Holler-Packel Value and Index – A New Characterization*

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Abstract

The Holler-Packel value and (non-normalized) index are given a new characterization by a *potential function*. The Holler-Packel potential of a general TU-game is the total value of all minimal crucial coalitions in the game; restricted to simple games it is simply the number of minimal winning coalitions. New economic interpretations follow from known equivalence results on existence of a potential function and balanced contributions, path independence, and Shapley blue print properties.

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1 Introduction

Hart and Mas-Colell (1988, 1989) introduced the concept of a *potential (function)* to cooperative game theory and applied it to give an elegant new characterization of the Shapley value. Based on a slightly modified definition, other authors have demonstrated that different (vector-valued) solution concepts for general transferable utility games (*TU-games*) and the subclass of *simple games* (which are frequently applied to analyze *power* in various economic and political decision bodies) can also be given characterizations with potential functions which provide additional interpretations and methods of computation. So far, potentials have been identified for the Banzhaf value (Dragan 1996, Ortmann 1998), the entire class of semivalues (Calvo and Santos 1997), and – using a generalized definition of a potential – weighted weak semivalues (Calvo and Santos 1997, 2000). The nucleolus, for example, does not admit a potential.

This paper addresses the question of whether the Holler-Packel value and its restriction to simple games (Holler 1982, Holler and Packel 1983, and Holler and Li 1995) admits a potential. This value considers only minimal winning coalitions in simple games (or so-called minimal crucial coalitions in general games) in line with Riker's (1962) size principle, and coincides with the member bargaining power measure proposed by Brams and Fishburn (1995) and Fishburn and Brams (1996) on individual simple games.

A potential function summarizes a game by one real number. A given player i's power or expected payoff in the game (as identified by some index or value with a potential) is simply the difference between this number and the corresponding number of the reduced game in which only players $j \neq i$ participate. That very complex games can be meaningfully condensed and players' role in them identified in this way is not only mathematically fascinating; it also allows for new interpretations of the value in question. The Holler-Packel *index* in its most popular *normalized* form does not admit a potential, but we identify the potential function of its non-normalized version and of the value.

The next section introduces our notation and briefly reviews common value concepts for cooperative games. Section 3 gives the definition and discusses some properties of the Holler-Packel value and (non-)normalized indices. Section 4 discusses the notion of a potential in more detail, and identifies the potential of the Holler-Packel value and its non-normalized simple game version. We conclude by discussing possible economic and political interpretations.

2 Notation, Values and Indices

A (transferable utility) cooperative *game* is a pair (N, v), where $N = \{1, \cdots, n\}$ is a finite set of players and $v\colon \mathcal{P}(N) \to \mathbb{R}$ is a *(characteristic) function* which assigns to each subset $S \subseteq N$ (called a *coalition*) a real number $v(S)$ (the *worth* of the coalition) with $v(\varnothing) = 0$. A game for which v only takes values in $\{0, 1\}$ is called a *simple game*. A *simple voting game* is a simple game in which all coalitions with $v(S) = 1$ – the *winning coalitions* – can be characterized by a weight w_i for each $i \in N$ and a quota $q > 0$ such

that $v(S) = 1 \iff \sum_{i \in S} w_i \geq q$.

A *solution* ψ (defined on the space of all games, $\mathcal{G}$, or a subclass of $\mathcal{G}$) maps each (N, v) to a vector $\psi(N, v) \in \mathbb{R}^n$ such that each player $i \in N$ is assigned his or her value in the game, $\psi_i(N, v)$. The latter is usually interpreted as player i's payoff expectation from playing the game or as an indicator of his or her importance or power in the game. A solution ψ on $\mathcal{G}$ is also referred to as a *value*. A value restricted to the class of simple games is called an *index*.

