Modified Power Indices for Indirect Voting

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Abstract

We consider here a voting situation such as that exemplified by the Electoral College (EC) in United States presidential elections. While our model should be suitable for other indirect elections (e.g., the EU Council of Ministers), most of our arguments will be directly based on the EC and on the results of recent elections. We show that the classical indices of voting power (Shapley-Shubik and Banzhaf-Coleman) give counter-intuitive results because they do not take differences among states, and correlations within these, into account. We show how these differences and correlations can be modeled.

Introduction

The Electoral College remains a controversial feature of U.S. political decision-making. After most U.S. presidential elections, there are calls for passage of a constitutional amendment to either abolish it or to "reform" it substantially. There are numerous complaints about the Electoral College, of which the most important is the potential for the winner of the Electoral College majority to be a popular vote loser. Consider three assertions that often surface in the debates about the political impact of the Electoral College.

First, the Electoral College is alleged to benefit the smaller states. Here the argument is simply that the failure of the Electoral College to satisfy the "one person, one

vote" standard by overweighting the seat shares of the smaller states disproportionately advantages those states in terms of their influence on presidential outcomes. An implication of this claim is that, *ceteris paribus*, candidates should spend more time and money campaigning in the smaller states than their populations would otherwise justify.

Second, the winner-take-all feature of statewide voting used in the Electoral College by 48 of the 50 states (and the District of Columbia) is alleged to <u>benefit the larger states</u>. Here, we have the argument, based on game theoretic ideas about pivotal power, that the Electoral College should disproportionately focus candidate attention on the largest states, since it is claimed that, *ceteris paribus*, the citizens in those states have a likelihood of being pivotal in the election in terms of turning a losing coalition of states into a winning one that is more than proportional to their state's share of Electoral College votes (Brams and Davis, 1974).

Third, it has recently been suggested that the Electoral College operates to <u>benefit the states experiencing close contests for the presidency</u>, by focusing candidate attention only on the relative handful of potentially competitive states, leaving much of the country barely aware that a presidential election is going on.

It might appear obvious that all these assertions cannot be true. In particular, it is far from intuitive how the Electoral College might structure incentives so as to simultaneously make it more likely that candidates would campaign in both the largest states and the smallest states at levels higher than the population of those states would seem to merit. Yet, as we will see, we can construct models in which this underrepresentation of the states of middling population can occur. However, unless closeness and size are perfectly correlated, or unless the effects of state size and level of competition on campaign investments act in a completely additive fashion, then we need to follow up on a point made in Brams and Davis [1974: 132] about the desirability of relaxing the restrictive assumption they make that each state's already decided voters are divided equally between the two parties on use of poll data about closeness.

An important distinction to make here is that between *a priori* voting power, which is based entirely on the laws and description of the voting process, and the *actual power* which depends on likely coalitions. Since we are specifically considering political

processes, it is clear that certain coalitions (e.g., coalitions among voters with similar ideology, or among voters living in a given area) are more likely than others. We discuss both types of power, but will give modifications mainly for the second (practical) case.

There are several power indices in the literature. The best known are the Shapley-Shubik [1954] and Banzhaf [1965]/[Coleman [1971] indices. Application of these indices shows that the larger states (California, New York, etc.) have substantially greater power than one would normally expect. Owen [1975, see also Owen 1995, p. 302] found that – on the basis of 1970 census figures – a voter in California had, a priori, 2.86 times as much power as a voter in North Dakota. This was so even though North Dakota had then 2.87 times as many electoral votes *per capita* as California. The question is whether this rather unintuitive result is reasonable; if not, we would like to suggest modifications.

We will limit ourselves to discussion and modifications of the Shapley-Shubik index. Other power indices give similar results; it is not necessary to discuss them here.

Multilinear Extensions

Our approach to the power index will be based on the multilinear extension [Owen 1972]. Let (v, N) be an n-person TU game in characteristic function form. Then the multilinear extension

$$(1) \qquad F(q_1, q_2, \ldots, q_n) = \sum_{S\subset N} \{\Pi_{j\in S}\, q_j\, \Pi_{j\in N-S}(1-q_j)\}\ v(S)$$

represents the expected worth, $\mathrm{E}[v(\varsigma)]$, of a random coalition ς, given that each player, i, has probability q_i of belonging to the coalition, and that all these probabilities are independent. The partial derivative $F_i = \partial F/\partial q_i$ represents the expected marginal contribution, defined as $v(\varsigma\cup\{i\})-v(\varsigma-\{i\})$, of player i to this random coalition.

Now the Shapley value can be obtained by the formula, from [Owen 1972],

$$(2) \qquad ш_i[v] = \int_0^1 F_i(t, t, \ldots, t)\, dt$$

in which the Russian letter ш stands for the Shapley value. This formula can be interpreted by the following parable: the n players in a game have agreed to meet in a given place, at a given time. Because of random fluctuations in watches, unforeseen

delays, etc., they in fact arrive in some random order. Each one's arrival time is a random variable, X_i; these n random variables are independent and have identical distribution. So long as this is a continuous distribution, there is no loss of generality in assuming that it is a uniform distribution over the unit interval.

We assume, as described in [Shapley 1953], that, on arrival, player i is paid his marginal contribution to the coalition consisting of those players who have already arrived. Then the value $u_i[v]$ is precisely player i's expected payment.

