Approval Balloting for Fixed-Size Committees [1]

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Abstract

Approval balloting is a natural approach to multi-winner elections because each ballot specifies a subset of the candidates, which is exactly what the election is to determine. The subset structure provides many different ways to aggregate approval ballots to determine a winning subset. In the most common multi-winner elections, the cardinality of the winning subset is specified in advance. Such elections and their variants are described, and the application of approval balloting is discussed. The first objective of this paper is to collect and compare systems for determining the winner based on the approval ballots. A secondary purpose is to identify properties of the procedures in the context of these most common multi-winner elections.

Keywords: Approval balloting, multi-winner election, committee election, fixed-size committee, approval voting

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1 Introduction

Approval voting is a well-known voting procedure for single-winner elections. Voters approve as many candidates as they like; a candidate wins if and only if no other candidate receives more approvals (Brams and Fishburn, 1978, 1983, 2005). But, as Merrill and Nagel (1987) point out, approval votes can be aggregated in different ways to serve different purposes, so it is reasonable to distinguish between *approval balloting*, in which each voter submits a ballot that identifies the voter's approved candidates, and *approval voting*, the single-winner procedure that selects the most-approved candidate(s).

Approval balloting is also appropriate in multi-winner elections, where the ballots, i.e., the voters' approvals, are used to identify a "best" subset of candidates. This application of approval balloting seems to capitalize on a natural correspondence between an approval ballot and a winning subset—both are subsets of the set of candidates. One possible approval-balloting procedure is plurality, in which each approval ballot is interpreted as a vote for a subset, so that a subset wins if and only if no other subset receives more votes. But this procedure treats all subsets as unrelated, making no inferences about gradations of preference for subsets similar but not identical to the one specified on the voter's ballot. The number of similar but not identical subsets can be very large for combinatorial reasons, which makes plurality unappealing and motivates a search for procedures that exploit this implied preference information. Eleven distinct procedures are identified by Kilgour (2010), who also describes some of their basic properties. But that study included only procedures and properties that arose in a context of broad applicability discussed below.

The objective of this paper is to present and discuss procedures for determining the winning subset(s) for one particular context for multi-winner elections, a context in which approval balloting is particularly appropriate. Its general characteristics are that (1) all *admissible* (or possibly winning) subsets are of specified size (cardinality), and (2) whether a subset is admissible is easily computable. To be explicit, whether a given subset of candidates is admissible can be determined in polynomial time, a condition that ensures that the winning subset under a well-behaved voting procedure is easy to compute. More generally, the computational effort to apply a procedure is determined by the procedure and not by the class of potentially winning subsets to which it is applied.

It is usual to study voting systems by identifying and comparing their properties (Arrow, Sen, and Suzumura, 2002; Brams and Fishburn, 2002); Ratliff (2003, 2006) follows this approach in investigating certain procedures for multi-winner elections that are not based on approval balloting. In this paper, the most common multi-winner elections will be described and discussed, reasonable procedures for conducting them using approval balloting will be elaborated, and the analysis and characterization of those procedures will be initiated. All of the procedures included here are anony-

mous (treat voters fairly) and neutral (treat candidates fairly), and share several other properties to be discussed later. The listing of properties of procedures is admittedly far from complete. Our intention is to make the procedures easy to compare, while postponing conclusions about which system is better for which purposes until more complete characterizations are available.

2 Multi-Winner Elections and Approval Balloting

We begin with basic terminology and notation for approval balloting and multi-winner elections. (Most of our notation and terminology is drawn from Kilgour (2010), where further references can be found.) Consider a multi-winner election with $n > 1$ voters and $m > 1$ candidates, and denote the set of all candidates by $[m] = \{1, 2, ..., m\}$. A voter's approval vote can be understood as the subset consisting of those candidates of whom the voter approves. Therefore, voter i's ballot can be represented as $V_i \subseteq [m]$, for $i = 1, 2, \ldots, n$. (Note that $V_i \in 2^{[m]}$, where $2^{[m]}$ is the "power set" of $[m]$, the set of all subsets of $[m]$. Voters may choose any subset in $2^{[m]}$, and in particular may approve no candidates, so $V_i = \emptyset$ is possible.) The ballot profile is $V = (V_1, V_2, \ldots, V_n)$, and the set of all possible ballot profiles is $\mathcal{V} = \left(2^{[m]}\right)^n$. We think of a voting procedure as taking a ballot profile V as input, and producing a winning subset as output.

A ballot profile is simply a list of the approval votes cast. Where no confusion is possible, we indicate subsets such as ballots without punctuation. In Example 1, for instance, there are $n = 4$ voters and $m = 3$ candidates. Voter 1 votes for candidate 1 only, voter 2 votes for candidates 1 and 2 (but not 3), voter 3 votes for candidates 1 and 3 (but not 2), and voter 4's ballot is identical to voter 3's. Then we write the ballots as $V_1 = 1$, $V_2 = 12$, and $V_3 = V_4 = 13$, and the ballot profile as $V = (1, 12, 13, 13)$. A tabular presentation of a ballot profile is simpler, as follows:

Example 1:

Voter	1	2	3	4
Ballot	1	12	13	13

Later we will describe ballot profiles using only the second row of tables like the one above, i.e. without naming the voters.

In general, any of the 2^m different subsets in $2^{[m]}$ might win a multi-winner election. But in practice there are usually *a priori* restrictions. A subset of candidates is called *admissible* if it is eligible to win the election; the set (or class) of all admissible subsets is denoted $\mathcal{A}$. We assume that $|\mathcal{A}| > 1$ (since otherwise there is no point in conducting an election). To define the special case that is the subject of this study, suppose that k is a fixed integer satisfying $1 \leq k \leq m - 1$, and define $\mathcal{A}_k = \{S \subseteq [m] : |S| = k\}$. Thus, $\mathcal{A}_k$ is the class of subsets of $[m]$ containing exactly k candidates. All of the elections studied here satisfy $\mathcal{A} \subseteq \mathcal{A}_k$, and are called k-elections. In fact, many of the k-elections we study have $\mathcal{A} = \mathcal{A}_k$.

Consider, for example, the case $k = 1$. Then a 1-election (with $\mathcal{A} = \mathcal{A}_1$) is a single-winner election. It is clear that approval balloting can be used for such elections; the approval voting procedure is well-studied—see Brams (2008) for recent references.

But if $k > 1$, a k-election is a multi-winner election in the sense that every candidate in the subset selected wins, and all other candidates lose. Nonetheless such an election could result in a tie of two or more admissible subsets. A voting procedure can be thought of as a function, possibly multi-valued, from the set of all possible ballot profiles, $\mathcal{V}$, to the class of admissible sets, $\mathcal{A}$. If the ballot profile is V, the winning subsets under a voting procedure $Proc$ might be denoted $Proc(V, \mathcal{A})$; note that $Proc(V, \mathcal{A}) \subseteq \mathcal{A}$ and $Proc(V, \mathcal{A}) \neq \emptyset$. In particular, we will write $Proc(V, \mathcal{A}_k) = Proc_k(V)$.

