Power in Social Influence Dynamics

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Summary. In a simple yet illustrative model basic notions of classical voting theory are extended to serve the goal of identifying the most influential actors in social influence dynamics. Feature of the model is that now the indirect power of those actors in democratic processes without institutional power (like insurgency and lobbyists) can be measured for a given social network structure.

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1 Introduction

Power in democracies is transfered and institutionalized through voting procedures. The voters have direct power in the decision-making process, whereas others have indirect power through influence relations with the voters. This paper contributes to the theoretical analysis of democratic processes by distinguishing and quantifying the role of all actors through their social relations. This paper advances on the traditional power analysis in two ways. Firstly, opposed to *a priori* analysis of voting bodies with independent voters, we explicitly allow for correlated voting behavior through an endogenous process of opinion formation. Secondly, an important feature of the model is the quantitative analysis of power of non-voters as participants in this process. For instance, despite the fact that big industries are not credited with direct voting power, strong activity of lobbyist provides them with indirect political power. Another example is insurgency, which is commonly understood as an unlawful movement, excluded from direct representative power. However, insurgency movements may be united with political parties in common goals – like in Ireland the IRA can be associated with the political Sinn Féin.

We focus on measures of what has been called ”I-power”, which is supposed to measure the influence of a voter over the outcome of a vote (in contrast to the idea of power as division of spoils). The classical measure of I-power proposed by Penrose(1946, 1952)[1] is based on a random- voting model in which each member votes for or against with equal probability independently of all other members. This is not a behavioral assumption but is a method of *a priori* analysis. The latter models the voting system as an abstract shell, without taking into consideration voters preferences, the range of issues over which a decision is taken or the degree of affinity between the voters. This abstraction seems to be necessary to evaluate the decision rule itself (for a more elaborated discussion on a priori voting power see e.g. Felsenthal and Machover 1998).

A common criticism of the widely used Penrose measure is that it fails to take account actual behaviour of voters in the particular voting bodies

[1] also known as the *absolute Banzhaf index*

under analysis. In their 1979, Dubey and Shapley generalize the Penrose measure ψ to the case where all voters vote independently. Let x_a denote the probability of the event that a votes 'yes'. Under the assumption that all x_i are arbitrary and independent for all $i \in N$ of the assembly N their equations (47) and (48) coincide with

$$\psi_a = \frac{\partial A}{\partial x_a}, \tag{1}$$

where A denotes the probability of the event that the outcome of a vote is positive (which Coleman (1971) called the ability of the collectivity to act). Thus ψ_a is the marginal contribution of the propensity of a to vote for a bill to the probability of the bills passing. The problem is now to generalise this idea to the situation when the assumption of independence is dropped. In this paper we seek to construct an empirically relevant power measure by relaxing the idea of independence and show how to replace it by the use of information about real or assumed voting patterns.

We seek a measure of the voting power taking into account exercise of *influence on* voting behaviour of others. To make this idea more precise we shall use social network theory to quantify it. Furthermore, we shall use empirical observations of voting patterns to estimate the architecture of a network. Social network analysis views social relationships in terms of nodes and ties. Nodes are the individual actors within the networks (which are generally individuals or organizations), and ties are the relationships between the actors. There can be many kinds of ties between the nodes as actors might be tied by one or more specific types of relations, such as values, visions, idea, financial exchange, friends, kinship, dislike, conflict, trade, etc. Research in a number of academic fields has shown that social networks operate on many levels, from families up to the level of nations, and play a critical role in determining the way problems are solved, organizations are run, and the degree to which individuals succeed in achieving their goals. The famous work *The Strength of Weak Ties* of Granovetter (1973) is considered as one of the most influential papers ever written in economic sociology. The notion *strength of a tie* is used as an assessment of the intensity of a bond between two actors. Some bonds are obviously stronger than others. Loosely speaking, we might refer to *strong* ties as those between

family members, good friends, colleagues we spend a lot of time with, business partners we happen to work with for a long time, political partners tied together by similar preferences or historical circumstances. In contrast *weak ties* link people who are just acquintances, say, someone we happen to work with ten years ago in Australia.[2] In our paper we show how to estimate the network architecture taking as a starting point the observed frequency distribution over the various theoretically possible possible voting profiles.

