Edge correlations in random regular hypergraphs

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The problem

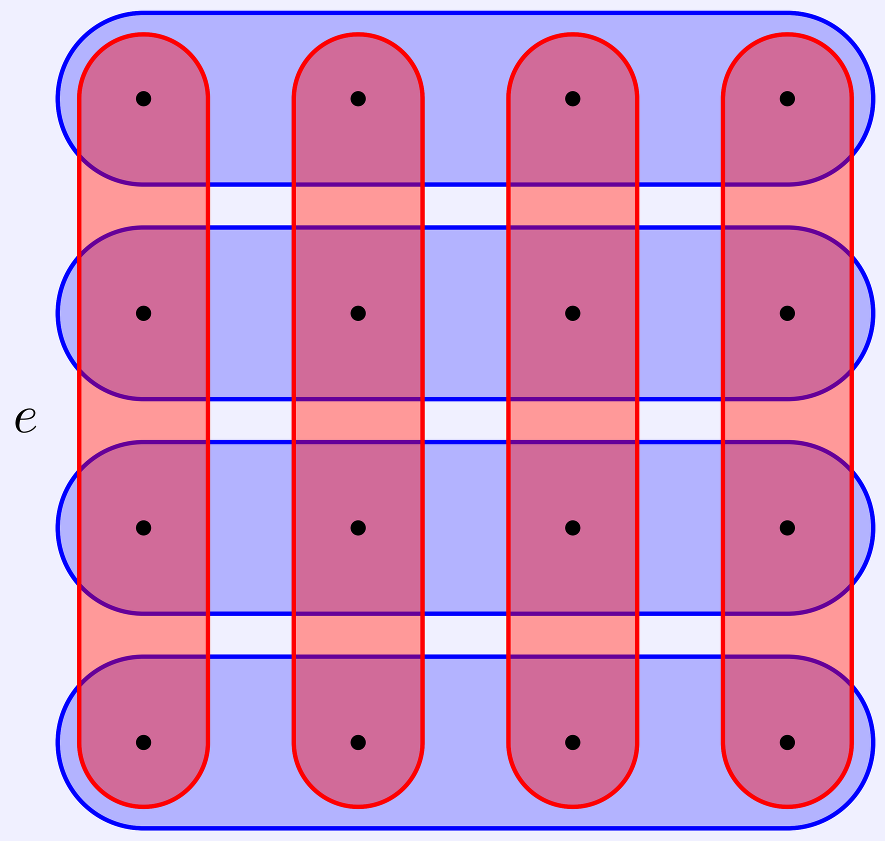
Random graphs have been an object of intense research for the past decades. The models introduced by Erdős and Rényi ($\mathcal{G}(n,p)$ and $\mathcal{G}(n,m)$) are quite well understood, whereas other models are not so much. In particular, results in the model of random regular graphs always lag behind $\mathcal{G}(n,p)$ or $\mathcal{G}(n,m)$. The problem when studying this model arises from the fact that there are correlations between the appearances of different edges. The same problem extends to regular r-uniform hypergraphs (r-graphs).