Values or indices are often defined in terms of players' *marginal contribution* $[v(S) - v(S \setminus i)]$ to the possible coalitions $S \subseteq N$.[1] The class of *probabilistic values* (Weber 1988) assigns to each player i his or her expected marginal contribution based on a possibly player-specific subjective probability distribution over the set of coalitions. Values in which expectations for all players are taken with the same probability measure p and in which $p(S)$ depends at most on the cardinality $s = |S|$ are called *semivalues* (Dubey, Neyman, and Weber 1981). Particularly prominent semivalues are the *Shapley value* φ (Shapley 1953) and the *Banzhaf value* β (Banzhaf 1965, Dubey and Shapley 1979) defined by

$$\varphi_i(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus i)]$$

and

$$\beta_i(N, v) = \sum_{S \subseteq N} \frac{1}{2^{n-1}} [v(S) - v(S \setminus i)],$$

respectively. φ is *efficient*, i. e. $\sum_{i \in N} \varphi_i(N, v) = v(N)$, but β is not.

A player i is *crucial* in $S \subseteq N$ if i makes a positive marginal contribution to S. A coalition S in which *every* member $i \in S$ is crucial is called a *minimal crucial coalition (MCC)* . In the context of simple games, MCCs are typically referred to as *minimal winning coalitions (MWCs)*. The set of all MCCs of (N, v) is denoted by $M(N, v)$. Players not belonging to any MCC are called *dummy players*.

3 Holler-Packel Value and Indices

The *Holler-Packel value (HPV)* η was first introduced on the class of simple games by Holler (1982), later axiomatized by Holler and Packel (1983)[2], and finally extended to general TU-games by Holler and Li (1995). It is defined by

$$\eta_i(N, v) = \sum_{\substack{S \in M(N,v) \\ i \in S}} v(S).$$

On the class of *simple games*, it can also be written as

$$\eta_i(N, v) = \sum_{S \in M(N,v)} [v(S) - v(S \setminus i)] = |\{S \in M(v) \colon i \in S\}|. \tag{1}$$

[1] We write i instead of $\{i\}$ where there is no danger of confusion.

[2] Also see Napel (1999).

Typically, η's *normalized* version, referred to as the (normalized) *Holler-Packel index (HPI)*,

$$\bar{\eta}_i(N,v) = \frac{\eta_i(N,v)}{\sum_{i\in N}\eta_i(N,v)}$$

is considered in applications. For the purposes of this paper, however, the HPV or non-normalized HPI is of greater interest.

It can be seen from (1) that on the class of simple voting games, to which both Banzhaf and Holler-Packel values are most commonly applied, β and η only differ (apart from the re-scaling by 2^{1-n}) in that the HPI restricts attention to *minimal* winning coalitions. This is motivated in Holler (1982) by arguing that – e. g. in a world where joining a coalition (endorsing a proposal) is costly – only coalitions where every member matters, i. e. is crucial, "will be *purposely* formed ('not by sheer luck')" (p. 267, italics added) and should be taken into account. This reflects the classic *size principle* advocated by Riker (1962).[3] The latter is derived from a game-theoretic model with assumptions (such as rationality, complete and perfect information, control over membership of coalitions) discussed in detail by Brams and Fishburn (1995) together with some empirics. Interestingly, the latter authors (also see Fishburn and Brams 1996) derive a measure of so-called *member bargaining power* whose restriction to single simple games coincides with the HPI.[4]

HPI and HPV have been characterized (for details see Holler and Packel 1983, Holler and Li 1995, and Napel 1999) by axioms which require from an index or value that it treats players anonymously (i. e. is invariant to permutations of N), assigns 0 to dummy players, is additive when a particular sum operation, $v_1 \oplus v_2$, is carried out with games having disjoint sets $M(v_i)$, and assigns the full coalition's value $v(S)$ to (at least) one member if this coalition S is the unique MCC (MWC) of the (simple) game.

It can easily be seen that, like the Banzhaf value, the HPV is not efficient, i. e. generally $\sum_{i\in N}\eta_i(N,v) \neq v(N)$. Several other properties of the Shapley value (and its index version, introduced by Shapley and Shubik 1954) are not shared by the HPV either. For example, the HPV (and HPI) can violate weak monotonicity in players' weights for simple voting games.[5] See Felsenthal and Machover (1998) on this and other 'paradoxes' which the HPI but also the Banzhaf and Shapley-Shubik indices may exhibit.