Wright (1990) studies the influence of lobbyists on policy decisions in the US House of Representatives. Information about which groups worked together on two controversial issues and which representatives they lobbied was obtained through personal interviews and a mail survey of professional lobbyists. He argues that the influence on representatives' policy decisions is best explained by and depending on the number of lobbying contacts from interest groups on each side of an issue. In particular, this study points out that campaign contributiors proved somewhat useful for explaining groups' lobbying patterns; but it appears to be lobbying, not money, that shapes and reinforces representatives' policy decisions. Wright's paper takes a different approach than the mainstream in the literature which focuses on the role of money. In his work, monetary investments may be represented by certain positions in the social network. We see our work on a posteriori power as complimentary to the work of Wright as it provides a measurement of power as influence of agents who are not necessarily part of the voting body, as in the case of lobbyists.

2 The model

Our model of absence of independent voting can be seen as a black box in which the independent voting probabilities enter as an input. The network determines the influence dynamics inside the black box such that the output is an interdependent voting pattern. Assume a finite set $N = \{1, ..., n\}$ of

[2] Moreover, graph theory (or the mathematical study of abstract representations of networks), can be extended to include negative ties such as animosity among persons. In the present account, however, we shall abstract from negative ties.

agents interacting and let $p = (p_1, ..., p_n)$ denote the vector of independent input approval probabilities, i.e. p_a is the probability that a voter votes 'yes' in absence of any social dynamics. Let $x(p) = (x_1(p), ..., x_n(p))$ denote the resulting probabilities as output voting pattern. In direct extension of 1) we measure power in presence of interdependence as

$$\psi_a = \partial A / \partial x_1 * \partial x_1 / \partial p_a + ... + \partial A / \partial x_n * \partial x_n / \partial p_a. \tag{2}$$

Hence power of a is the ability to influence a voter k who in turn exerts power. Note that this definition doesn't require a to be part of the voting body. Here, the term $\partial x_k / \partial p_a$ measures the sensitivity with which k reacts to changes in a's opinion. In order to compute a power measure we need to specify the terms in (2). As a possible application, we demonstrate specifying the influence terms $\partial x_j / \partial p_a$ using network theory.

REMARK 2.1. *In his 1957 paper, Robert A. Dahl defined power of one individual over another as the extent to which the first can get the other to do something he would not otherwise do, minus the extent that the second can similarly impose his will on the first. While his model is different to ours it is still possible to integrate this idea in form of a rescalation of (2) to*

$$\psi_a = \sum_{i=1}^{n} \left(\partial A / \partial x_k * \partial x_k / \partial p_a - \partial A / \partial x_a * \partial x_a / \partial p_k \right). \tag{3}$$

2.1 Agents and Interaction

Assume the agents interact according to a social network as given by a directed graph. The interaction pattern determining the influence dynamics are captured through an *interaction matrix*. This is a $n \times n$ matrix $W = \left[w_{ij}\right]_i^j$, the elements of which are understood as *influence parameters*; w_{ij} is the weight that agent i places on the opinion of agent j. Note that the matrix may be directed such that $w_{ij} > 0$ while $w_{ji} = 0$. We shall normalize

the weights such that the weights voter i assigns to others sum up to one. Formally this means that the rows of W sum up to one.[3]

2.2 Influence Dynamics

Let $p \in [0,1]^n$ denote the initial approval probability vector. Starting with initial approval probability p_k, voter k updates this probability after one time step according to

$$w_{k1}p_1 + ... + w_{kn}p_n.$$

In general, after t time steps, the approval vector is given by

$$p(t) = W^t p.$$

DEFINITION 2.1. *A matrix W is convergent if $\bar{W} = \lim_{t\to\infty} W^t p$ exists for all vectors p.*

