Voting power paradoxes revisited*

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Abstract

Power indices are heterogeneously grounded either in a particular formula with a more or less clear interpretation, and/or a set of -as a rule non compelling- axioms. In view of this unsatisfactory situation, and in order to test and compare them, some authors have proposed 'natural' properties that the power indices 'should' satisfy. The violation of these properties by the indices are called 'paradoxes.'

In this paper three general power measures based on both the voting rule and the voters' voting behavior (which include only as particular cases the most popular power indices, as well as some extensions of this concept), and some of these paradoxes test each other. As a result the consistence of these measures is satisfactorily checked, and new light on the non-paradoxical character of some paradoxes is shed.

Key words: Voting power measures, voting rules, voting behavior, paradoxes.

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1 Introduction

Different power indices have been proposed to assess the a priori distribution of power in voting situations. That is, the distribution of power among the voters for a given decision rule. Since the only recently vindicated Penrose (1946) and the later but much more popular Shapley and Shubik's (1954) and Banzhaf's (1965, 1966) indices, some other power indices have been proposed: the Coleman (1971, 1986) indices, the Deegan and Packel (1978) index, the Johnston (1978) index, and the Holler and Packel (1983) index. See also Felsenthal and Machover (1998) for a critical overview. There are also to be found in the cooperative game theoretic literature some solution concepts, as probabilistic values (Weber, 1979), semivalues (Weber, 1979 and 1988), and weak semivalues (Calvo and Santos, 2000) that can be seen as generalizations of the concept of power index when restricted to simple games.

The results given by these indices may differ quite widely. In order to test and compare these indices, some authors have proposed 'natural' properties that a power measure 'should' satisfy. The violation of these properties are called 'paradoxes'. There exists a variety of paradoxes in the literature. See for instance Brams (1975), Brams and Affuso (1976, 1985a, 1985b), Deegan and Packel (1982), Kilgour (1974), Fisher and Schotter (1978), and Dreyer and Schotter (1980). More recently, Felsenthal and Machover (1995, 1998) and Felsenthal, Machover and Zwicker (1998) have critically discussed some of these paradoxes and proposed new ones, like 'the bloc paradox', 'the donation paradox' and 'the bicameral paradox'.

All variations of the traditional power index concept formally take the voting rule as the only explicit input for the assessment of power. That is to say, power indices map voting procedures, usually modeled as simple games, onto vectors whose coordinates are interpreted as the 'power' of the corresponding voter. These power measures leave aside the voters' voting behavior, and whatever personal characteristics might condition it, as preferences over the issues or their interpersonal relations, etc. Consequently, the lack of basis for a positive or descriptive interpretation of these indices is obvious, as pointed out by some authors, as Garrett and Tsebelis (1999), because no information about the voters' behavior enters the model.

In Laruelle and Valenciano (2001a) a general *positive* or *descriptive* concept of voting power measurement based on *two* independent inputs in any real-world voting situation, the voting rule and the *voting behavior* of the voters, is provided. The voters' voting behavior, dependent on their preferences over the issues at stake, etc., is summarized by a probability distribution over the vote configurations. Then a priori voting power is defined as the *probability* of the different voters to 'exert power', that is, being decisive when a

decision is to be made according to a given voting rule, in three precise different senses. As shown in Laruelle and Valenciano (2001a), these general measures include as particular cases the Shapley-Shubik index and the (non normalized) Banzhaf index, as well as some game theoretic solution concepts.

In this paper we test 'against each other' some of the best-established voting power paradoxes and the three general measures of voting power introduced in Laruelle and Valenciano (2001a). This reciprocal test sheds some new light on the meaning of these so-called paradoxes and helps to understand better the concept of power in voting situations. In particular it shows how the voters' behavior influences their voting power. The coherence of the three alluded notions of voting power comes out reinforced by this test.

The rest of the paper is organized as follows. Section 2 contains the basic framework concerning voting rules. In section 3 the three measures of power based on the voting rule and the voting behavior introduced in Laruelle and Valenciano (2001a) are presented. In Section 4 we discuss the principle according to which 'the better the seat, the more the power.' Two paradoxes are considered, the 'dominance paradox' and the 'preference for blocker paradox'. Section 5 deals with the effect of transferring some weight in weighted majorities from one voter to another. It is proved that neither of the three measures of power displays the 'donation paradox'. Section 6 discusses the difficulties underlying the 'paradox of quarrelling members' and the 'bloc paradox'. Section 7 deals with bicameral systems and the 'weak bicameral paradox'. Finally, Section 8 sums up with some concluding comments.