Suppose, however, that the n players' arrival times are not identically distributed. (One is habitually tardy; another is an early riser, etc.) We let g_i be the cumulative distribution for i's arrival time, i.e., $g_i(t) = \text{Prob}\{X_i \leq t\}$, and assume these variables are independent and absolutely continuous (so each can be represented by a density function g_i'). Then

(3) $$\psi_i = \int_{A}^{B} F_i(g_1(t), \ldots, g_n(t))\, g_i'(t)\, dt$$

is the expected payment to i under these assumptions.
[Note: In formula (3), A and B should be chosen so that all $g_i(A) = 0$, and all $g_i(B) = 1$. Since this may not be practical (e.g., some distribution may have infinite support), we merely require that they be close to 0 and 1 respectively.]

To see how this works, we give two easy examples, with three players each, and normal distributions for their arrival times.

Example 1. Consider a three-person situation, where any two of the voters form a winning coalition. In this case, the multilinear extension is given by

$$F(q_1, q_2, q_3) = q_1q_2 + q_1q_3 + q_2q_3 - 2q_1q_2q_3 .$$

The partial derivatives here are

$$F_1 = q_2 + q_3 - 2q_2q_3$$

and similarly for the other two.

Let the three voters' times of arrival be normally distributed, with means and standard deviations

$$\mu_1 = .4,\ \sigma_1 = .1$$
$$\mu_2 = .5,\ \sigma_2 = .2$$
$$\mu_3 = .7,\ \sigma_3 = .1$$

We will let

$$g_i(t) \;=\; \text{Prob}\ \{X_i \leq t\} \;=\; \Phi(\ (t\text{-}\mu_i)/\sigma_i)$$

where Φ is the standard normal distribution function. Thus

$$g_1(t) \;=\; \Phi(10t\text{-}4)$$
$$g_2(t) \;=\; \Phi(5t\text{-}2.5)$$
$$g_3(t) \;=\; \Phi(10t\text{-}7)$$

Note that, for all three, we have $g_i(0)$ very close to 0, and $g_i(1)$ very close to 1. Thus it should suffice to let A = 0 and B = 1 in our integration formula above. (If a more precise result were necessary we could let A = -1 and B = 2.)

From the above, we obtain the densities (derivatives)

$$g_1'(t) \;=\; 10\ \varphi(10t\text{-}4)$$
$$g_2'(t) \;=\; 5\ \varphi(5t\text{-}2.5)$$
$$g_3'(t) \;=\; 10\ \varphi(10t\text{-}7)$$

where φ is the normal density function, $\varphi(x) = (2\pi)^{-½} \exp\{-x^2/2\}$.

The integration formula leads to the result ψ = (.323, .490, .187). Thus, player 2, whose expected time of arrival is in the middle, has the advantage, though he will frequently (more than half the time) shift out of the middle position. Player 1, whose expected position is more moderate than that of player 3, does in fact considerably better than 3.

Note that this effectively assumes motion "from left to right", with a coalition forming as the left-most members (those closest to 0) join first, then those in the middle, and finally those on the right (closest to 1). We can imagine as well motion from right to left (the reverse order), but in fact this gives the same results as before. This is to be expected where the voting game is *decisive*: for every coalition *S*, either *S* or *N-S* (but not both) is a winning coalition. For such games, an order and the reverse order give the same result.

Example 2. Consider a similar three-person situation, with the same winning coalitions and the same multilinear extension. The difference will be in the three voters' times of arrival, now characterized by

$$\mu_1 = .5,\ \sigma_1 = .1$$

$$\mu_2 = .5,\ \sigma_2 = .2$$

$$\mu_3 = .5,\ \sigma_3 = .05$$

We continue as in Example 1. The integration formula now leads to the result ψ = (.369, .166, .465). In this case, we find that player 3 is favored, mainly because his smaller variance means he will generally be closer to the center of the distribution, and thus more frequently in the middle, between the other 2. But note that, if the voting game required unanimity, the situation would be quite different: in this case, we would find ψ = (.316, .417, .267).

Thus, where the expected times of arrival are different (as in example 1), those players with expected positions near the median will be advantaged. Where the expected arrival times of the players are all equal (as in example 2), and a simple majority of the votes is necessary to win, the player with smaller variance is generally advantaged. (On the other hand, with a supermajority necessary, the situation may well be different.) What is not obvious from the example is that, when there are many players, the advantage will (asymptotically) be inversely proportional to the square root of the variance.

The Electoral College

Let us see how this applies to the Electoral College. There are n players (states), with differing numbers of electoral votes, depending on the state's population. Let v be the n-person game among the states. Let m_j be the number of voters in state j, and let r_j be the number of votes needed to determine the state's electors, which is in this case $(m_j+1)/2$. We shall let w_j represent the simple game, with m_j players, in which the minimal winning coalitions are precisely those with exactly r_j members. Let $G_j(y_1, \ldots, y_m)$ be the multilinear extension of game w_j, and define

(4) $$g_j(t) = G_j(t, \ldots, t).$$

It is easy to see that $g_j(t)$ is, in this case, the probability that a binomial random variable with parameters m_j and t be <u>at least</u> equal to r_j. What this means is that, if each of the m_j voters arrives according to a uniform distribution in the unit interval, $g_j(t)$ is the probability that a majority (or more of these has arrived not later than time t: in other words, it is the probability that state j will have its "time of arrival" not later than t. See [Owen 1975] for this.