Most of the procedures we discuss will select all admissible subsets that maximize (or minimize) some function, called a *score*. More specifically, a score Sc is a function $Sc : \mathcal{V} \times \mathcal{A} \longrightarrow \mathbb{R}$. Note that, for any admissible subset $S \in \mathcal{A}$, the value of a score function, $Sc(V, S) = Sc(S)$, is assumed to depend only on the vote profile, $V \in \mathcal{V}$. Usually, $Sc(S)$ is a measure of the suitability of the admissible set S to win the election. In this case, the outcome of a voting procedure $Proc$ based on the score Sc is

$$Proc(V, \mathcal{A}, Sc) = \arg\max_{S \in \mathcal{A}} Sc(V, S).$$

[For scores that measure unsuitability rather than suitability, the above maximization (arg max) is replaced by a minimization (arg min).] As indicated above, we also write $Proc(V, \mathcal{A}_k, Sc) = Proc_k(V, Sc)$.

Procedures based on maximization or minimization of a score share some important fundamental properties, in addition to the "fairness" properties of anonymity and neutrality. The relative scores of two admissible subsets S_1 and S_2, $Sc(V, S_1)$ and $Sc(V, S_2)$, depend only on V, and of course S_1 and S_2, not on any other subsets in the admissible class. Thus if $S_1, S_2 \in \mathcal{A}^0$, and S_1 wins and S_2 does not, then S_2 can never win in any admissible class $\mathcal{A}$ that contains both S_1 and S_2. It follows that score-based procedures satisfy Independence of Irrelevant Alternatives and the Weak Axiom of Revealed Preference (Nurmi, 2002).

Before discussing specific procedures, we turn to restrictions that are commonly applied in multi-winner elections conducted using approval balloting.

3 Common k-Elections

This paper concerns a special class of multi-winner elections distinguished by their admissible sets: k-elections, in which the class of admissible sets contains only sets of exactly k candidates. Further restrictions are common. For example, representation constraints typically designate subsets of the

candidates and mandate the election of some prespecified number of candidates from such subsets.

The most common of these elections are committee elections, where voters determine the membership of a committee with k members. Sometimes the committee is also required to have a minimum number of members, or a specific number, from one or more subsets of the set of candidates. For example, a basketball all-star team requires one center, two forwards, and two guards. Many committee elections require the committee to contain at least one woman, or equal numbers of men and women, or members of each of several defined subgroups. University committees are typically required to include representatives of various faculties, schools, etc.

Brams (2008) suggested that this problem be addressed by Constrained Approval Balloting, in which all subsets in $\mathcal{A}_k$ are considered admissible and are measured not only by their levels of support but also by the extent to which they fulfil representational requirements. This method allows for a tradeoff between support and representativeness, and may indeed be appropriate for complex representational problems in which individual candidates belong to several subsets so that one candidate may serve to represent more than one designated group, for example. But Fishburn and Pekeč (2004) and Kilgour (2010) take another approach, simply building representativeness constraints into the class of admissible sets. For instance, if a committee of v men and w women is required, then a $(v + w)$-election is conducted in which the only admissible subsets contain exactly v men and w women. This approach is appropriate, and much more efficient, when distributional requirements are simpler and easier to satisfy.

Ballot restrictions are often imposed in fixed-size committee elections, masking the fact that they are conducted under approval balloting, and imposing strategic constraints on the voters. For example, a ballot that names more than k candidates may be declared *spoiled*, and discarded. An analogous restriction reflects representation conditions, declaring spoiled any ballot with more than the required number of representatives of a designated group. The justification for these rules is uncertain, but their effects are clear; they preclude the common approval voting strategy of voting against one or a few candidates by supporting all others. Thus, with a restricted ballot all ballots are "positive" rather than "negative." However, even if there are ballot restrictions, all ballots are approval ballots, and winners can be determined by any of the procedures discussed below.

4 k-Election Procedures and their Properties

We now introduce procedures for k-elections, where the admissible sets satisfy $\mathcal{A} \subseteq \mathcal{A}_k$ for $1 \leq k < m$. Thus we always require that admissible sets contain k candidates, and sometimes we will require that any such subset is admissible, i.e. that $\mathcal{A} = \mathcal{A}_k$.

We begin with approval voting in single-winner elections. Given a vote profile V, the single-

winner approval voting winners are the members of

$$AV(V) = \arg \max_{j=1,2,\ldots,m} |\{i : j \in V_i\}|.$$

Note that $AV(V)$ is a non-empty subset of $[m]$, consisting of the most approved candidates; the interpretation is that all of these candidates tie for winner of the Approval Voting election.

4.1 Generalized Approval Procedures

A natural generalization of approval voting is based on the idea of a *rep sequence*, which (in the context of k-elections) is a k-vector $r = (r(1), r(2) \ldots, r(k))$ with the properties that $r(1) \geq 0$ and $r(j) \geq r(j-1)$ for $j = 2, 3, \ldots, k$. Thus the rep coefficients $r(j)$ form a non-negative, non-decreasing sequence. The idea is that, from the point of view of voter i, the suitability of an admissible subset S to win the election is $r(j)$, where $j = |S \cap V_i|$. Thus the score of S will equal $r(1)$ times the number of voters who approved of one candidate in S, plus $r(2)$ times the number of voters who approved of two candidates in S, plus $r(3)$ times …, etc.

Definition 1: For fixed k and any $\mathcal{A} \subseteq \mathcal{A}_k$, the *Generalized Approval* procedure based on the rep sequence $r = (r(1), r(2) \ldots, r(k))$ chooses as winning subset(s)

$$GA_k(r, V) = \arg \max_{S \in \mathcal{A}} f_r(S),$$

where $f_r(\cdot)$ is the *rep score*, $f_r(S) = \sum_i r(|S \cap V_i|)$.

To repeat, in a Generalized Approval election, the rep score of each admissible subset $S \in \mathcal{A}$ is determined by adding the contributions from each voter: S gains $r(j)$ if the voter approved of exactly j members of S, for $j = 1, 2, \ldots, k$. Of course, a voter who approved of no members of S contributes nothing to the rep score of S. The Generalized Approval winners are the members of $\mathcal{A}$ that achieve maximum score.

Many approval balloting procedures identified by Kilgour (2010) can be expressed as Generalized Approval procedures using an appropriate rep sequence. The table below is easy to verify.

Procedure	Symbol	$r(j)$
Simple Approval	AV	j
Proportional Approval	PAV	$\sum_{\ell=1}^{j} \frac{1}{\ell}$
p Representatives	REP_p	$\begin{cases} 0 & \text{if } j < p \\ 1 & \text{if } j \geq p \end{cases}$

In the REP_p procedures, p is assumed fixed, $1 \le p < k$. The Simple Approval score of a subset S will be denoted $Sc_{AV}(S)$, the Proportional Approval score $Sc_{PAV}(S)$, and the p-Representative Approval score $Sc_{REP_p}(S)$. Of course, each of these scores is a sum of individual voters' scores. For voter i, the score contribution of an admissible subset S equals the number of members of S approved by i. Under Proportional Approval, it equals 1 if i approved of one member of S, $1 + \frac{1}{2}$ if i approved of two members of S, $1 + \frac{1}{2} + \frac{1}{3}$ if i approved of three members of S, etc. Under p-Representative Approval, it equals 1 if i approved of at least p members of S, and 0 if i approved of fewer than p members of S.