2 Voting procedures

A *voting rule* is a well-specified procedure to make decisions by the vote of any kind of committee of a certain number of members. If the number of voters is n, the different *seats* will be labelled, and N will denote the set of labels, where usually $N = \{1, ..., n\}$. Once a proposal is submitted to the committee, votes are cast. A *vote configuration* is a possible or conceivable result of a vote, that is, a list of the votes cast from the different seats. We will consider only the case where voters do not abstain: every voter will be assumed to vote either 'yes' or 'no'. Under this assumption there are 2^n possible vote configurations, and each configuration can be represented by the set of seats' labels from which the 'yes' votes are cast. So, for each $S \subset N$, we refer as the *vote configuration* S to the result of a vote where the vote from the seats in S are 'yes', while the vote from the seats outside S are 'no'. We will drop i's brackets in $S\backslash\{i\}$ or $S \cup \{i\}$, and will denote by s the cardinal of S.

An N-voting rule specifies which vote configurations lead to the passage of a proposal

and which ones to its rejection. Thus an N-voting rule can be represented by the set of vote configurations (i.e., subsets of N) that would lead to the passage of a proposal. These configurations will be called *winning configurations*. In what follows $\mathcal{W}_N$ (or $\mathcal{W}$ when N is clear from the context) will denote the set of winning configurations representing an N-voting rule.

It will be assumed that a voting procedure satisfies these requirements. 1: The unanimous 'yes' configuration leads to the passage of the proposal: $N \in \mathcal{W}$. 2: The unanimous 'no' configuration leads to the rejection of the proposal: $\emptyset \notin \mathcal{W}$. 3: If a vote configuration is winning, then any other configuration containing it is also winning: If $S \in \mathcal{W}$, then $T \in \mathcal{W}$ for any T containing S. 4: If a vote configuration leads to the passage of a proposal, the configuration $N \backslash S$ will not: If $S \in \mathcal{W}$ then $N \backslash S \notin \mathcal{W}$.

Let VR_N denote the set of all such N-voting rules, each of them identified with the set $\mathcal{W}$ of winning configurations that specifies it. Some particular voting rules that will be considered later are the following. In a *weighted majority* rule, a 'weight' $w_i \geq 0$ is associated with each seat i, and a certain 'quota' $Q > 0$, such that $\frac{1}{2} \sum_{i \in N} w_i < Q \leq \sum_{i \in N} w_i$, is given. After a vote, the weights from the seats where 'yes' votes were cast are summed up. The proposal is passed if the sum is greater or equal to the quota. This is the voting rule specified by

$$\mathcal{W} = \{S \subseteq N : \sum_{i \in S} w_i \geq Q\}.$$

In a *dictatorship* the final outcome always coincides with the vote from a given seat. We will refer to the voter sitting in that seat as the *dictator*. The dictatorship of seat i is the voting rule $\mathcal{W} = \{S \subseteq N : i \in S\}$. We will also distinguish particular seats, such as a 'veto seat'. In voting rule $\mathcal{W}$, seat j is a *seat with veto* if for any $S \in \mathcal{W}$, $i \in S$. A voter sitting in a seat with veto will be referred to as a *vetoer* or a *blocker*, as the vetoer can 'block' a decision by voting against it.

In voting rule $\mathcal{W}$, seat j *dominates* seat i (denoted $j \succeq_{\mathcal{W}} i$) if for any configuration of votes S such that $i, j \notin S$,

$$S \cup i \in \mathcal{W} \ \Rightarrow \ S \cup j \in \mathcal{W}.$$

If j dominates i, but i does not dominates i, then j *is more desirable than* i ($j \succ_{\mathcal{W}} i$). The intuition of this notion (first proposed by Isbell, 1958) is that the seat j is more desirable that the seat i, for starting from vote configurations where i and j vote 'no', more often the vote from seat j will be able to turn out the outcome from a 'no' to a 'yes'. A seat with veto dominates any other. Note that the domination relation is not complete, seats cannot always be compared. In a weighted majority, the relation of domination is complete though: a seat with a larger weight will dominate those with a smaller weight.

3 Measurement of voters' voting power

All variations of the power index notion alluded to in the introduction formally take the voting rule as the only explicit input for the assessment of power. That is to say, they are defined as a map: $\varphi : VR_N \to R^n$, where for each voting rule $\mathcal{W} \in VR_N$, and any $i \in N$, $\varphi_i(\mathcal{W})$ is interpreted as voter sitting on seat i's a priori capacity to influence the outcome of a vote in voting rule $\mathcal{W}$. For some 'power indices' a more or less clear interpretation in probabilistic terms can be given. Nevertheless these probabilistic stories can only provide normative support -or evidence the lack of it- for some indices. But the lack of basis for a positive or descriptive interpretation of these indices is obvious, as pointed out by some authors, as Garrett and Tsebelis (1998), because no information about the voters' behavior enters the model[1].