The marginal contribution of voter i in any given coalition S is always nondecreasing in weight w_i. It follows that the HPV cannot be expressed as an expected marginal contribution for a probability distribution which depends on N (and possibly the player i) but not the characteristic function v. In other words, the HPV is *not* a probabilistic value.

[3] Another power measure based on the size principle has been proposed by Deegan and Packel (1978). The Deegan-Packel index weights contributions in MWC S by $|S|^{-1}$ (instead of 1 for the HPI).

[4] The Holler-Packel index is often associated with coalition outcomes interpreted as or relating to a public good and even introduced as *public good index* (see, e. g., Holler 1982, 1998).

[5] Consider (N,v) defined by $w = (35,20,15,15,15)$ and $q = 51$: $w_2 > w_3$ but $2 = \eta_2(N,v) < \eta_3(N,v) = 3$. – Whether the possible violation of monotonicity for some particular w and q is a fatal problem or rather an advantage for a voting power index is a matter of debate; see e. g. Brams and Fishburn (1995) and Holler and Napel (2004) for support of the latter view.

4 Potential of the Holler-Packel Value

Given a game (N, v), let the game $(N \setminus i, v)$ be defined by restricting the domain of the characteristic function, $\mathcal{P}(N)$, to $\mathcal{P}(N \setminus i)$. Coalitions involving player i are 'deleted' from the game and all other coalitions $S \subseteq N \setminus i$ simply retain their old worth $v(S)$. Note that if (N, v) is a simple voting game, $(N \setminus i, v)$ is characterized by the same quota q and weights $w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n$.[6]

Characterizing a value ψ by a potential function amounts to the provision of a mapping P from the space of all games $\mathcal{G}$ or a subclass $G \subseteq \mathcal{G}$ closed under the above removal operation, to the real numbers such that any player i's value $\psi_i(N, v)$ is for any (N, v) equal to the difference between the *potential* $P(N, v)$ of the considered game and the potential $P(N \setminus i, v)$ of the restricted game resulting from dropping player i (letting the remaining players 'play amongst themselves'). In other words, if (and only if) a value $\psi\colon G \to \mathbb{R}^n$ admits a potential function $P\colon G \to \mathbb{R}$, one can view (and calculate) $\psi_i(N, v)$ as i's *marginal contribution*

$$\Delta_i(N, v) \equiv P(N, v) - P(N \setminus i, v)$$

to the game (N, v), where the latter is evaluated by P.[7] Provided it exists, ψ's potential in view of $\psi_i(N, v) = \Delta_i(N, v)$ satisfies the recursive equation

$$P(N, v) = \frac{1}{n} \left[\sum_{i=1}^{n} \psi_i(N, v) + \sum_{i=1}^{n} P(N \setminus i, v) \right] \tag{2}$$

for all $N \neq \varnothing$ and becomes *uniquely* determined after making the definition

$$P(\varnothing, v) \equiv 0. \tag{3}$$

Hart and Mas-Colell (1988, 1989) were the first to consider the notion of a potential – which has a long tradition in physics (see Ortmann 1998 for a detailed discussion) – in the context of games. They showed that the Shapley value is the unique value which is efficient and admits a potential;[8] namely,

$$P^{\varphi}(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S)$$

Dragan (1996) and Ortmann (1998) first showed that the Banzhaf value admits a potential, too. It is given by

$$P^{\beta}(N, v) = \sum_{S \subseteq N} \frac{1}{2^{n-1}} v(S).$$

[6]Slightly uncustomary, our definition of a simple game has *not* required $v(N) = 1$. This keeps the space of simple games closed under the considered removal operation.