To illustrate, consider Example 2, treating separately the k-elections with $k = 1, 2,$ and 3. In each cell of the results table below, the score of the winning subset(s) is given in parentheses. Note that "all" means that every admissible subset is a winner.

Example 2:
$n = 6$ voters; $m = 5$ candidates

12	23	35	45	123	345

	AV	PAV	REP_1	REP_2
$\mathcal{A} = \mathcal{A}_1$	3 (4)	3 (4)	3 (4)	all (0)
$\mathcal{A} = \mathcal{A}_2$	23, 35 (7)	23, 25, 35 (6)	25 (6)	12, 23, 35, 45 (2)
$\mathcal{A} = \mathcal{A}_3$	235 (10)	235 (8)	125, 134, 135, 234, 235, 245 (6)	235 (4)

It is easy to check that if single-winner approval voting is applied to the vote profile in Example 2, the unique winner is candidate 3, with four approvals. In a 1-election, such as reported in the $\mathcal{A} = \mathcal{A}_1$ row of this table, the admissible sets consist of single candidates. Observe that the results of these elections match the approval voting results for all procedures except REP_2. Of the four procedures shown, REP_2 is the only one that fails the next property.

Definition 2: A procedure for k-elections is *based on approval voting* iff its winning subsets when $\mathcal{A} = \mathcal{A}_1$ are singletons containing exactly the candidates who win in single-winner approval voting.

In a Generalized Approval procedure applied to an election with $\mathcal{A} \subseteq \mathcal{A}_1$, a candidate's score increases by either $r(1)$ or 0 for each voter, according as the voter approved the candidate or not. From this fact, it is easy to show that any Generalized Approval procedure is based on approval voting iff $r(1) > 0$. A Generalized Approval procedure with a rep sequence satisfying this condition is called *proper*. We have shown that all REP_p procedures with $p > 1$ are improper.

Next we consider how the scores of subsets of different sizes are related within a procedure. For Simple Approval in Example 2, it is easy to check for subsets in $\mathcal{A}_1$ that $SC_{AV}(j) = 2, 3, 4, 2, 3$ for $j = 1, 2, 3, 4, 5$. Among subsets in $\mathcal{A}_2$, the maximum AV score, 7, is achieved by 23 and 35 only,

and among subsets in $\mathcal{A}_3$, the maximum AV score, 10, is achieved uniquely by 235. Observe that the score of any subset of candidates is the sum of the scores of the candidates (treated as subsets of size 1).

Definition 3: For fixed $k \geq 1$ and $\mathcal{A} \subseteq \mathcal{A}_k$, a score is *candidate-wise* iff it satisfies $Sc(S) = \sum_{j \in S} Sc(j)$, where $Sc(j)$ is the score of j when $\mathcal{A} = \mathcal{A}_1$.

For instance, it is easy to verify, and confirmed by the calculations above, that the Simple Approval score is candidate-wise. We say that a procedure based on a candidate-wise score is a candidate-wise procedure. Proportional Approval is not a candidate-wise procedure, for observe that $Sc_{PAV}(j) = 2, 3, 4, 2, 3$ for $j = 1, 2, 3, 4, 5$ (since $Sc_{PAV}(S) = Sc_{AV}(S)$ when $|S| = 1$). Thus, $Sc_{PAV}(2) = 3$ and $Sc_{PAV}(3) = 4$, but $Sc_{PAV}(23) = 5 \neq Sc_{PAV}(2) + Sc_{PAV}(3)$. More generally, it can be shown that a Generalized Approval Voting procedure with rep sequence $r = (r(1), r(2), \ldots, r(k))$ is candidate-wise if and only if $r(j) = jr(1)$ for $j = 1, 2, \ldots, k$. Thus, Simple Approval is essentially the only Generalized Approval procedure that is candidate-wise.

In the context of $\mathcal{A}_k$ elections, candidate-wise procedures are the analogues of the additive procedures discussed by Kilgour (2010). They are important because they make it easy to determine all winning subsets.

Theorem 1. In a candidate-wise procedure that maximizes {minimizes} a score $Sc(S)$, the winning subsets in the election with $\mathcal{A} = \mathcal{A}_k$ are exactly the subsets of candidates with the k largest {respectively, smallest} values of $Sc(j), j = 1, 2, \ldots, m$.

Proof: In a candidate-wise procedure, the score of a set S is $Sc(S) = \sum_{j \in S} Sc(j)$. Thus any set of k candidates has maximum {minimum} score iff it contains the k candidates with the largest {smallest} individual scores. □

For example, for AV in Example 2, we noted that $Sc_{AV}(j) = 2, 3, 4, 2, 3$ for $j = 1, 2, 3, 4, 5$, and the unique winner when $k = 1$ is 3. Since $Sc_{AV}(j) = 3$ for $j = 2$ and 5 and $Sc_{AV}(j) < 3$ for $j = 1$ and 4, Theorem 1 implies that the winning subsets when $k = 2$ are 23 and 35. Similarly, the unique winning subset when $k = 3$ is 235, as 2, 3, and 5 form the unique set of three highest scoring candidates.

The relationships among the AV winners of Example 2 suggested above are important. Every winning subset in the $\mathcal{A}_k$ election can be constructed by adjoining a candidate to some winning subset in the $\mathcal{A}_{k-1}$ election. For example, the winners of the $\mathcal{A}_2$ election are 23 and 35, obtained by adding either 2 or 5 to 3, the unique winner of the $k = 1$ election. Similarly, the winner of the $\mathcal{A}_3$ election is 235, constructed by adjoining 2 to 35 (which is one winner when $k = 2$) or by adding 5 to 23 (the other winner when $k = 2$).

Definition 4: A procedure is *upward-accretive* at $k \geq 1$ iff for every winning set $S \in \mathcal{A}_{k+1}$, there

exists $j \in S$ such that $S - j$ is a winning set in $\mathcal{A}_k$.

Likewise, it is possible to ask whether the winners of the election in $\mathcal{A}_{k-1}$ can be constructed from the winners in $\mathcal{A}_k$.

Definition 5: A procedure is *downward-accretive* at $k \geq 1$ iff for every winning set $S \in \mathcal{A}_k$, there exists $j \notin S$ such that $S \cup j$ is a winning set in $\mathcal{A}_{k+1}$.

Thus, a procedure is upward-accretive iff every winning subset in the (larger) $(k+1)$-election can be obtained by adding a candidate to some winning subset in the (smaller) k-election. Consequently, *every* winning subset in the larger election differs by exactly one candidate from *some* winning subset in the smaller election. A procedure is downward-accretive iff every winning subset in the smaller election can be obtained by deleting a member from some winning subset in the larger election. Consequently, *every* winning subset in the smaller election differs by exactly one candidate from *some* winning subset in the larger election. In particular, every candidate who belongs to some winning subset in the smaller election must be a member of some winning subset in the larger election.