The Appendix provides a characterization of convergence following Golub and Jackson (2007). For what follows we shall assume that W is convergent. We put

$$x(p) = \bar{W}p \tag{4}$$

and hence

$$\partial x_k / \partial p_a = \bar{w}_{ka}. \tag{5}$$

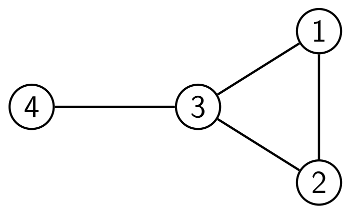
Note that the terms defined by (5) cover exercise of influence in direct but also indirect ways through the network. E.g. it is possible that $w_{ka} = 0$ such that a and k are not directly linked but $\bar{w}_{ka} > 0$. The latter is taken to mean

[3]In general parlance W is a stochastic matrix.

that there is a *path* between a and k such that a can influence k by a cascade of influencing intermediaries.

Example 1: Consider 4 agents connected by the following network:

$$W = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$



Here,

$$\bar{W} = \lim_{t \to \infty} W^t = \begin{pmatrix} 1/4 & 1/4 & 1/3 & 1/6 \\ 1/4 & 1/4 & 1/3 & 1/6 \\ 1/4 & 1/4 & 1/3 & 1/6 \\ 1/4 & 1/4 & 1/3 & 1/6 \end{pmatrix}$$

Here, agent 4 doesn't attach any weight to the opinion of agent 1 as indicated by $w_{41} = 0$. However, even though he attaches some weight to agent 3 who is in turn connected to 1 and 2. As a result agent 4 is in fact influenced by agent 1 as indicated by $\bar{w}_{41} = 1/4 > 0$.

It now remains to specify the terms $\partial A / \partial x_k$ in equation (2). In absence of any information apart from the network put $p_k = 1/2$ for all $k = 1, ..., n$.[4] Since $\bar{W}$ is a stochastic matrix it is easy to see from (4) that $x_k = 1/2$ for all

[4]Here, we follow the Bernoullian Principle of Insufficient Reason which claims that each of the alternatives should have equal probability if there is no known reason for assigning unequal ones.

$k = 1, ..., n$. However, at the latter point the term $\partial A / \partial x_k$ coincides with the (a priori) absolute Banzhaf measure β_k such that (2) follows as

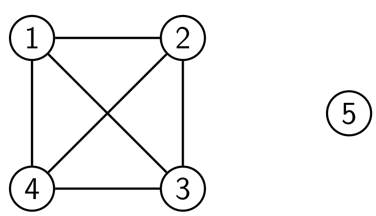
$$\psi_a = \beta_1 * \bar{w}_{1a} + ... + \beta_n * \bar{w}_{na} \tag{6}$$

Equation (6) is a direct extension of (1) in absence of independence. Power of a is the sum taken over the ability to exert power through other agents k; the ability to influence a voter k,as measured by $\bar{w}_{ka}$, who in turn exerts power as measured by β_k.

Example 1: Here, since all rows of $\bar{W}$ are given by $(1/4, 1/4, 1/3, 1/6)$ power of *any* voter a follows as

$$\psi_a = 1/4\,\beta_1 + 1/4\,\beta_2 + 1/3\,\beta_3 + 1/6\,\beta_4 \tag{7}$$

Example 2 (Machover (2007)): Consider the canonical simple majority decision rule with an assembly of 5 voters: $I_5 = \{1, 2, 3, 4, 5\}$. Let P be the probability distribution that assigns probability 0 to the 20 divisions of I_5 in which the positive camp contains exactly two or exactly three voters; and equal probability of $1/12$ to each of the remaining 12 divisions. The question is now how to translate this voting patter P into our framework. Can we take P to say something about the architecture of interaction, here, the network? In fact, in network terms P gives rise to the following structure