[7]Ortmann (2000) introduced the related notion of a *multiplicative* potential function: a solution ψ admits a multiplicative potential iff there exists a function $P\colon G \to \mathbb{R}$ such that $\frac{P(N,v)}{P(N \setminus i, v)} \equiv \psi_i(N, v)$.

[8]Actually, Hart and Mas-Colell originally included efficiency in their definition of a potential function.

More generally, as proven by Calvo and Santos (1997), *any* semivalue ψ^{sv} admits a potential. It is[9]

$$P^{\psi^{sv}}(N,v) = \sum_{S \subseteq N} p_s v(S),$$

where p_s denotes the formation probability of coalitions with s members.

Since the (normalized) HPI for simple games is by definition efficient, Hart and Mas-Colell's result implies that it does *not* admit a potential. However, the HPV is neither efficient nor a semivalue. So, the question of whether it admits a potential or not has so far to our knowledge not been answered. Before we give our affirmative answer and discuss the potential of HPV and non-normalized HPI, let us point to several equivalence results that underline the relevance of the question. Namely, a value admits a potential (see Hart and Mas-Colell 1989, Ortmann 1998, Calvo and Santos 1997, and Dragan 1999) if and only if it has

1. the *balanced contribution property* (or *preserves differences*),
2. the *path independence property* (or *is conservative*), or
3. the *Shapley blue print property*.[10]

The property of *balanced contributions* has been defined by Myerson (1980) and intuitively requires that for any two players the gains or losses that one imposes on the other (according to some value ψ) by leaving the game is equal for both. Formally, ψ defined on G satisfies the balanced contribution property iff

$$\forall (N,v) \in G\colon \forall i,j \in N\colon\ \ \psi_i(N,v) - \psi_i(N \setminus j, v) = \psi_j(N,v) - \psi_j(N \setminus i, v).$$

ψ satisfies *path independence* if, intuitively speaking, one could sequentially 'buy off' players from the game such that they leave one by one in exchange for getting 'paid' their value $\psi_i(N',v)$ of the game involving all remaining players $N' \subseteq N$, thereby exhausting a *total* amount of money that does *not* depend on the order in which players are paid to leave. Formally, denote the set of all permutations $\omega\colon N \to N$ by $\Omega(N)$ and the set of players preceding i in permutation ω by N_i^ω. Then, ψ satisfies path independence iff

$$\forall (N,v) \in G\colon \forall \omega, \omega' \in \Omega(N)\colon\ \ \sum_{i \in N} \psi_i(N_i^\omega \cup i, v) = \sum_{i \in N} \psi_i(N_i^{\omega'} \cup i, v).$$

The (order-independent) total amount spent equals the potential of the game (N,v), namely

$$\begin{aligned} P(N,v) &= P(N,v) - P(\varnothing, v) \\ &= \big(P(N,v) - P(N \setminus i_1, v)\big) + \big(P(N \setminus i_1, v) - P(N \setminus \{i_1, i_2\}, v)\big) + \\ &\qquad\qquad \ldots + \big(P(i_n, v) - P(\varnothing, v)\big) \\ &= \psi_{i_1}(N,v) + \psi_{i_2}(N \setminus i_1, v) + \ldots + \psi_{i_n}(i_n, v) \end{aligned}$$

[9] Again, it is uniquely determined up to the constant which we chose to be zero in (3).

[10] Dragan (1999) also proves equivalence to a *recursive formula property* which generalizes the set of equations provided by Sprumont's (1990) recursive definition of the Shapley value. It leads to a recursive definition of semivalues, but does not seem to provide additional interpretational insights generally.

with $i_k = \omega(k)$ for any given $\omega \in \Omega(N)$.