Theorem 2. A candidate-wise procedure is upward-accretive and downward-accretive.

Proof: By Theorem 1, the winning subsets in a k-election conducted using a candidate-wise procedure consist of any k highest-scoring candidates, and the winning subset in a $k+1$-election consist of any $k+1$ highest-scoring candidates. Any subset containing the $k+1$ highest-scoring candidates must include a subset containing k highest scoring candidates, so the procedure must be upward-accretive. Similarly, the procedure must be downward-accretive because every subset containing k highest-scoring candidates is contained in some subset containing $k+1$ highest scoring candidates. $\square$

We have already observed that AV is upward-accretive in Example 2, and can confirm directly that it is downward-accretive as well. Theorem 2 indicates that this is not a surprise. However, the results for Example 2 indicate that PAV is downward-accretive but not upward-accretive, as 25 is a winning subset in the 2-election, but neither 2 nor 5 wins the 1-election. Moreover, REP_1 is neither upward- nor downward-accretive, as 3 is the unique winner of the 1-election and 25 is the unique winner of the 2-election. Likewise, REP_2 is not downward-accretive, since 45 wins the 2-election, but 4 does not belong to any subset that wins the 3-election.

Theorem 3. Let $1 \leq k_1 < k_2 \leq m$ and consider elections with vote profile V conducted according to a procedure $Proc$.

(a) Suppose that $Proc$ is upward-accretive and that $S_2 \in Proc(V, \mathcal{A}_{k_2})$. Then S_2 has a subset that is a member of $Proc(V, \mathcal{A}_{k_1})$.

(b) Suppose that $Proc$ is downward-accretive and that $S_1 \in Proc(V, \mathcal{A}_{k_1})$. Then S_1 has a superset

that is a member of $Proc(V, \mathcal{A}_{k_2})$.

Proof: Suppose that $Proc$ is upward-accretive and that $S_2 \in Proc(V, \mathcal{A}_{k_2})$. Then $|S_2| = k_2$, and S_2 must have a subset S_2^1 of size $k_2 - 1$ that belongs to $Proc(V, \mathcal{A}_{k_2-1})$. This completes the proof if $k_2 - 1 = k_1$. Otherwise, S_2^1 must have a subset of size $k_2 - 2$, S_2^2, that belongs to $Proc(V, \mathcal{A}_{k_2-2})$. Moreover, $S_2^2 \subseteq S_2^1 \subseteq S_2$. Continue this iteration for $k_2 - k_1$ steps to obtain a subset $S_2^{k_2-k_1}$ of S_2, of cardinality k_1, that belongs to $Proc(V, \mathcal{A}_{k_2-(k_2-k_1)}) = Proc(V, \mathcal{A}_{k_1})$. This completes the proof of (a). The proof of (b) is similar.□

Recall that a proper Generalized Approval procedure has rep sequence satisfying $0 < r(1) \leq r(2) \leq \ldots \leq r(k)$. Our next example, in combination with Theorem 3, determines one condition under which a proper Generalized Approval procedures may fail to be either upward- or downward-accretive. Consider the ballot data summarized by

Example 3:

$n = 6$ voters; $m = 4$ candidates

12	13	14	2	3	4

It is easy to verify that under any Generalized Approval procedure, the $\mathcal{A}_1$ election must be won by candidate 1, since $Sc_r(1) = 3r(1)$, whereas $Sc_r(j) = 2r(1)$ for $j = 2, 3$, and 4. For the $\mathcal{A}_3$ election, the score of any subset containing 1 is equal to $Sc_r(123) = 2r(2) + 3r(1)$, whereas the score of the only subset excluding 1 is $Sc_r(234) = 6r(1)$. Clearly, the $\mathcal{A}_3$ election is won by 234 uniquely iff $6r(1) > 2r(2) + 3r(1)$, which is equivalent to $r(2) < \frac{3}{2}r(1)$. We conclude that, if a rep sequence satisfies this condition, then the associated Generalized Approval procedure is neither upward- nor downward-accretive. This conclusion can be generalized using other examples, but we note that, since AV is accretive, the greatest possible extent of the generalization is $r(2) < 2r(1)$.

Before turning to other procedures, we introduce one more property that a procedure might possess. A (b, g)-election is an election in which each candidates belongs to one of two disjoint classes, which for convenience we call 'boys' and 'girls.' In a (b, g)-election, the objective is to select a subset consisting of exactly b boys and exactly g girls. There are two natural ways to conduct a (b, g)-election:

- Conduct a $(b+g)$-election in which the only admissible sets are sets containing exactly b boys and exactly g girls;
- Conduct a b-election among the boys and a g-election among the girls and then combine the winning subsets.

The next property specifies that these two processes produce exactly the same (combined) subset(s).

Definition 6: Assume that the set of candidates is $[m] = B \cup G$ where $B \cap G = \emptyset$, $0 < b < |B|$, and $0 < g < |G|$. Define $\mathcal{A}_B = \{S \in B : |S| = b\}$, $\mathcal{A}_G = \{T \in G : |T| = g\}$ and $\mathcal{A}_{BG} = \{R \subseteq [m] : R =$

$S \cup T, S \in \mathcal{A}_B$ and $T \in \mathcal{A}_G\}$. Then a procedure *Proc* satisfies *composition* iff for any vote profile V on $B \cup G$,

$$Proc_{b+g}(V, \mathcal{A}_{BG}) = \{R \subseteq [m] : R = S \cup T, S \in Proc_b(V, \mathcal{A}_B) \text{ and } T \in Proc_g(V, \mathcal{A}_G)\}.$$

Thus, the property of composition implements the idea that when a committee is to represent two different groups (in any proportion), it is immaterial whether the representation requirement is built into the class of admissible sets or whether it is guaranteed by means of separate elections.

Theorem 4. A candidate-wise procedure satisfies composition.

Proof: Assume *Proc* is candidate-wise. By Theorem 1, $Proc_b(V, \mathcal{A}_B)$ contains all sets of b highest scoring candidates in B, and $Proc_g(V, \mathcal{A}_G)$ contains all sets of g highest scoring candidates in G. The union of two such sets must produce a member of $Proc_{b+g}(V, \mathcal{A}_{BG})$, since all of these subsets have the same score, and any $(b+g)$-subset of $[m]$ with a higher score could not possibly be admissible, for otherwise a higher-scoring subset could be found in either $\mathcal{A}_B$ or $\mathcal{A}_G$. $\square$

4.2 Majority Threshold Procedures

Among the threshold procedures proposed by Fishburn and Pekeč (2004), Kilgour (2010) singled out Majority Threshold (MT) and Strict Majority Threshold (SMT) as particularly attractive. A subset of candidates is defined to represent a voter if and only if a majority (or strict majority) of members of the subset are approved by the voter; a winner is an admissible set that represents the most voters. In the context of k-elections, the representation condition is met whenever the subset contains $p = \frac{k}{2}$ (MT) or at least $p = \frac{k+1}{2}$ (SMT) candidates approved by the voter.