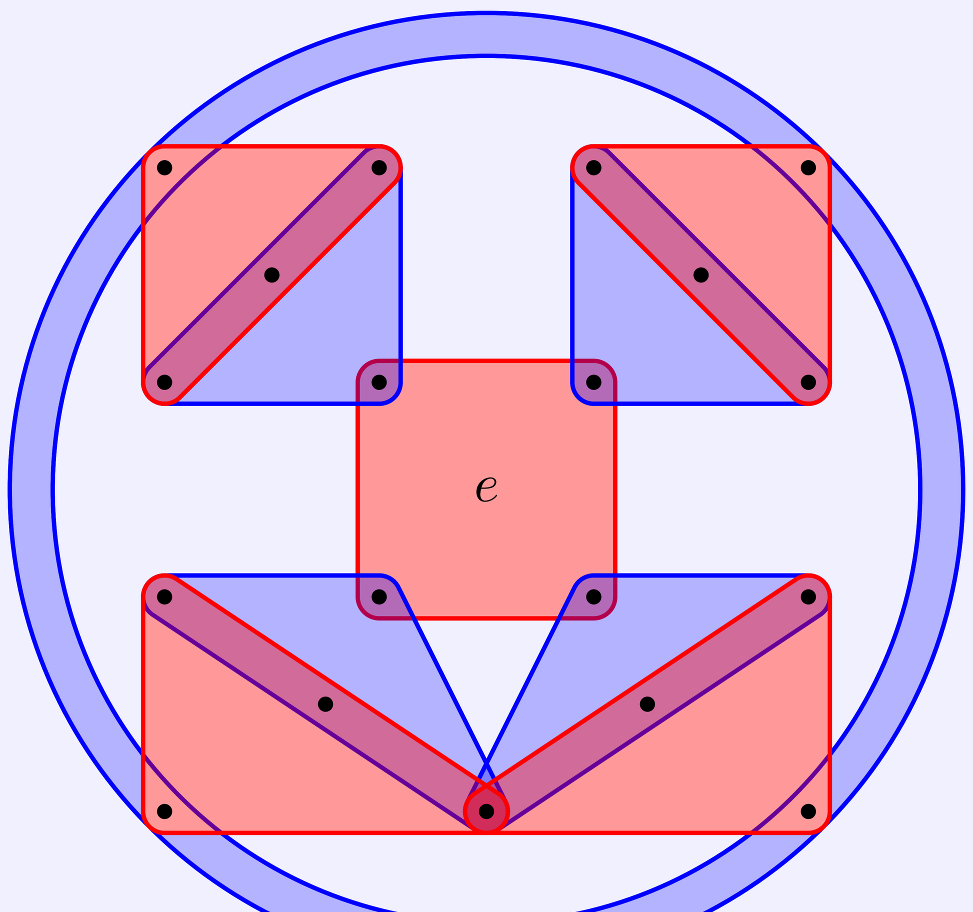
Notation: We let $\mathcal{G}^{(r)}_{n,d}$ be the set of all d-regular r-graphs on n vertices, and $G^{(r)}_{n,d}$ be an r-graph chosen uniformly at random from $\mathcal{G}^{(r)}_{n,d}$. We use $\mathcal{G}^{(r)}_{n,d,H,H'}$ to denote the set of all r-graphs $G \in \mathcal{G}^{(r)}_{n,d}$ such that $H \subseteq G$ and $H' \subseteq \overline{G}$.

Estimating edge correlations in random regular hypergraphs via switchings

Lemma (Espuny Díaz, Joos, Kühn, Osthus, 2018$^+$)**.** *Let $r \geq 2$ be a fixed integer. Suppose that $d = \omega(1)$ and $d = o(n^{r-1})$. Let $H, H' \subseteq \binom{V}{r}$ be two edge-disjoint r-graphs such that $\Delta(H), \Delta(H') = o(d)$. Then, for all $e \in \binom{V}{r} \setminus (H \cup H')$ we have $\mathbb{P}\left[e \in G^{(r)}_{n,d} \mid \mathcal{G}^{(r)}_{n,d,H,H'}\right] = (1 \pm o(1))(r-1)!\, d/n^{r-1}$.*

Sketch of the proof:





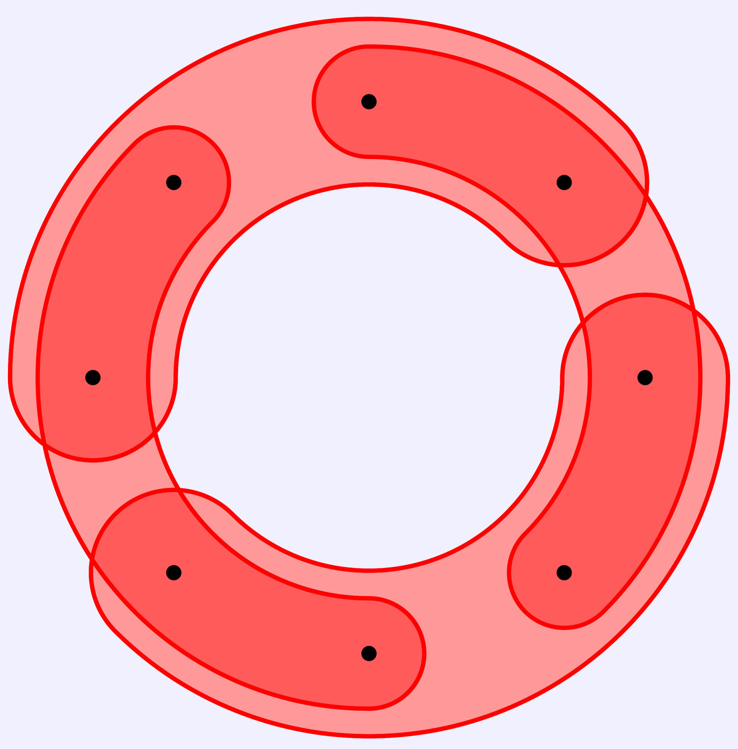
- Partition $\mathcal{G}^{(r)}_{n,d,H,H'}$ into $\mathcal{F}_e$ ($e \in G$) and $\mathcal{F}_{\overline{e}}$ ($e \notin G$).
- Use switchings to relate $|\mathcal{F}_e|$ and $|\mathcal{F}_{\overline{e}}|$. Switchings preserve degree-sequences and are reversible.
- Define a bipartite graph $\Gamma = (\mathcal{F}_e, \mathcal{F}_{\overline{e}}, E)$ where $(G_1, G_2) \in E$ if there is a switching going from one to the other.
- Suppose

$$\forall\, G \in \mathcal{F}_e,\ \#\left\{\begin{array}{c}\text{switchings that transform}\\ G \text{ into a graph in } \mathcal{F}_{\overline{e}}\end{array}\right\} \leq a,$$

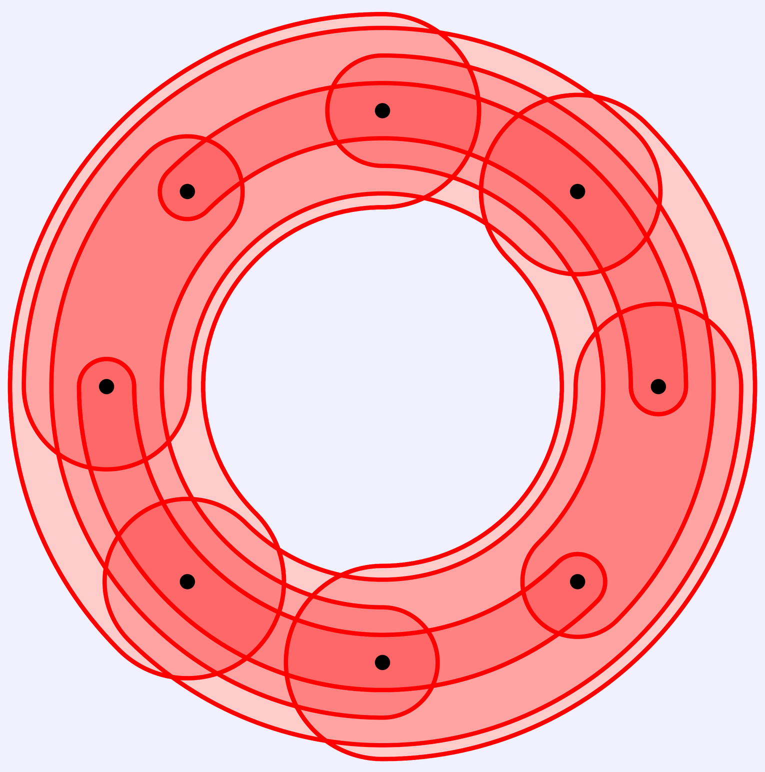
$$\forall\, G \in \mathcal{F}_{\overline{e}},\ \#\left\{\begin{array}{c}\text{switchings that transform}\\ G \text{ into a graph in } \mathcal{F}_{e}\end{array}\right\} \geq b.$$

Then $a|\mathcal{F}_e| \geq b|\mathcal{F}_{\overline{e}}|$ by double counting the edges in Γ.

Hamiltonicity of random regular hypergraphs



A 2-overlapping 4-cycle



A 3-overlapping 4-cycle

In [3], the authors show that $G^{(r)}(n, p_d) \subseteq G^{(r)}_{n,d}$ a.a.s., where p_d is very close to the density of $G^{(r)}_{n,d}$, for a wide range of d. Using this, one can translate results from $\mathcal{G}^{(r)}(n,p)$ to $\mathcal{G}^{(r)}_{n,d}$. They use it to prove an existence result for Hamilton cycles.