The *Shapley blue print property*, defined by Dragan (1999), requires a value ψ applied to all games (N, v) to equal the Shapley value φ applied to 'related games' (N, v_ψ). In particular, for any given (N, v), the game (N, v_ψ) is defined by

$$v_\psi(S) \equiv \sum_{i \in S} \psi_i(S, v), \tag{4}$$

i. e. the worth of a coalition S in the 'related game' – the so-called *power game of* (N, v) (cf. Dragan 1999) – is the sum of the ψ-values of its members in the reduction of the original game to player set S.[11] Then a value ψ has a potential iff $\psi(N, v) \equiv \varphi(N, v_\psi)$. The Shapley blue print property and, in particular, the mapping $v \mapsto v_\psi$ allow to connect a value ψ to the many interpretations and results concerning the Shapley value (see e. g. Winter 2002). Value ψ may be viewed as the 'standard solution' (that is: Shapley's, which has diverse cooperative and non-cooperative foundations) to coalition formation and distribution problems for a game related to (N, v) in a particular way determined by ψ.

That not only the Banzhaf value or other semivalues but indeed also the HPV has the properties listed under 1.–3. follows from

PROPOSITION 1 *$P^\eta \colon G \to \mathbb{R}$ with $P^\eta(\varnothing, v) = 0$ and otherwise*

$$P^\eta(N, v) = \sum_{S \in M(N,v)} v(S) \tag{5}$$

is the potential of the Holler-Packel value.

Proof:
Partition the set of (N, v)'s MCC into

$$M_i(N, v) = \{S \in M(N, v) \colon i \in S\}$$

and

$$M_{-i}(N, v) = \{S \in M(N, v) \colon i \notin S\}.$$

Then, (5) can be written

$$P^\eta(N, v) = \sum_{S \in M_i(N,v)} v(S) + \sum_{S \in M_{-i}(N,v)} v(S).$$

One also has

$$P^\eta(N \setminus i, v) = \sum_{S \in M(N \setminus i,v)} v(S) = \sum_{S \in M_{-i}(N,v)} v(S)$$

[11] Clearly, $(N, v_\psi) \equiv (N, v)$ iff ψ is efficient.

because $M(N \setminus i, v) = M_{-i}(N, v)$. Hence,

$$\Delta_i^\eta(N,v) \equiv P(N,v) - P(N \setminus i, v) = \sum_{S \in M_i(N,v)} v(S) = \sum_{\substack{S \in M(N,v) \\ i \in S}} v(S) = \eta_i(N,v).$$

Note that, by (2), P^η is unique up to the additive constant $c \equiv P^\eta(\varnothing, v)$ chosen to be zero. □

The potential function of HPV's restriction to simple games (N,v), the non-normalized HPI, can also be written as

$$P^\eta(N,v) = |M(N,v)|,$$

i. e. a simple game's potential is simply the number of MWC in it.

Example As illustration consider the 7-person simple game (N,v) with

$$M(N,v) = \{\{1,2,3,4\},\{3,4,5,6\},\{3,4,5,7\},\{3,5,6,7\},\{4,5,6,7\}\}$$

analyzed by Holler and Li (1995). The potential of the game is $P^\eta(N,v) = 5$. From

$$M(N \setminus 1, v) = \{\{3,4,5,6\},\{3,4,5,7\},\{3,5,6,7\},\{4,5,6,7\}\}$$

one obtains $P^\eta(N \setminus 1, v) = 4$, i. e. player 1 contributes a potential of 1 (one MWC) to the game and consequently has a HPV or non-normalized HPI $\eta_1(N,v) = 1$. Analogously, $\eta_2(N,v) = 1$, $\eta_3(N,v) = \eta_4(N,v) = \eta_5(N,v) = 4$, and $\eta_6(N,v) = \eta_7(N,v) = 3$.