Thus, in the context of fixed-size elections, both MT and SMT are equivalent to a REP_p procedures for the appropriate value of p. Moreover, k-elections held under MT and SMT are identical whenever k is odd, for then a subset contains a majority if and only if it contains a strict majority.

Because MT and SMT are REP_p procedures, they are also Generalized Approval procedures, and we have already identified some of their properties. For completeness, we mention that MT fails composition, as can be seen by setting $B = \{1, 2\}$ and $G = \{3, 4\}$ in Example 4 and considering a $(1, 1)$ election. The unique winner of the B-election is 1, the unique winner of the G-election is 3, but both 14 and 23 win the combined election.

Example 4: $n = 6$, $m = 4$

Voter	1	2	3	4	5	6
Ballot	12	12	24	34	13	13

Another interesting example for MT and SMT elections is Example 5, below. Note that the $k = 1$ and $k = 3$ elections are identical, but the $k = 2$ elections are different.

Example 5:
$n = 5$ voters; $m = 4$ candidates

1	12	14	123	234

	MT	SMT
$\mathcal{A} = \mathcal{A}_1$	1 (4)	1 (4)
$\mathcal{A} = \mathcal{A}_2$	12, 13, 14 (5)	12, 23 (2)
$\mathcal{A} = \mathcal{A}_3$	124 (4)	124 (4)

Note that these calculations are evidence that SMT is neither upward- nor downward-accretive. To test for composition, set $B = \{1, 2\}$ and $G = \{3, 4\}$, and consider a $(1, 1)$ election. The unique winner of the B-election is 1 (score 4), 3 and 4 tie in the G election, with score 2, and the unique winner of the combined $(1, 1)$ election is 23, with a score of 2.

4.3 Satisfaction-Related Procedures

Now we define three related procedures which, as we will show later, cannot be represented as Generalized Approval procedures. The first is included in Kilgour (2010), and discussed in detail in Brams and Kilgour (2010). We believe that the second and third are new.

Definition 7: For fixed k and any $\mathcal{A} \subseteq \mathcal{A}_k$, and for any vote profile V, the following three *satisfaction-related* procedures choose as winning subset(s) $\arg\max_{S\in\mathcal{A}} f(S)$:

Procedure	Symbol	Score $f(S)$
Satisfaction Approval	SAV	$f_{SAV}(S) = \sum_i \frac{\lvert S\cap V_i\rvert}{\lvert V_i\rvert}$
Capped Satisfaction Approval	CSA	$f_{CSA}(S) = \sum_i \frac{\lvert S\cap V_i\rvert}{\lvert S\rvert}$
Modified Satisfaction Approval	MSA	$f_{MSA}(S) = \sum_i \frac{\lvert S\cap V_i\rvert}{\min\{\lvert V_i\rvert,\lvert S\rvert\}}$

By convention, a fraction is taken to equal 0 whenever its denominator is 0.

To illustrate these definitions, consider a new example, with results table below.

Example 6:
$n = 9$ voters; $m = 4$ candidates

12	12	2	13	3	3	14	4	4

	SAV	CSA	MSA
$\mathcal{A} = \mathcal{A}_1$	3, 4 (2.5)	1 (4)	1 (4)
$\mathcal{A} = \mathcal{A}_2$	34 (5)	12, 13, 14 (3.5)	34 (5)
$\mathcal{A} = \mathcal{A}_3$	134, 234 (7)	123, 124, 134 (3.33)	134, 234 (7)

The results for Example 6 are direct evidence that SAV is not based on approval voting, that CSA is not candidate-wise, and that MSA is not accretive. To complete the details, note that 1 is the unique approval winner so, of the three satisfaction-related procedures, only SAV is not based on approval voting. Also $f_{CSA}(1) = 4$, $f_{CSA}(2) = 3$, and $f_{CSA}(12) = 3.5$, so $f_{CSA}(12) \neq f_{CSA}(1) + f_{CSA}(2)$, demonstrating that CSA is not candidate-wise. The facts that $MSA_1(V) = 1$ uniquely and $MSA_2(V) = 34$ uniquely imply that MSA is neither upward- nor downward-accretive. The results are is also indirect evidence that MSA is not candidate-wise; if it were, it would be accretive by Theorem 2. Finally, we note that the calculations on which this table is based make it easy to show that the none of these procedures is a Generalized Approval procedure.

Except for not being based on approval voting, SAV is a very well-behaved procedure.

Theorem 5. SAV is a candidate-wise procedure.

Proof. For any $S \in \mathcal{A}_k$ and for any voter i, observe that $|S \cap V_i| = \sum_{j \in S} |j \cap V_i|$. Therefore

$$f_{SAV}(S) = \sum_i \frac{|S \cap V_i|}{|V_i|} = \sum_i \frac{1}{|V_i|} \sum_{j \in S} |j \cap V_i| = \sum_{j \in S} \sum_i \frac{1}{|V_i|} |j \cap V_i| = \sum_{j \in S} f_{SAV}(j),$$

as required. The reversal of the order of summation is justified because all sums are finite. □

As a corollary to Theorem 5, SAV is both upward- and downward-accretive, by Theorem 2.

Although the Capped Satisfaction Approval procedure, CSA, is not candidate-wise, it does turn out to be accretive, as demonstrated next.

Theorem 6. For all k such that $1 \leq k < m$ and all vote profiles V, $CSA_k(V) = AV_k(V)$. In particular, CSA is both upward- and downward-accretive, and satisfies composition.

Proof. For fixed $k > 0$ and any $S \in \mathcal{A}_k$, $f_{CSA}(S) = \sum_i \frac{|S \cap V_i|}{|S|} = \frac{1}{k} \sum_i |S \cap V_i| = \frac{1}{k} f_{AV}(S)$. It follows that any subset $S \in \mathcal{A}_k$ that maximizes $f_{CSA}(S)$ also maximizes $f_{AV}(S)$, and vice versa. Since AV is both upward- and downward-accretive, so is CSA. □

By Theorem 4, candidate-wise procedures such as AV and SAV satisfy composition. So must CSA, since its winning sets are the same as those of AV. In particular, Theorem 6 shows that it is not necessary to be candidate-wise in order to be accretive. Of the satisfaction-related procedures, only MSA fails to satisfy composition, as shown by the following variation on Example 6.

Example $6'$:
$n = 9$ voters; $m = 4$ candidates
$B = \{1,3\}, G = \{2,4\}$

12	12	2	13	3	3	14	4	4

Consider a $(1,1)$-election in Example $6'$ conducted according to MSA. It is easy to verify that a separate 1-election in B, i.e. with admissible sets $\mathcal{A}_B = \{1,3\}$ has unique winner 1, with a score of 4. (The score of 3 is 3.) Similarly, a separate 1-election in G, i.e. with admissible sets $\mathcal{A}_G = \{2,4\}$ produces a tie between 2 and 4, both with score 3. But a 2-election with admissible sets $\mathcal{A}_{BG} = \{12,14,23,34\}$ has unique winner 34, with a score of 5. (The scores of 12, 14, and 23 are 4, 4.5, and 4.5, respectively.) Thus, 1 is the unique winner in the separate election in B, but 1 does not a member of the unique winning subset in the combined election.

