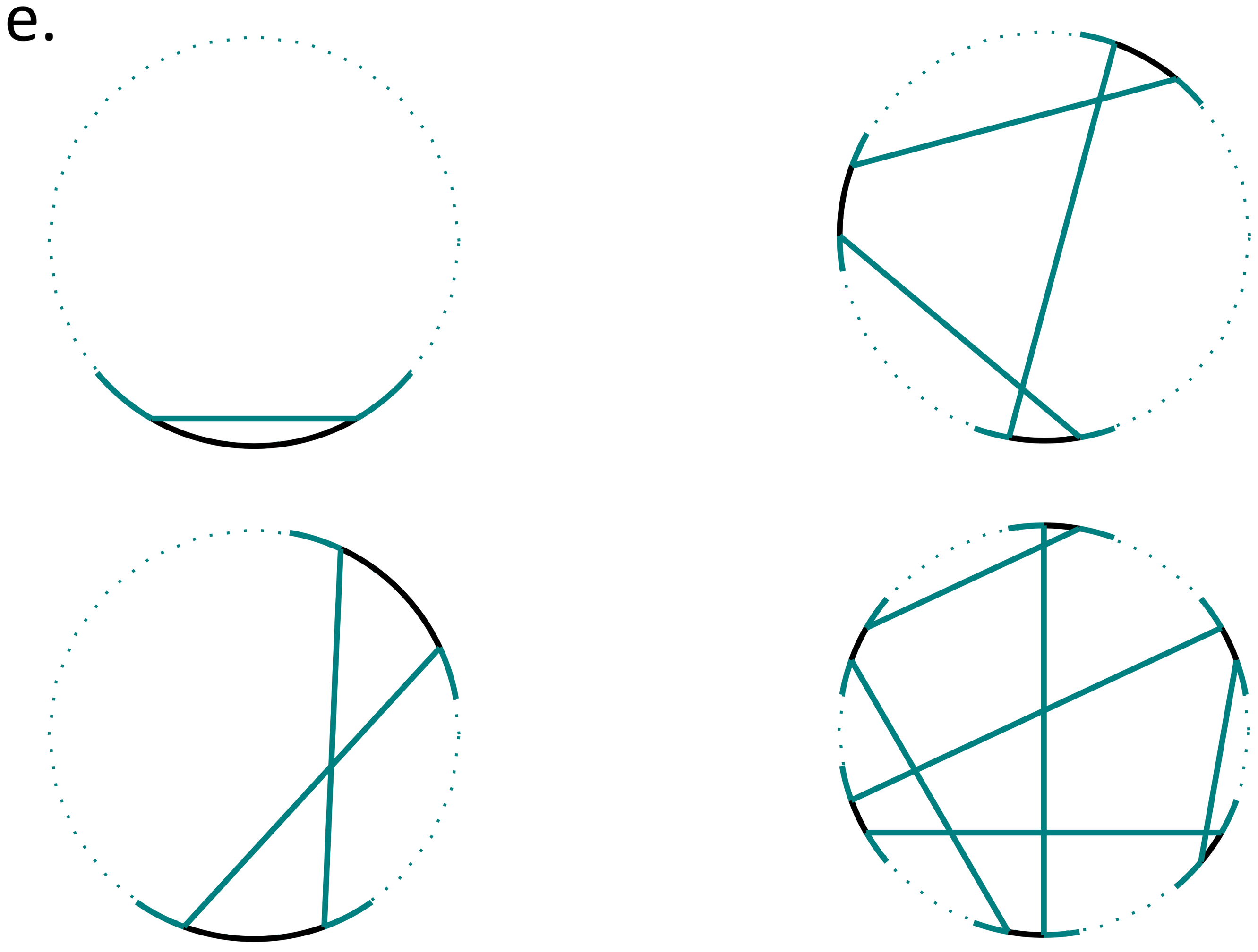
We call a vertex *problematic*, if it has too low degree. An argument of Keevash and Sudakov [2] shows that it is enough to find one long cycle that contains all problematic vertices, called a *contradicting cycle*.

We consider arcs on the Hamilton cycle which do not contain problematic vertices. We find that edges between arcs in certain configurations imply the existence of a contradicting cycle.

Contradicting configurations

The circle represents the Hamilton cycle, the not dotted parts are the considered arcs and teal shows the contradicting cycle.





A key lemma

We want to show that sparse graphs have a high independence number while dense graphs contain one of the contradicting configurations.

We call arcs *good* if they help us find one of the contradicting configurations. We say the *size* of an arc-system is the number of arcs in it while the *length* is the number of vertices of each arc in it.

Lemma. There exists a_p and b_p such that given a simple arc-system $\mathcal{A}$ of size $b_p x^{p(p-1)/2}$ and length $a_p x$, there is either an independent set of size $x^p + 1$ in $G[\mathcal{A}]$ or there is a non-empty subset of $\mathcal{A}$ that has only good arcs in it.

References

[1] Choongbum Lee and Benny Sudakov. Hamiltonicity, independence number, and pancyclicity. *European J. Combin.*, 33(4):449–457, 2012.

[2] Peter Keevash and Benny Sudakov. Pancyclicity of Hamiltonian and highly connected graphs. *J. Combin. Theory Ser. B*, 100(5):456–467, 2010.