Now consider, e. g., the coalition $S = \{1,2,3,4\}$. In the restricted game (S,v), the HPV evaluates to $\eta_1(S,v) = \eta_2(S,v) = \eta_3(S,v) = \eta_4(S,v) = 1$. So the worth of S in the power game (N, v_η) induced by (N,v) is $v_\eta(S) = \sum_{i \in S} \eta_i(S,v) = 4$. Analogously, one obtains $v_\eta(S', v) = 0$ for $S' = S \setminus 1$. So the marginal contribution of player 1 to coalition S in (N, v_η) is $[v_\eta(S) - v_\eta(S \setminus 1)] = 4$. Weighted with $\frac{3!3!}{7!}$ this contributes to player 1's Shapley value in (N, v_η) – which equals player 1's HPV in (N,v). □

Let $\pi(N,v)$ denote the sum of the worths of the MCCs for all players in (N,v), i.e.,

$$\pi(N,v) = \sum_{i \in N} \sum_{\substack{T \in M(N,v) \\ i \in T}} v(T).$$

Let us say that an arbitrary value ψ defined on $G \subseteq \mathcal{G}$ *distributes the worths of MCCs* iff

$$\forall (N,v) \in G \colon \forall S \subseteq N, M(S,v) \neq \varnothing \colon \sum_{i \in S} \psi_i(S,v) = \pi(S,v).$$

This is similar to the efficiency property and requires ψ – in any restricted version of a given game (N,v) – to assign a total value to the participating players which is equal to the sum of the individual total worths experienced by these players in all their minimal

crucial coalitions. The HPV distributes the worths of MCCs. But so does, for example, the value ψ defined by

$$\psi_i(N,v) = \begin{cases} \pi(N,v) & \text{if } i = \min N \\ 0 & \text{otherwise.} \end{cases}$$

In analogy to Hart and Mas-Colell's characterization of the Shapley value (as the unique value which is efficient and admits a potential) and Dragan's and Ortmann's characterization of the Banzhaf value (as the unique value which "distributes the marginalities" and admits a potential – see Dragan 1996 and Ortmann 1998), we have

PROPOSITION **2** *The Holler-Packel value is the unique solution which distributes the worths of MCCs and admits a potential.*

Proof:
Let ψ be an arbitrary value with potential P and distributing the worths of MCCs, i.e. $\psi_i(S,v) = \Delta_i P(S,v)$ and $\sum_{i \in S} \Delta_i P(S,v) = \pi(S,v)$. By rearranging,

$$P(S,v) = \frac{1}{s}[\pi(S,v) + \sum_{i \in S} P(S \setminus i, v)].$$

Consider any coalition $S \subseteq N$ with $|S| = 1$, i.e. $S = \{i\}$ for some $i \in N$. We have $P(S,v) = v(i) = P^\eta(S,v)$. Now, $P \equiv P^\eta$ (and thus $\psi \equiv \eta$) follows by induction: suppose $P(S,v) = P^\eta(S,v)$ for all $S \subseteq N$ with $|S| = s$ and consider $T \subseteq N$ with $|T| = t = s+1$. Then, using that the HPV distributes the values of MCCs,

$$\begin{aligned} P(T,v) &= \frac{1}{t}[\pi(T,v) + \sum_{i \in T} P(T \setminus i, v)] \\ &= \frac{1}{t}[\sum_{i \in T}(P^\eta(T,v) - P^\eta(T \setminus i, v)) + \sum_{i \in T} P^\eta(T \setminus i, v)] \\ &= \frac{1}{t}\sum_{i \in T} P^\eta(T,v) = P^\eta(T,v) \end{aligned}$$

□

In general, the Shapley value of a game (N,v) may or may not be an element of (N,v)'s core. In the former case, the interpretation of $\varphi(N,v)$ as a distribution of (expected) payoffs in (N,v) is particularly robust because no coalition could increase own payoffs by breaking away from the grand coalition. A characterization of those games for which the Shapley value lies in the core has been provided by Inarra and Usategui (1993) and Izawa and Takahashi (1998). Since the value ψ of (N,v) is just the Shapley value of the *power game* (N, v_ψ) defined by (4) if ψ admits a potential, the question of when the Shapley value of this power game lies in the power game's core is of interest. It makes the Shapley

blue print property of a value more meaningful in that the interpretation of ψ as the well-founded Shapley solution for a game related to (N, v) via ψ becomes more relevant because the latter is coalitionally rational and does not conflict with the (also well-founded) core solution of this game.