Assuming m_j large, we can approximate this binomial probability by the normal distribution with mean $t\, m_j$ and variance $t(1-t)m_j$. Approximately, then,

(5) $$g_j(t) = \Phi\left(\frac{t\, m_j - r_j}{\sqrt{t(1-t)m_j}}\right)$$

Next, we calculate the function $F(g_1, \ldots, g_n)$. Let state j have w_j electoral votes. Then, since $g_j(t)$ is the probability that state j arrives on or before time t, then the number of

electoral votes that state j will have contributed by time t can be thought of as a random variable with mean $w_j g_j(t)$, and variance $w_j^2 g_j(t)(1-g_j(t))$. It will follow that the number of electoral votes Y in the random coalition ζ has mean

$$M_Y(t) = \sum_j w_j g_j(t)$$

and variance

$$\sigma_Y^2(t) = \sum_j w_j^2 g_j(t)(1-g_j(t)).$$

Given the number of states, it is possible to approximate Y by a normal random variable having the same mean and variance. Thus F can be expressed in terms of the normal distribution function Φ, and the calculation is then a straightforward problem in integration (easily carried out with current computer packages, using Simpson's rule). The reader is invited to read [Owen 1975] for details of this integration.

It can also be seen that, since the ratios r_j / m_j are all nearly equal to ½, then the change of variable

$$\tau = (t - \tfrac{1}{2}) / \sqrt{[t(1-t)]} \tag{6}$$

gives us the much simpler expression

$$g_j = \Phi(\tau \sqrt{m_j}) \tag{7}$$

where Φ is the normal distribution function. Thus, the effect of difference in population translates into a difference in variance: the time of arrival of state j is now a normal random variable with mean 0 and variance $1/m_j$.

(It should be noted that, under this change of variable, the values 0 and 1 for the original variable, t, transform into $-\infty$ and $+\infty$, respectively, for τ. The integral (3) becomes an improper integral, but, in practice, it should suffice to let τ run from $-K$ to $+K$, where K is large enough so that $K\sqrt{m_i}$ is of the order of 3 for the smallest of the constituencies.)

Now, the Shapley value assumes that (a) all voters have their "time of arrival" identically distributed and (b) these are independent, so that, in fact, formula (1) above holds. We will call this the **hypothesis of universal population homogeneity and independence (HUPHI)**. This means that all the **state** positions have identical means, with variance inversely proportional to the population. The effect of this is that the larger states are more likely to be near the center, and, as in Example 2 above, are more likely to be pivotal **if a simple majority of the constituency weights is necessary to win**. If, on the other hand, a super-majority (say, two thirds) is necessary, then these larger states are less likely to be pivotal (though of course they might still be stronger, simply because their voting weights are greater).

First Modification: Introduction of Undecided Voters

Suppose, now, that the HUPHI does not hold. Different states have different distributions for their populations. We shall use a relatively simple model for this; there are of course several other possibilities.

We assume that, in each state, part of the population is definitely on the left, part is definitely on the right, and the remaining voters (the undecideds) are the ones in play. In our "time of arrival" parable, the left wing arrives immediately at time 0, the right wing arrives at time 1, and the undecideds arrive according to a uniform distribution in the unit interval. as before, we wish to find g_j, the probability distribution of X_j, state j's time of arrival.

Let m_j be the voting population of constituency j, and let a_j and b_j be the population of the left- and right-wing blocs respectively. Then the undecideds are $c_j = m_j - a_j - b_j$. Assume, as before, that $r_j = (m_j+1)/2$ votes are needed to carry the constituency; then the left-wing party requires $s_j = r_j - a_j$ votes from among the undecideds.

(It may, of course, happen that $a_j \geq r_j$. In such case the left-wing party will certainly carry the state, X_j will be equal to 0, g_j will be constantly equal to 1, and the voters in this state will, in our analysis, have zero power. Similarly, if $b_j \geq r_j$, then the right-wing party will certainly win the state, X_j will be equal to 1, g_j will be constantly equal to 0, and once again the voters here have no power.)

We will therefore consider only states for which r_j is greater than both a_j and b_j. Continuing as above, we would then find that the state "arrives" when exactly s_j of the c_j undecideds have arrived:

$$(7) \qquad g_j(t) = \Phi\left(\frac{tc_j - s_j}{\sqrt{t(1-t)c_j}}\right)$$

If we let $\gamma_j = s_j / c_j$, we obtain

$$(8) \qquad g_j(t) = \Phi\left(\frac{(t-\gamma_j)\sqrt{c_j}}{\sqrt{t(1-t)}}\right)$$

The density is then given by

$$(9) \qquad g_j'(t) = \frac{(t + \gamma - 2\gamma t)\sqrt{c_j}}{2(t-t^2)^{3/2}} \ \varphi\left(\frac{(t-\gamma_j)\sqrt{c_j}}{\sqrt{t(1-t)}}\right)$$

To obtain the modified power, we would then modify the calculations in [Owen 1975, see also 1995, pages 298-299], using these values for the functions g_j. In principle, there is no great difficulty – given the existence of mathematical packages for computers – in carrying out the necessary integration. Some care must of course be taken to avoid values of t which are too close to 0 or 1, but such values would be of importance only if s_j is extremely close to either 0 or c_j, and these states will, in our model, have negligible power. We wish, however, to look at some qualitative properties of this modified index, making a simplification similar to the one above (equation 7).