4.4 The Maximum Representation Procedure

We now turn to other procedures catalogued in Kilgour (2010). The one called Representativeness is here termed $MAXREP$; its origins are in a proposal by Monroe (1995), adapted to approval balloting by Potthoff and Brams (1998). The following definition is equivalent to theirs.

For any $S \in \mathcal{A}_k$, let $X(S)$ be the set of 0–1 $n \times m$ matrices $x_{i,j}$ defined by the following conditions:

(MR1) For each $i = 1, 2, \ldots, n$, $\sum_{j \in S} x_{i,j} = 1$,
(MR2) For each $j \notin S$, $x_{i,j} = 0$ for all $i = 1, 2, \ldots, n$,
(MR3) For each $j \in S, L \le \sum_{i=1}^{n} x_{ij} \le U$,

where $L = \lfloor \frac{n}{k} \rfloor$ and $U = \lceil \frac{n}{k} \rceil$. Note that if $\frac{n}{k}$ is an integer, then $L = U$, which means that condition (3) can be replaced by

(MR3$'$) For each $j \in S, \sum_{i=1}^{n} x_{ij} = \frac{n}{k}$.

Thus, given any $S \in \mathcal{A}_k$, each $x \in X(S)$ is a 0–1 matrix with the properties that each row contains exactly one 1; if $j \notin S$, every entry in column j is 0; and if $j \in S$, column j contains either L or $U = L + 1$ 1's (unless n is a multiple of k, in which case there are exactly $\frac{n}{k}$ 1's in each column $j \in S$). Thus, any $x \in X(S)$ is a 0–1 matrix containing exactly n 1's, one in each row, all in columns corresponding to S, with (approximately) equal numbers in each such column.

Definition 8: For fixed k, let $\mathcal{A} = \mathcal{A}_k$. For any vote profile V, the *Maximum Representation* procedure ($MAXREP$) chooses as winner(s) the subset(s)

$$\arg\max_{S \in \mathcal{A}_k} \left\{ \max_{x \in X(S)} \sum_{j \in S} \sum_{i=1}^{n} x_{ij} |j \cap V_i| \right\}.$$

To interpret the constraints, note that $x_{i,j} = 1$ indicates that candidate j is elected and is

assigned to represent voter i. Each elected candidate is assigned to at least L and at most U voters, making the representation as close to balanced as possible. (Specifically, each elected candidate represents at least L and at most $U = L + 1$ voters if $\frac{n}{k}$ is not an integer, and $\frac{n}{k}$ if it is.) Every voter is represented, but not necessarily by a candidate the voter approved; the double summation represents the number of voters represented by approved candidate. Potthoff and Brams (1998) present an integer program to determine the choice of x to achieve the maximum required for $MAXREP$.

It is easy to verify directly that the $MAXREP$ procedure is based on approval voting. However Example 4 (from Kilgour (2010)) demonstrates that it is neither candidate-wise nor accretive (in any sense), as $MAXREP_1 = 1$ but $MAXREP_2 = 23$.

Moreover, $MAXREP$ fails composition, as can be seen from Example 4 by setting $B = \{1, 2\}$ and $G = \{3, 4\}$ and considering a $(1, 1)$ election. The unique winner of the election within B is 1 and the unique winner of the election within G is 3, but the unique winner of the combined election is 23.

4.5 Centralization Procedures

Centralization procedures for committee elections with approval balloting are adaptations of an approach used in many problems: Since each voter's ballot can be considered to propose a committee, the most representative committee is the one that is "closest" to the ballots. These ideas were adapted to voting in multi-winner elections by Kilgour, Brams, and Sanver (2006) and Brams, Kilgour, and Sanver (2007). The presentation below is complete, but is original only in that it depends on ballot data (as opposed to compressed ballot data).

The crucial concept is that of distance between subsets. For $S, T \subseteq [m]$, the *Hamming distance* between S and T, $d(S, T)$, is defined by

$$d(S, T) = |S \Delta T| = |S - T| \cup |T - S| = |\,(S \cap T^c) \cup (S^c \cap T)\,|.$$

Thus, the distance between two sets of candidates, S and T, equals the number of candidates in one of S and T but not the other.

For a ballot profile, V, and any $S \in 2^{[m]}$, define $d(S, V) = \sum_i d(S, V_i)$. Then $d(S, V)$ represents the total distance from S to the collection of all ballots. Brams, Kilgour, and Sanver (2004) proved that any committee $S \in 2^{[m]}$ that minimizes $d(S, V)$ must contain every candidate who is supported on more than half the ballots, and cannot contain any candidate who is supported on fewer than half the ballots. The implication is that a Candidate-by-Candidate Majority procedure mentioned can be implemented using a total distance minimization criterion, which was called "minisum." But

such a procedure does not account for admissibility, so it was not included by Kilgour (2010), and will not be considered here.

Since centralization properly takes place within the space of ballots rather than the space of voters, it is appropriate to shift the calculation space to one based on the distinct ballots that are cast in a ballot profile V, and the number of times each one is cast, rather than to remain in a space indexed by voters. Let $W_1, W_2, \ldots, W_{2^m}$ represent any enumeration of $[m]$; let c_h denote the number of times ballot W_h appears in the ballot profile V. Any 2^m-vector $w = (w_1, w_2, \ldots, w_{2^m})$ satisfying $w_h \geq 0$ will be called a *ballot weight vector*. Note that these ballot weights are not normalized; to use them to find a weighted mean, normalization would be required. However, we use weighted distances only for comparison, so we can normalize weights for convenience.

Definition 9: For fixed k, let $\mathcal{A} \subseteq \mathcal{A}_k$. For any vote profile V and any ballot weight w, the *Weighted Minisum* procedure ($MSUM_w$) chooses as winner(s) the subset(s)

$$\arg\min_{S \in \mathcal{A}} \sum_{h=1}^{2^m} w_h d(S, W_h),$$

and the *Weighted Minimax* procedure ($MMAX_w$) chooses as winner(s) the subset(s)

$$\arg\min_{S \in \mathcal{A}} \max_{h=1,2,\ldots,2^m} w_h d(S, W_h).$$

Note that the Minisum and Minimax can be thought of as procedures based on a score, but it is a score that measures unsuitability rather than suitability, and therefore must be minimized.

Theorem 7. For any weight vector, Weighted Minisum is upward- and downward-accretive and satisfies composition.