Conjecture ([3])**.** *If $2 \leq \ell < r$ and $d \ll n^{\ell-1}$, then a.a.s. a random d-regular r-graph contains no ℓ-overlapping Hamilton cycle.*

Theorem (Espuny Díaz, Joos, Kühn, Osthus, 2018$^+$)**.** *The conjecture is true.*

Our result determines the threshold for Hamiltonicity and also gives a counting result above the threshold.

Other counting results: perfect matchings, spanning lattices.

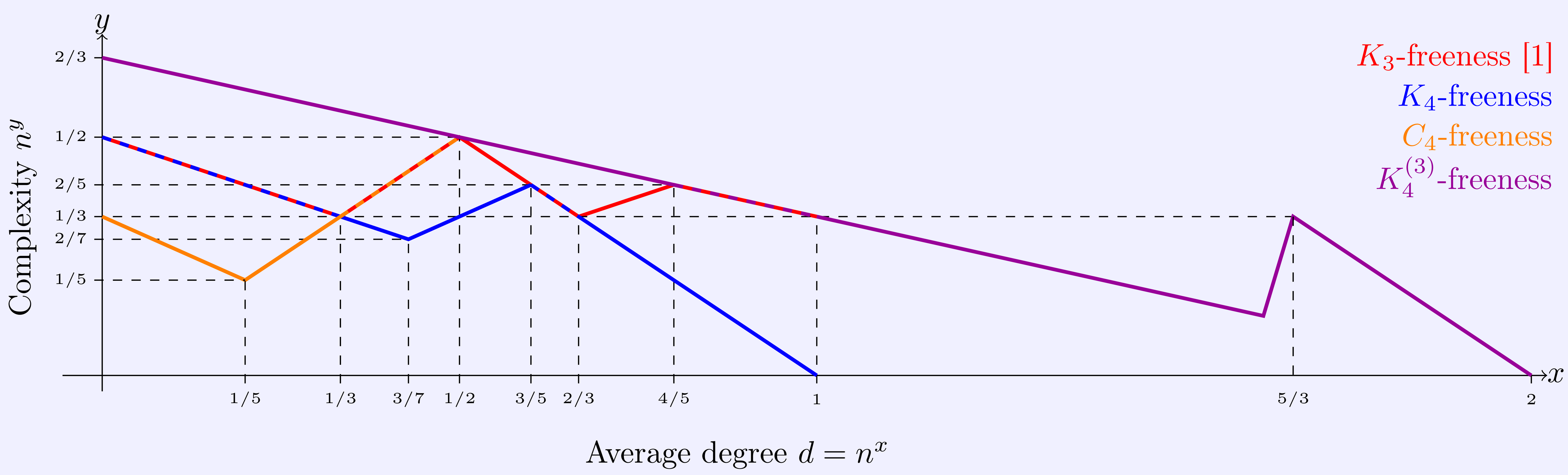
Main limitation: need results in $\mathcal{G}^{(r)}(n,p)$ or $\mathcal{G}^{(r)}(n,m)$.

Applications to property testing

Goal: to develop ultraefficient algorithms in exchange for precision in the answer. Does $G \in \mathcal{P}$, or is it "far" from $\mathcal{P}$?

We consider testing subgraph-freeness in the general (hyper) graphs model (best suited for applications). We provide lower bounds on the query complexity of testing subgraph-freeness.

The bounds rely strongly on the results for random regular hypergraphs.



References

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[2] D. Altman, C. Greenhill, M. Isaev and R. Ramadurai, A threshold result for loose Hamiltonicity in random regular uniform hypergraphs, *arXiv:1611.09423* (2016).

[3] A. Dudek, A. Frieze, A. Ruciński and M. Šileikis, Embedding the Erdős-Rényi hypergraph into the random regular hypergraph and Hamiltonicity, *J. Combin. Theory Ser. B* **122** (2017), 719–740.

[4] A. Espuny Díaz, F. Joos, D. Kühn and D. Osthus, Edge correlations in random regular hypergraphs and applications to subgraph testing, *arXiv:1803.09223* (2018).