Unfortunately, since the γ_j are not all equal, the simplifying transformation (6) is not available. We note, however, that $g_j(\gamma_j) = ½$, so that the median entry time for constituency j is γ_j.

The calculations at this point become somewhat complicated. Nevertheless, it is not too difficult to prove that, if two constituencies, i and j, have $\gamma_i < \gamma_j$, and undecided populations $c_i < c_j$ respectively, then

$$\text{Prob}\{X_i > X_j\} \leq \Phi\left([\gamma_i - \gamma_j]\sqrt{(2c_i)}\right).$$

If we further assume that c_i is of the order of 20,000 (a rather small number given the size of today's electorates), we find that

$$\text{Prob}\{X_i > X_j\} \leq \Phi\left(200[\gamma_i - \gamma_j]\right).$$

Now, we know that Φ is a strictly increasing function, with $\Phi(-2) = 0.023$. Thus, if $\gamma_j - \gamma_i > 0.01$, we find that $\text{Prob}\{X_i > X_j\} < 0.023$. We conclude that, unless the quantities γ_i are very close for several states, the probability of an out-of-order arrival is extremely low, and thus almost all of the voting power will reside in a small number (possibly two or three) of median states.[1]

Discussion of the Above Indices

As we have seen, the classical Shapley value (or for that matter the Banzhaf-Coleman index) gives excessive *a priori* value to the larger constituencies, essentially because the HUPHI (hypothesis of universal population homogeneity and independence) causes all constituencies to have the same expected time of arrival, with smaller variances for the larger ones. When the voting game requires a simple majority (i.e., one half plus one) of the weighted votes, a smaller variance increases the probability that a state lie in pivotal position. (See example 2 above.) It is of course also true that, for the Shapley value – though not the B-C index – a requirement for a super-majority will weaken the state with smaller variance.

On the other hand, the inclusion of undecideds, etc., which we mentioned above, will give differing expected times of arrival for the states. However, it allots almost all the power to a very few constituencies – those with expected time of arrival nearest the median. It is as if, in this last (2004) presidential election, we need only consider four states: Ohio with 20 votes, Nevada, 5, New Mexico, 5, and Iowa, 7. To see this, note that Ohio was in fact the pivotal state. According to our analysis above, only states with a two-party division of the vote within 1% of Ohio would be considered possible pivots: these are precisely the four states mentioned. Among the remaining states, Bush had 249 votes, while Kerry had 252. To win, Bush needed 269 votes, and Kerry, 270. (This

difference is due to the fact that, in case of an Electoral College tie, the election would be resolved in the House of Representatives, where the Republicans had a sizable majority of the states.) Thus whoever carried Ohio would win the election. We should perhaps include Florida and Wisconsin, where the division differed from that of Ohio by about 1.4 %. Then Ohio would not have all the power, but (as discussed above) the probability that Florida would end up to the left of Ohio, or Wisconsin to the right, are of the order of less than 2%. We would conclude that Ohio had perhaps 98% of the total power; the remaining 2% would be divided among Florida, Wisconsin, and the three other states mentioned above. The other 44 states, and the District of Columbia, would have truly negligible power.

Now, it is our belief that the greatest problem with the indices thus far considered is that they fail to take into account correlation among the voters of a given state. To see why correlation is important, consider the following straw man:

Suppose that, at birth (or on naturalization) each citizen of the United States were assigned a two-letter symbol. There are 51 of these, and they are assigned according to a certain probability distribution. For example, the symbol CA has probability 0.12, the symbol WY has probability 0.017, etc. Otherwise the assignment is totally random: an individual's symbol has no relation to his family or place of birth, and even twin siblings can have symbols as totally different as MA and UT.

These symbols seem to be quite meaningless, except that they are kept in the electoral rolls. Then, for a presidential election, votes are totaled according to symbol, Each symbol is then given a number of "supervotes", proportional to the original probability distribution, and these will be assigned (on a winner-take-all basis) to the candidate with most votes from among those with a given symbol.

Now, it is hard to imagine that anyone would espouse this proposal. So how is it different from the Electoral College? The answer is that in the Electoral College, people are grouped together by their state of residence rather than by some meaningless pair of letters (akin perhaps to the last two digits in the social security number). Because of this, there is meaningful correlation (of opinion) among voters who are grouped together in

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[1] The authors will be happy to provide details of this analysis.

the Electoral College system, but not in our straw man proposal. However, the classical voting indices essentially assume that voters' decisions are totally independent and thus no different from the situation in the straw man proposal.

An Alternative Approach

Granted that the two methods described above give counterintuitive results, we feel that a different approach, assuming substantial correlation among nearby voters, and especially among voters within a given state, should be considered. With such a correlation, the variance of the state arrival times would be considerably larger than given by our model – certainly much larger than $1/m_i$ or even $1/c_i$. As Gelman/Katz/Bafumi [2004] point out, empirical evidence suggests that the variance is closer to $m_i^{-0.2}$.

To model the correlation, we represent this by a partial differential equation, where we represent the voters by points, x, on a line, and time by t. The coordinate x corresponds to physical location: if $|x-y|$ is small, then voters x and y are "neighbors" who can talk to each other. Assume that there are two parties, which we generically call the right-of-center and the left-of-center. We let $u(x, t)$ represent voter x's state of mind (his feeling towards the two parties) at time t. Specifically, we assume that x has a "usual" state. Then $u(x, t) > 0$ means that, at time t, x is more likely than usual to vote for the right-of-center party; similarly, $u(x, t) < 0$ means he is more likely than usual to vote for the left-of-center.