Proof. Let $S \subseteq [m]$ and suppose that $j_1 \in S$ and $j_2 \notin S$. Then $d((S - j_1) \cup j_2, V_i) = d(S, V_i) + \delta(V_i, j_1, j_2)$, where

$$\delta(V_i, j_1, j_2) = \begin{cases} 0 & \text{if } j_1, j_2 \in V_i \\ 0 & \text{if } j_1, j_2 \notin V_i \\ 2 & \text{if } j_1 \in V_i \text{ and } j_2 \notin V_i \\ -2 & \text{if } j_1 \notin V_i \text{ and } j_2 \in V_i \end{cases}$$

Assume that j_1 achieves a Weighted Minisum score when $\mathcal{A} = \mathcal{A}_1$ that is strictly lower than the score of j_2. Then

$$\sum_{i=1}^{n} w_i d(j_1, V_i) < \sum_{i=1}^{n} w_i d(j_2, V_i) = \sum_{i=1}^{n} w_i (d(j_1, V_i) + \delta(V_i, j_1, j_2)) = \sum_{i=1}^{n} w_i d(j_1, V_i) + \sum_{i=1}^{n} w_i \delta(V_i, j_1, j_2),$$

so $\sum_{i=1}^{n} w_i\delta(V_i, j_1, j_2) > 0$. It follows that

$$\sum_{i=1}^{n} w_i d((S - j_1) \cup j_2), V_i) = \sum_{i=1}^{n} w_i(d(S, V_i) + \delta(V_i, j_1, j_2))$$
$$= \sum_{i=1}^{n} w_i d(S, V_i) + \sum_{i=1}^{n} w_i\delta(V_i, j_1, j_2)) > \sum_{i=1}^{n} w_i d(S, V_i).$$

This is to say that the weighted sum of a subset of size k will be increased if any member of the subset is replaced by another candidate whose score when $k = 1$ is greater. Thus, the winning subset at size k must be a subset consisting of k candidates with the lowest Weighted Minisum scores when $k = 1$. As in Theorems 2 and 4, it follows that Weighted Minisum is accretive and satisfies composition. □

It remains to define appropriate weights. First, the *count weight* vector is defined by $c = (c_1, c_2, \ldots, c_{2^m})$. One procedure recommended by Brams, Kilgour, and Sanver (2007), Minisum Unit ($MSUM_c$), can be based on application of this natural weight vector, which ensures that each voter receives equal treatment. By Theorem 7, $MSUM_c$ is accretive and satisfies composition. The next result shows that $MSUM_c$ is based on approval voting.

Theorem 8. $MSUM_c$ is based on approval voting.

Proof. Let $\mathcal{A} = \mathcal{A}_1$. For any vote profile V, the $MSUM_w$ winners are the candidates $j \in [m]$ satisfying

$$\arg\min_{j \in [m]} \sum_{h=1}^{2^m} c_h d(j, W_h).$$

For any ballot W_h and any candidate j,

$$d(j, W_h) = \begin{cases} |W_h| - 1 & \text{if } j \in W_h \\ |W_h| + 1 & \text{if } j \notin W_h \end{cases}$$

Because $\sum_{h=1}^{2^m} c_h|W_h| = \sum_{i=1}^{n} |V_i|$, it follows that

$$\sum_{h=1}^{2^m} c_h d(j, W_h) = \sum_{i=1}^{n} |V_i| + n - 2|\{i : j \in V_i\}|.$$

Clearly, the minimum of this sum occurs for those values of j that maximize $|\{i : j \in V_i\}|$. This set is exactly $AV(V)$. □

But Kilgour, Brams, and Sanver (2006) found that extreme voters could have a substantial influence on the outcome of a $MMAX_w$ election carried out using count weights, since their influence

is unchanged no matter how many voters cast ballots near the median. They were therefore motivated to reduce the weight of ballots cast by extreme or isolated voters by using *proximity weights*, defined by

$$p_h = \frac{c_h}{\sum_{\ell=1}^{2^m} c_\ell d(W_h, W_\ell)}.$$

The proximity weight for a ballot is a fraction with the count of the ballot (the number of times it was cast) in the numerator, and the total distance to all other ballots, weighted by their counts, in the denominator. Note that the proximity weights of extreme voters are small, because they are farther away from the other voters, and the proximity weight of a ballot is proportional to the number of times it was cast. We recommend only $MSUM_u$ and $MMAX_p$ procedures, since both of which give more influence to commonly-cast ballots cast often, and less influence to extreme ballots.

We illustrate the centralization procedures using Example 4. The ballots with positive counts are 12, 24, 34, 13, and the respective counts are 2, 1, 1, 2. Since $\sum_{h=1}^{2^4} c_h d(12, W_h) = 10$, it follows that the proximity weight of 12 is $\frac{2}{10}$. The complete set of proximity weights is $\frac{2}{10}, \frac{1}{14}, \frac{1}{14}, \frac{2}{10}$. It is convenient to multiply these weights by 70 to normalize them to integers; they become $14, 5, 5, 14$. Then it follows that

	$MSUM_c$	$MMAX_p$
$\mathcal{A} = \mathcal{A}_1$	1 (10)	1 (15)
$\mathcal{A} = \mathcal{A}_2$	12, 13 (10)	12, 13, 14, 23 (28)
$\mathcal{A} = \mathcal{A}_3$	123 (10)	123 (15)

The example shows however that $MMAX_p$ is not upward-accretive, since 23 wins at $k = 2$ but neither 2 nor 3 wins at $k = 1$. It is also not downward-accretive, since 14 wins at $k = 2$ but there is no winning set containing 4 at $k = 3$. The $MMAX_p$ procedure also fails composition, as can be seen by setting $B = \{1, 2\}$ and $G = \{3, 4\}$. Then the B-election produces 1 uniquely, the G-election results in a tie between 3 and 4, but the combined election produces a tie, including not only 13 and 14, but also 23. It is conjectured that $MMAX_p$ is not based on approval voting. In contrast, $MSUM_c$ is based on approval voting, accretive, and satisfies composition, although it is not candidate-wise.

4.6 Sequential Procedures

The first sequential procedure, now known as Sequential Proportional Approval, was proposed by Thiele (c.1890). The essential idea is based on Approval Voting with weighted voters. The idea is to build a committee one member at a time, weighting voters according to how many candidates

they support who are already on the committee. Note that, to produce a k-member committee, k Approval Voting elections must be held, each with new voter weights.

Our contribution to the development of this voting procedure is to note that Approval Voting is not the only way to conduct a single-winner election using approval ballots. In particular, any procedure not based on Approval Voting may produce different winners; we illustrate using Sequential Approval (the original procedure of Thiele (c.1890), based on Approval Voting) and Sequential Satisfaction Approval (in which the individual weighted elections are conducted using a weighted Satisfaction score).

First we define weighted scores of candidates. Let $w = (w_1, w_2, \ldots, w_n)$ be a candidate weight vector, and set

$$f_{w,AV}(j) = \sum_{i=1}^{n} w_i |j \cap V_i| \ ; \ f_{w,SAV}(j) = \sum_{i=1}^{n} w_i \frac{|j \cap V_i|}{|V_i|}$$

for each $j \in [m]$. Thus, the approval or satisfaction of higher-weight voters is emphasized in the scores. As usual, the winner is any candidate who achieves maximum score.