PROPOSITION **3** *The Shapley value of (N, v)'s power game induced by the Holler-Packel value, (N, v_η) – and thus the Holler-Packel value of (N, v) itself – lies in the core of (N, v_η).*

Proof:
By Corollary 4.1 in Dragan and Martinez-Legaz (2001) a solution ψ with potential P is in the core of the power game of (N, v) (actually even of each of its subgames (T, v)) iff

$$\begin{aligned} \forall S \subset T \subseteq N: \qquad & \sum_{i \in S}[P(S,v) - P(S \setminus i, v)) \leq \sum_{i \in S}[P(T,v) - P(T \setminus i, v)] \\ \iff & \sum_{i \in S} \psi_i(S,v) \leq \sum_{i \in T} \psi_i(T,v). \end{aligned}$$

Again let

$$M_i(N,v) = \{S \in M(N,v) \colon i \in S\}$$

and consider subgames (T, v) and (S, v) of (N, v) with $S \subset T \subseteq N$. Then

$$\forall i \in N: M_i(S,v) \subseteq M_i(T,v).$$

This implies that for any $i \in N$

$$\sum_{S' \in M_i(S,v)} v(S') \leq \sum_{S' \in M_i(T,v)} v(S')$$

and hence

$$\eta_i(S,v) \leq \eta_i(T,v),$$

which implies

$$\sum_{i \in S} \eta_i(S,v) \leq \sum_{i \in T} \eta_i(T,v).$$

□

5 Concluding Remarks

Already Hart and Mas-Colell (1989, p. 590) have remarked that "[a]lthough the potential is in its essence just a technical tool, it is . . . a powerful and suggestive one." Equivalence of its existence with established properties like preservation of differences and path independence is a case in point.

The potential of a given game can be regarded as a summary of all players' joint opportunities as captured by the corresponding value or index. These opportunities typically increase when a new player joins; the difference of potentials measures this increase and hence the contribution of the new player. One can start with the empty coalition, then successively draw additional players into the game and see joint opportunities increase exactly by the player's value in the resulting game.

In simple games the Holler-Packel potential can be regarded as the non-weighted number of decision-making opportunities the players collectively have (under Riker's size principle). Assuming, first, that the characteristic function v of a simple game just characterizes winning coalitions (not levels of transferable utility) and, second, that players get utility from having opportunities, one may interpret coalitions' worths (and players' values) in the power game v_η as sums of transferable utility. Because total opportunities derived from membership in minimal winning coalitions come with economies of scale, i. e. they expand with increasing marginals when new players are added, the power game is convex. Hence, evaluating players' overall contributions to opportunities in the game under consideration, i. e. determining their Holler-Packel 'power' or 'value' in it, amounts to assigning them a particularly distinguished (expected) utility vector in the associated power game's core: the Shapley value.

Based on the possible interpretation of the Shapley or Banzhaf values as an expected payoff, Ortmann (1998, p. 424) argues that potential "has got the interpretation of the ability to obtain utility" in analogy to potential (energy) in physics which is the ability of doing work. In the same vein, the Holler-Packel potential captures the aggregate ability to perform and benefit from collective actions (via minimal winning coalitions or minimal crucial coalitions). It can be interpreted as a measure of collective power or even freedom in the game under consideration.[12]

Note that the sum of players' (non-normalized) Holler-Packel values typically exceeds the worth of the grand coalition. If one regards players as receiving utility from having opportunities in the form of belonging to minimal crucial coalitions, the worths of non-singleton minimal crucial coalitions are 'distributed' several times (to all their members) in the power game. This concurs with the public good interpretation often given for Holler-Packel value and, in particular, Holler-Packel index.

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[12]See Braham (2004) on the close relationship between freedom and power, and corresponding 'opportunity' and 'exercise' concepts. Also see Laruelle and Valenciano (2004a, fn. 12; 2004b) for an explicit link between Coleman's (1971) *power of the collectivity to act* and a generalized potential function.

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