This implies that four agents, say 1,2,3 and 4, are strongly correlated and also vote coordinated as a bloc while agent 5 is isolated - which explains probability 0 of divisions in which the positive camp contains exactly two or exactly three voters. Hence the bloc of 1 to 4 vote independently from

agent 5 such that P describes the probability of in fact two independent voters which happen to vote similarly by mere chance. In absence of any further information we put

$$W = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $W = \lim_{t\to\infty} W^t$. Further note that the canonical simple majority decision rule implies the a priori abolute Banzhaf measures $\beta_a = 3/8$ for all $a \in I_5$. Hence (6) simplifies to

$$\begin{aligned} \psi_a &= 3/8\,[w_{1a} + ... + w_{5a}] && (8) \\ &= 3/8 && (9) \end{aligned}$$

for any a. At first sight, it might come as a surprise that power is the same for all voter although the network is asymmetrical. Note, however, this is due to the symmetric weight distribution within the bloc $\{1,2,3,4\}$ which has a counterbalancing effect. Every voter within this bloc loses power by being influenced but equally gains power by exerting influence.

Example 3: Consider again the canonical simple majority decision rule with five players but now with

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/10 & 7/10 & 1/10 & 1/10 & 0 \\ 1/10 & 1/10 & 7/10 & 1/10 & 0 \\ 1/10 & 1/10 & 1/10 & 7/10 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Here,

$$\bar{W} = \lim_{t\to\infty} W^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which yields

$$\begin{aligned} \psi_1 &= (3/8) * 4 = 3/2 && (10) \\ \psi_2 &= \psi_3 = \psi_4 = 0 && (11) \\ \psi_5 &= 3/8 && (12) \end{aligned}$$

Here, voter 1 absorbs all the power within the bloc $\{1,2,3,4\}$. Note that $\psi_1 > 1$ due to the high sensitivity with which all voters within this bloc react to 1's influence.

3 Appendix - Characterization of Convergence

3.1 Walks, paths and cycles

The following are standard graph-theoretical definitions applied to the directed graph of connections induced by the interaction matrix W.

A *path* in W is a sequence of nodes $B = i_1, i_2, \ldots, i_K$, not necessarily distinct, such that $W_{i_\ell i_{\ell+1}} > 0$ for all $\ell \in \{1,2,\ldots,K\}$. Then B is called a path from i_1 to i_K. B is a *simple path* if each of its nodes occurs only once. A *cycle* is a path $i_1, i_2, \ldots, i_K$ such that $i_1 = i_K$. The *length* of the cycle is defined to be $K-1$. A cycle is *simple* if the only node appearing twice in the sequence is the starting (ending) node. The matrix W is *strongly connected* if there is

a path relative to W from any node to any other node. Similarly, we say that a group of agents $T \subset N$ is *strongly connected* if the restriction of W to T is strongly connected. Then this is equivalent with the fact that there is a cycle in W containing only the nodes in T. A group of nodes $T' \subset T$ is *closed* relative to W if $i \in T'$ and $W_{ij} > 0$ implies $j \in T'$. A closed group of nodes T' is *minimal* relative to W if no nonempty strict subset is closed. Observe that any minimal closed group is strongly connected.

DEFINITION 3.1. *A matrix W is* convergent *if $\lim_{t\to\infty} W^t x$ exists for all vectors x.*

DEFINITION 3.2. *The matrix W is* aperiodic *if the greatest common divisor of the lengths of its simple cycles is 1.*

It is well-known that if W is strongly connected (also referred to as being irreducible) and aperiodic, then it is convergent (e.g., see Meyer (2000)). Golub and Jackson (2007) offer an example to see what can go wrong when aperiodicity fails.

Example:

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here,

$$W^t = \begin{cases} W \text{ if } t \text{ is odd,} \\ I \text{ if } t \text{ is even.} \end{cases}$$

In fact, Golub and Jackson (2007) show that for strongly connected W aperiodicity is also a necessary condition.

THEOREM 3.1. ***(Golub and Jackson (2007))*** *If a stochastic matrix W is stongly connected, then it is convergent if and only if it is aperiodic.*

However, since most social interactions will not involve strong connections the authors provide the following important generalization.

DEFINITION 3.3. *The matrix W is* strongly aperiodic *if it is aperiodic when restricted to any closed group of nodes.*

THEOREM 3.2. *A stochastic matrix W is convergent if and only if it is strongly aperiodic.*

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