For k such that $1 \leq k < m$, a Sequential Procedure for an election with $\mathcal{A} = \mathcal{A}_k$ can be set out as an iterative procedure (modelled on Kilgour (2010)):

- Begin by setting $\mathcal{C}_0 = [m]$ and defining the weight w^1 by setting $w_i^1 = 1$ for all voters i. Find the weighted score $f_w(j)$ for all $j \in \mathcal{C}_0$. The first candidate added to the winning subset is any candidate $j_1 \in \mathcal{C}_0$ that maximizes $f_w(j)$. Now set $\mathcal{C}_1 = \mathcal{C}_0 - j_1$.

- Suppose that $1 < h \leq k$ and that candidates $j_1, j_2, \ldots, j_{h-1}$ have already been added to the winning subset. The set of remaining candidates is $\mathcal{C}_{h-1}$. Reweight the voters so that the weight of voter i is

$$w_i^h = \frac{1}{1 + |V_i \cap \{j_1, j_2, \ldots, j_{h-1}\}|}.$$

  Now find the weighted score $f_w(j)$ for all $j \in \mathcal{C}_{h-1}$. The h^{th} candidate added to the winning subset is any candidate $j_h \in \mathcal{C}_{h-1}$ that maximizes $f_w(j)$. If $h = k$, stop. Otherwise set $\mathcal{C}_h = \mathcal{C}_{h-1} - j_h$ and repeat.

This sequential process is called Sequential Approval (SEQ_{AV}) if the candidate score used is the Approval score, $f_{w,AV}(j)$, and Sequential Satisfaction (SEQ_{SAV}) if the candidate score used is the Satisfaction Approval score, $f_{w,SAV}(j)$.

Note that any stage, h, of a sequential procedure may produce a tie for the next candidate to be added to the committee, j_h. If so, any of the tied candidates may be added to the committee, resulting in a different committee. Moreover, if $h < k$, the set of candidates for stage $h+1, \mathcal{C}_h$ depends on the candidate added at stage h, producing the possibility of "branching." In general,

any k-subset produced by this process is a sequential committee. To find all sequential committees, one must consider all possible resolutions of any tie at any stage.

We use Example 6 to illustrate that SEQ_{AV} and SEQ_{SAV} are different. In choosing a 2-person committee, SEQ_{AV} selects 1 (score 4) in the first stage, and SEQ_{SAV} selects either 3 or 4 (tied at 2.5). In the second stage, SEQ_{AV} produces a tie between 3 and 4 (second stage scores 2.5), so the 2-committee chosen by SEQ_{AV} is either 13 or 14. In the second stage, SEQ_{SAV} produces 34, whether by starting with 3 and adding 4 (score 2.5) or vice versa. This confirms that SEQ_{AV} and SEQ_{SAV} are different, and that SEQ_{SAV} is not based on approval voting.

By construction, any sequential procedure is both upward- and downward-accretive. But one variation is of interest. If there is a tie at stage $h < k$, the standard procedure insists that any resolution of this tie can be used as a basis for future stages. The modification proposes that, at stage $h + 1$, the candidates who could be added to each of the different versions of C_h should be compared, and among those candidates any associated with lower values of $f_w(j)$ (in stage $h + 1$) should be discarded. It can be shown that, with this change, Sequential procedures remain upward-accretive, but can fail to be downward-accretive. In other words, this example demonstrates that the two directions of accretion are not equivalent.

5 Comparison of Procedures

Table 1: Procedures and Properties

	Based on Approval	Candidate-wise	Upward-accretive	Downward-accretive	Composition
AV	Yes	Yes	Yes Thm. 2	Yes Thm. 2	Yes Thm. 4
PAV	Yes	No Ex. 2	No Ex. 2	No Ex. 6	No Ex.2: $B = \{1,2\}, G = \{3,4,5\}$
REP_1	Yes	No Ex. 2	No Ex. 2	No Ex. 2	No Ex.2: $B = \{1,2\}, G = \{3,4,5\}$
REP_2	No Ex. 2	No Thm. 2	No Ex. 7	No Ex. 7	—
MT	Yes	No Thm. 2	No Ex. 5	No Ex. 5	No Ex. 5
SMT	Yes	No Thm. 2	No Ex. 5	No Ex. 5	No Ex. 5
SAV	No Ex. 6	Yes Thm. 5	Yes Thm. 2	Yes Thm. 2	Yes Thm. 4
CSA	Yes	No Ex. 6	Yes Thm. 6	Yes Thm. 6	Yes Thm. 6
MSA	Yes	No Thm. 2	No Ex. 6	No Ex. 6	No Ex. 6'
$MAXREP$	Yes	No Thm. 2	No Ex. 4	No Ex. 4	No Ex. 4
$MSUM_u$	Yes	No Ex. 4	Yes Thm. 7	Yes Thm. 7	Yes Thm. 7
$MMAX_p$	??	No Thm. 2	No Ex. 4	No Ex. 4	No Ex. 4
SEQ_{AV}	Yes	—	Yes	Yes	??
SEQ_{SAV}	No Ex. 6	—	Yes Yes	Yes Yes	??

6 Conclusions

This paper has surveyed methods of using approval ballots in elections to choose a k-subset of a set of candidates. The selection of a representative committee, with or without other qualifications on membership, is a very common application of approval balloting, with or without restrictions on the ballots. We have begun the identification and classification of approval-balloting procedures for this purpose, and the study of properties that are relevant in this context.

The systems listed here are classed as Generalized Approval procedures, Majority Threshold procedures, Satisfaction-related procedures, the Maximum Representation procedure, Centralization procedures and Sequential procedures. Of the fourteen specific procedures discussed here, two (AV and CSA, were shown (Theorem 5) to always produce identical winners, and another $MSUM_c$, is already known (Kilgour, 2010) to be identical to the Net Approval procedure, one that is related to AV but not included here because it is difficult to interpret in the context of k-elections. As can be seen from Table 1, there is a good base of knowledge of these procedures, though there remain a few details to fill in.

It is also noteworthy that many of the procedures appearing in Table 1 possess identical combinations of properties—at least, of the properties we have studied. Since the procedures are all different, this suggests that more properties must be identified to characterize the procedures. All of the properties studied here, for instance, depend only on one vote profile, rather than compare the results for two related vote profiles, as is required, for example, to define monotonicity. Nonetheless, the construction of a table such as Table 1 seems the right way to proceed, since it facilitates a direct study of procedures. Moreover, combinations of procedures have been recommended (Brams, Kilgour, and Sanver (2007) to break ties. Yet it seems more efficient to study procedures singly, in order that the conclusions be as simple and direct as possible.

It is to be hoped that the theoretical analysis of procedures can be mirrored, eventually, by studies of their performance on large-scale data sets, as initiated, for example, by Brams and Kilgour (2010). This kind of study would facilitate assessment of the likely representativeness of the winners under various procedures, and of their computational requirements. Such comparisons invite analysis of whether the idea of specifying in advance the number of members on a committee—could one do better by making the size of the committee, as well as the membership, an output of the election?

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