Now, we assume that voter x is influenced by the voters near him, and has a tendency to move in the same way that they do. We will represent this by the equation

$$(10) \qquad \partial u/\partial t \; = \; k\, \partial^2 u/\partial x^2 \; + \; f(x, t)$$

$$u(x, 0) \; = \; g(x)$$

$$-\infty < x < \infty,\ 0 < t < \infty$$

where k is a constant of proportionality, corresponding to the speed with which political news and opinions spread among neighbors, and where the forcing term, f, represents

external events, and may even be split into two parts, one corresponding to random events, and the other to efforts by one or the other of the two parties.

To estimate the correlation, assume there is no variation until time 0. Assume then a unit shock, concentrated at some point $x = x_o$, at time $t = 0$. This corresponds to an initial condition

$$u(x, 0) = \delta(x-x_o)$$

where δ is the Dirac delta function. Assume also that there is no forcing term, i.e. $f = 0$.

The knowledgeable reader will recognize the above as the heat equation that appears in most courses in partial differential equations. It has the solution

$$u(x, t) = (4\pi kt)^{-1/2} \exp\{-(x-x_o)^2 /4kt\} \tag{11}$$

Of course this shock could happen at any point x_o within the state. There may be several such shocks, at possibly different times, but we assume these are uncorrelated. Thus the correlation between different voters can be given by the autocorrelation of this function. Some analysis tells us that the relative correlation is

$$\rho(x, y; t) = \exp\{-(x-y)^2/8kt\}. \tag{12}$$

Now, for small values of t, this will be a very sharp curve, with a strong maximum at $x = y$, and falling to 0 very quickly. For large t, on the other hand, his will be a very flat curve.

Suppose, now, that the population of a constituency occupies a line segment of length c, which we can assume, without loss of generality, to be the interval $[0, c]$. Assume also that each individual's political stance has a variance 1. Then the variance of the sum of their positions is given by

(13) $$\int_0^c \int_0^c \exp\{-(x-y)^2/8kt\}\,dx\,dy$$

which, after some calculations, reduces to

(14) $$\mathrm{Var} \;=\; c\,(8\pi kt)^{½}\mathrm{erf}\{c(8kt)^{-½}\} - 8kt\,[1 - \exp\{-c^2/8kt\}]$$

and the variance of the mean is this same quantity divided by c^2.

[Note: if each individual's political stance has a variance of σ^2 rather than 1, then the above result should be multiplied by σ^2.]

We find here that for small c (small compared to kt), then Var, as given above, is almost equal to c^2. Thus the variance of the mean is only slightly smaller than σ^2, the variance of position for an individual voter. On the other hand, for large c, Var can be significantly smaller than c^2, and is asymptotically proportional to c. To see this behavior, we calculate the quantities Var and Var/c^2 for several values of c, and for kt = 12500:

c	Var	Var / c^2
100	9983	.9983
200	39735	.9934
300	88674	.9853
500	240080	.9603
1000	861524	.8615
2000	254663	.6367
4000	6089800	.3806
6000	9634700	.2676
10000	16724500	.1672
50000	87622500	.03505
100000	176245000	.01762

As may be seen, Var/c^2 decreases very slowly until about $c = 500$. Afterwards, it decreases much more rapidly, until for $c \geq 10{,}000$, it can be approximated reasonably well by $1760/c$. **In effect, this means that the variance behaves *as though* the *c* undecided voters arrived, not independently, but, rather, *in independent bunches of* 1760. This will hold so long as the actual number of undecideds is at least of the order of 10,000.**

Of course, the numerator 1760, the size of the independent bunches, depends on both k and t. Specifically, **for large *c*, it will be proportional to the square root of *kt*.** It is difficult to decide how large this should be chosen, but in the next section we will try to justify a value chosen. (See footnote 1, below.)

Statistical Analysis

We will make the assumption that Republican candidate i's share of the two-party vote in state j can be approximated by

(15) $$q_{ij} = \mu_i + z_j + e_{ij}$$

where μ_i represents the candidate's personal popularity, z_j is state j's Republican tendency (low for MA and DC, high for UT and WY), and e_{ij} is an additional (random) term, representing perhaps good or bad luck for the candidate in that state. Typically, μ_i would be of the order of 0.1 for a very popular candidate (say, Nixon in 1972), -0.1 for an unpopular candidate (Goldwater in 1964) and in between for other candidates. In fact, we will not worry about the μ_i, since they do not affect the states' rankings, and we are only interested in the probability that a given state be a pivot.

We estimate z_j by looking at the Republican share of the vote in the last five elections (1988-2002). Since positions evolve over time, we have discounted past elections by a factor of 0.8 for each cycle. Thus, for each state, we calculate the quantity

(16) $$z_j = \frac{s_{2004,j} + 0.8\, s_{2000,j} + 0.64\, s_{1996,j} + 0.512\, s_{1992,j} + 0.4096\, s_{1988,j}}{3.3616}$$

where $s_{k,j}$ is the Republican share of the two-party vote in state j, in the year k presidential election. This allows us to give a position to each state. (The numbers should be divided by 1000.) We obtain an ordering for the states, from most Democratic to most Republican. The electoral votes, w_j, are those of the 2000 reapportionment:

State	z_j	γ_j	w_j	Cumulative EV
DC	102	0	3	3
MA	368	0	12	15
RI	370	0	4	19
NY	395	0	31	50
VT	418	0.09	3	53
HI	420	0.10	4	57
MD	438	0.19	10	67
IL	440	0.20	21	88
CT	441	0.205	7	95
CA	445	0.225	55	150
ME	452	0.26	4	154
WA	454	0.27	11	165
DE	456	0.28	3	168
MN	458	0.29	10	178
NJ	458	0.29	15	193
OR	469	0.345	7	200
MI	472	0.36	17	217
PA	474	0.37	21	238
IA	477	0.385	7	245

WI	482	0.41	10	255
NM	488	0.44	5	260
WV	496	0.48	5	265
AR	497	0.485	6	271
MO	501	0.505	11	282
NH	504	0.52	4	286
OH	506	0.53	20	306
FL	517	0.585	27	333
NV	518	0.59	5	338
CO	520	0.60	9	347
LA	523	0.615	9	356
TN	530	0.65	11	367
AZ	536	0.68	10	377
VA	541	0.705	13	390
GA	549	0.745	15	405
NC	549	0.745	15	420
KY	551	0.755	8	428
MT	568	0.84	3	431
AL	570	0.85	9	440
SC	573	0.865	8	448
IN	574	0.87	11	459
SD	575	0.875	3	462
TX	577	0.885	34	496
MS	587	0.935	6	502

KS	596	0.98	6	508
OK	600	1	7	515
ND	605	1	3	518
AK	632	1	3	521
NE	637	1	5	526
WY	644	1	3	529
ID	658	1	4	533
UT	686	1	5	538

We next assume that 20% of the voters in each state are undecideds, and that they have, in the past, divided evenly between the two candidates. Thus, for state j, where the Republican's share of the vote is y_j, we assume that $d_j = 0.9 - y_j$ are the Democratic stalwarts, and $e_j = z_j - 0.1$ are the Republican stalwarts. Then, if $z_j < 0.4$, the state is not in play, as the Democrats have a majority even without any of the undecideds. Similarly, if $z_j > 0.6$, then the Republicans have a certain majority. In our "time of arrival" parable, the former of these arrive at time 0, while the latter arrive at time 1. Thus, there are 7 safe states for the Republicans, and 3 safe states, as well as the DC, for the Democrats.

There are 40 states with $0.4 < z_j < 0.6$. These states are theoretically in play, though such states as Kansas, at .596, and Vermont, at .418, seem safe for their parties except in case of a landslide of historical proportions. We then note that, for a state to "arrive", at least a fraction γ_j of its undecided voters must join, where

(17) $$\gamma_j = 5z_j - 2.$$

As discussed above, this γ_j is also the median value of X_j, state j's arrival time. As may be seen, Arkansas is the median state in this ordering, with $z_{AR} = .497$. Then $\Gamma = \gamma_{AR} = 0.485$ is a good approximation to the median time at which a winning coalition forms.

It is still necessary to get a distribution for these times of arrival. As mentioned above, the effect of correlation is to decrease the variance of arrivals. Thus, with c_j undecideds, and independence, the variance is σ^2/c_j, where σ^2 is the variance of each individual's time. However, we saw that (under certain assumptions), the group variance was *as if the undecideds arrived in bunches of 1760 at a time*, these bunches coming independently. Moreover, this bunch size of 1760 depends on certain parameters which are admittedly uncertain. Let us assume, then, that the size of the bunches is, rather, 2500.

Now, if we assume that 35% of the state population vote, and that 20% of these are undecided, we find that the undecideds are 7% of p_j, the state population. However, these arrive in bunches of 2500. Thus the number of undecided bunches would be

$$(18) \qquad c_j = .07\, p_j/2500 = .000028\, p_j$$

We call c_j , the *virtual number of undecided bunches*.[2]

Of these virtual bunches, a fraction γ_j must arrive for state j to arrive. Thus we need

$$(19) \qquad s_j = \gamma_j c_j$$

of these bunches. Then, in equation (4), $g_j(t)$ is equal to the probability that a binomial variable, with parameters c_j and t, be at least equal to s_j. Approximating the binomial by a normal variable, we have the cumulative distribution function g_j and its density, g_j' as given by equations (8) and (9) above, which we repeat here as (20) and (21):

$$(20) \qquad g_j(t) = \Phi\left(\frac{(t-\gamma_j)\sqrt{c_j}}{\sqrt{t(1-t)}}\right)$$

$$(21) \qquad g_j'(t) = \frac{(t+\gamma-2\gamma t)\sqrt{c_j}}{2(t-t^2)^{3/2}}\ \varphi\left(\frac{(t-\gamma_j)\sqrt{c_j}}{\sqrt{t(1-t)}}\right)$$

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[2] To see that this is a reasonable number, note that, for a standard-sized electoral district of 500,000 inhabitants, this gives us a total of 14 virtual bunches. As mentioned above, Gelman, Katz and Bafumi (2004) suggest that the variance should behave as the –0.2 power of population, i.e., **as if the population should be replaced by its fifth root**. But the fifth root of 500,000 is 13.8. This is not to say that our analysis is exact, only that it is not unreasonable.

Unfortunately, X_j is not normally distributed. The distribution g_j is given in terms of the normal distribution, but its "variance" unfortunately depends on t. Nevertheless, we note that, since $t(1\text{-}t) \leq 0.25$, then, for all $t > \gamma_j$, we must have

$$(22) \qquad g_j(t) > \Phi\,[2(t\text{-}\gamma_j)\surd c_j]$$

while the opposite inequality will hold if $t < \gamma$. Now, we know that $\Phi(1.96) = .975$, while $\Phi(-1.96) = 0.025$. It will follow that

$$(23) \qquad g_j(\gamma_j + 0.98/\surd c_j) > .975, \qquad g_j(\gamma_j - 0.98/\surd c_j) < .025,$$

and we conclude that $\gamma_j \pm 1/\surd c_j$ gives us better than a 95% interval for X_j.

We now simplify matters by assuming that we can disregard anything outside the 95% interval, $\gamma_j \pm 1/\surd c_j$. We calculate this for all but the 10 states, and the DC, mentioned above as always safe for one or the other of the two parties. The several states' populations, p_j, are the July 2005 Census Bureau estimates, obtained from the web site www.factmonster.com.

State	γ_j	p_j(thousands)	c_j	$\gamma_j \pm 1/\surd c_j$	Cumulative EV
DC	0	551	15.43	0	3
MA	0	6399	179.17	0	15
RI	0	1076	30.13	0	19
NY	0	19255	539.14	0	50
VT	0.09	623	17.44	[0, .33]	53
HI	0.10	1275	35.70	[0, .27]	57

MD	0.19	5600	156.80	[.11, .27]	67
IL	0.20	12763	357.364	[.15, .25]	88
CT	0.205	3510	98.28	[.10, .31]	95
CA	0.225	36132	1011.70	[.19, .26]	150
ME	0.26	1322	37.02	[.10, .42]	154
WA	0.27	6288	176.06	[.19, .35]	165
DE	0.28	844	23.63	[.07, .49]	168
MN	0.29	5133	143.72	[.21, .37]	178
NJ	0.29	8718	244.10	[.23, .35]	193
OR	0.345	3641	101.95	[.25, .44]	200
MI	0.36	10121	283.39	[.30, .42]	217
PA	0.37	12430	348.04	[.31, .43]	238
IA	0.385	2966	83.05	[.28, .49]	245
WI	0.41	5536	155.01	[.33, .49]	255
NM	0.44	1928	53.98	[.30, .58]	260
WV	0.48	1817	50.88	[.34, .62]	265
AR	0.485	2779	77.81	[.37, .60]	271
MO	0.505	5800	162.40	[.43, .58]	282
NH	0.52	1310	36.68	[.35, .69]	286
OH	0.53	11464	320.99	[.47, .59]	306
FL	0.585	17790	498.12	[.54, .63]	333
NV	0.59	2415	67.62	[.47, .71]	338
CO	0.60	4665	130.62	[.51, .69]	347
LA	0.615	4524	126.67	[.53, .70]	356

TN	0.65	5963	166.96	[.57, .73]	367
AZ	0.68	5939	166.29	[.60, .76]	377
VA	0.705	7567	211.88	[.64, .77]	390
GA	0.745	9073	254.04	[.68, .81]	405
NC	0.745	8683	243.12	[.68, .81]	420
KY	0.755	4173	116.84	[.66, .85]	428
MT	0.84	936	26.21	[.64, 1]	431
AL	0.85	4558	127.62	[.76, .94]	440
SC	0.865	4255	119.14	[.77, .96]	448
IN	0.87	6272	175.62	[.79, .95]	459
SD	0.875	776	21.73	[.66, 1]	462
TX	0.885	22860	640.08	[.85, .92]	496
MS	0.935	2921	81.79	[.82, 1]	502
KS	0.98	2745	76.86	[.87, 1]	508
OK	1	3548	99.34	1	515
ND	1	637	17.84	1	518
AK	1	664	18.59	1	521
NE	1	1759	49.25	1	526
WY	1	509	14.25	1	529
ID	1	1429	40.01	1	533
UT	1	2470	69.16	1	538

As may be seen, if we consider only the times within each state's 95% interval, a majority (270 electoral votes) can arrive no sooner than time 0.38 (Arkansas's early time), and will certainly have arrived by time 0.59 (Missouri's late time). Thus, only

those states whose interval overlaps [.38, .59] can be pivots under our scheme. We can put the states in five categories:

a) Safe for the Democrats : DC, RI, MA, NY, with 50 votes;
b) Almost safe for the Democrats : VT, HI, MD, IL, CT, CA, WA, MN, NJ, with 136 votes;
c) In play : ME, DE, OR, MI, PA, IA, WI, NM, WV, AR, MO, NH, OH, FL, NV, CO, LA, TN, with 181 votes;
d) Almost safe for the Republicans : AZ, VA, GA, NC, KY, MT, AL, SC, IN, SD, TX, MS, KS, with 141 votes;
e) Safe for the Republicans: OK, ND, AK, NE, WY, ID, UT, with 30 votes.

Only the 18 states "in play" have a non-negligible probability of being pivot. Since 186 votes are going to arrive early, the pivot will be that state (among these 18) to complete 84 votes. Thus, for example, Maine, with its 4 votes, must arrive at a time when other "in play" states with at least 80, and not more than 83, electoral votes, have arrived. This number of votes is of course an integer, but we will approximate via a continuous distribution, Then, if $Y_j(t)$ is the number of votes (not counting those of state j) to have arrived by time t, we find that the j^{th} partial derivative (where j refers to Maine) of the multilinear extension F, evaluated at $(g_1(\mathrm{t}), \ldots, g_n(t))$, is given by

(24) $$F_j(\mathbf{g}(t)) = \text{Prob}[79.5 \leq Y_j(t) \leq 83.5].$$

Thus the probability that ME is in fact the pivot will be given by

(25) $$\psi_j = \int \text{Prob}[79.5 \leq Y_j(t) \leq 83.5]\, g_j'(t)\, dt,$$

where the integral is taken over a sufficiently large interval, for example, ME´s 95% interval, [.10, .42].

For other states, the probability is given by a similar integral; the probability in the integrand must however be replaced by

(26) $$F_j(\mathbf{g}(t)) \;=\; \text{Prob}[83.5 - w_j \leq Y_j(t) \leq 83.5],$$

where w_j is state j's electoral weight (number of votes). The integral in (25) should be modified accordingly.

It remains to determine the probability that $83.5 - w_j \leq Y_j(t) \leq 83.5$. In fact, at time t, state k has arrived with probability $g_k(t)$. Thus it will have contributed w_k votes with probability g_k, and 0 votes with probability $1-g_k$. Thus the number of votes it has contributed is a random variable with mean $w_k g_k$, and variance $w_k^2 g_k(1-g_k)$. Now Y_j is the sum of these variables, for all states k other than j. The states are assumed to arrive independently, and thus Y_j has mean

(27) $$M_j \;=\; \textstyle\sum_{k\neq j} w_k g_k,$$

and variance

(28) $$V_j \;=\; \textstyle\sum_{k\neq j} w_k^2 g_k(1-g_k).$$

Note that, since all the g_k depend on t, so do M_j and V_j.

We will now assume that the number of states in play is sufficiently large that we can approximate Y_j by a normal random variable with the given mean and variance. If that is so, then we have the approximation

(29) $$F_j \;\approx\; \Phi\left(\frac{83.5 - M_j}{\sqrt{Vj}}\right) \;-\; \Phi\left(\frac{83.5 - w_j - M_j}{\sqrt{Vj}}\right)$$

Finally, we must determine the individual voters' power. We note that a resident, k, of state j will be the pivot if (1) j is the pivot among the states, and (2) k is the median voter in state j. Since (apart from states in the two "safe" categories) only undecided voters will be in median position, we divide the state's pivot probability, ψ_j , by the

number of undecided voters, which we had calculated as .07 p_j. This gives us the voting power of individuals in state j..

The table gives the approximate values. (We used Maple 10 to carry out the several integrations.) Entries in the last column should be multiplied by 10^{-9}.

State	γ_j	p_j(thousands)	E. V.	Piv. prob. ψ_j	Power ψ_j / .07 p_j
DC	0	551	3	0	0
MA	0	6399	12	0	0
RI	0	1076	4	0	0
NY	0	19255	31	0	0
VT	0.09	623	3	0	0
HI	0.10	1275	4	0	0
MD	0.19	5600	10	0	0
IL	0.20	12763	21	0	0
CT	0.205	3510	7	0	0
CA	0.225	36132	55	0	0
ME	0.26	1322	4	.0021	22.7
WA	0.27	6288	11	0	0
DE	0.28	844	3	.0059	99.9
MN	0.29	5133	10	0	0
NJ	0.29	8718	15	0	0
OR	0.345	3641	7	.0067	26.3
MI	0.36	10121	17	.0026	3.7
PA	0.37	12430	21	.0024	2.7

IA	0.385	2966	7	.0360	173.3
WI	0.41	5536	10	.0636	164.1
NM	0.44	1928	5	.0838	623.6
WV	0.48	1817	5	.1017	799.6
AR	0.485	2779	6	.1472	756.7
MO	0.505	5800	11	.1984	488.7
NH	0.52	1310	4	.0606	660.9
OH	0.53	11464	20	.1872	233.3
FL	0.585	17790	27	.0583	46.9
NV	0.59	2415	5	.0243	143.7
CO	0.60	4665	9	.0102	31.3
LA	0.615	4524	9	.0051	16.1
TN	0.65	5963	11	.0021	5.0
AZ	0.68	5939	10	0	0
VA	0.705	7567	13	0	0
GA	0.745	9073	15	0	0
NC	0.745	8683	15	0	0
KY	0.755	4173	8	0	0
MT	0.84	936	3	0	0
AL	0.85	4558	9	0	0
SC	0.865	4255	8	0	0
IN	0.87	6272	11	0	0
SD	0.875	776	3	0	0
TX	0.885	22860	34	0	0

MS	0.935	2921	6	0	0
KS	0.98	2745	6	0	0
OK	1	3548	7	0	0
ND	1	637	3	0	0
AK	1	664	3	0	0
NE	1	1759	5	0	0
WY	1	509	3	0	0
ID	1	1429	4	0	0
UT	1	2470	5	0	0

As may be seen, the power is concentrated among the states near the middle of the order. West Virginia voters are the most powerful, with Arkansas, New Mexico and New Hampshire close behind. Missouri and Ohio seem to be the most important states. There seem to be certain anomalies, e.g. Oregon seems to be more powerful than Michigan, though the latter is more populous and more centrally located. The explanation seems to be that it is easier to "move" the population of Oregon towards the center (there are fewer Oregonians than Michiganders) and so Oregon becomes an easier prize than Michigan.

Admittedly these results depend quite heavily on the size of the "virtual bunches." In a subsequent article we will try to get a better statistical handle on these.

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