In Laruelle and Valenciano (2001a) the assessment of voting power is explicitly based on *two* independent inputs, the voting rule and the *voting behavior* of the voters. That is to say, a general *positive* or *descriptive* measure of voting power is defined as a map: $\Phi : VR_N \times \mathfrak{P}_N \to R^n$, where VR_N is the set of all N-voting rules and $\mathfrak{P}_N$, still to be specified formally, is the set of all conceivable voting behaviors of n (N-labelled) voters. Thus for any voter $i \in N$, $\Phi_i(\mathcal{W}, p)$ is voter i's a priori capacity of being decisive in the voting rule $\mathcal{W} \in VR_N$ if the voters' voting behavior is described by $p \in \mathfrak{P}_N$. By 'a priori' we mean prior to the vote is cast, but once the voters occupy the seats.

The voters' voting behavior, dependent on their preferences over the issues at stake, etc., is summarized by a probability distribution over the vote configurations. Formally, $p : 2^N \to R$ will denote a distribution of probability that associates with each vote configuration S its probability of occurrence $p(S)$, where $0 \leq p(S) \leq 1$ for any $S \subseteq N$, and $\sum_{S \subseteq N} p(S) = 1$. That is, $p(S)$ gives the probability that voters whose labels are in S vote 'yes', while voters whose labels are outside S vote 'no'. We assume that voters vote 'yes' and vote 'no' with strictly positive probability[2]. In the following $\alpha_i(p)$ will denote the probability that voter i votes 'yes', $1 - \alpha_i(p)$ being then the probability that voter i votes 'no' as there is no abstention. We have thus

$$\alpha_i(p) = \sum_{T : i \in T} p(T) \quad \text{with} \quad 0 < \alpha_i(p) < 1.$$

$\mathfrak{P}_N$ will denote the set of all such distributions of probability over 2^N. This set can be

[1] Only in special cases, like in dictatorships, the information embodied in the decision-making rule is sufficient to a priori assess the voters' capacity to influence the outcome.

[2] This rules out the case of voters who will always vote 'yes' or will always vote 'no'. These cases can be dealt with by eliminating this voter and its seat, and redefining the rule and the probability distribution for the remainder voters.

interpreted as the set of all conceivable voting behaviors of n (N-labelled) voters (yes/no voters, in fact, according to the simplifying assumption made).

Observe that voter i's vote is not necessarily independent of other voters' vote. This is not a loss of generality but the opposite, because the independence is only a special case. In this particular case the probability of a vote configuration is fully determined by the vector $\alpha_N = (\alpha_i)_{i\in N}$, where α_i gives voter i's probability of voting 'yes'. Namely, in this particular case, denoting p_{α_N} the corresponding probability distribution, we have

$$p_{\alpha_N}(S) = \prod_{i\in S} \alpha_i \prod_{j\in N\setminus S} (1-\alpha_j).$$

Now let a voting situation in which an N-labelled set of n voters whose voting behavior is described by $p \in \mathfrak{P}_N$ is going to make decisions by means of voting rule $\mathcal{W}$. A voter's a priori voting *power* can be defined as the probability that the outcome of a vote coincides with her or his vote and this vote is decisive for it. We will previously distinguish the conditional probability of pushing a decision and that of blocking it. More precisely, i's *positive power* is the conditional probability of a vote configuration that leads to the passage of the proposal and i's vote is decisive for it, given that i votes 'yes'. Similarly, i's *negative power* is the conditional probability of a vote configuration that leads to the rejection of the proposal and i's vote is decisive for it, given that i votes 'no'.

The following definitions formalize the previous notions of voting power.

Definition 1 *(Laruelle and Valenciano, 2001a) For a given decision-making procedure $\mathcal{W} \in VR_N$ and a distribution of probability over the vote configurations $p \in \mathfrak{P}_N$,*

(i) voter i's positive voting power in voting situation $(\mathcal{W}, p)$ is given by:

$$\Phi_i^+(\mathcal{W}, p) := P(i \text{ is decisive} \mid i \text{ votes 'yes'}) = \sum_{\substack{S: i\in S\\ S\in\mathcal{W}\\ S\setminus i\notin\mathcal{W}}} \frac{p(S)}{\alpha_i(p)}, \tag{1}$$

(ii) voter i's negative voting power in voting situation $(\mathcal{W}, p)$ is given by:

$$\Phi_i^-(\mathcal{W}, p) := P(i \text{ is decisive} \mid i \text{ votes 'no'}) = \sum_{\substack{S: i\notin S\\ S\notin\mathcal{W}\\ S\cup i\in\mathcal{W}}} \frac{p(S)}{1-\alpha_i(p)}, \tag{2}$$

(iii) voter i's (general) voting power in voting situation $(\mathcal{W}, p)$ is given by:

$$\Phi_i(\mathcal{W}, p) := P(i \text{ is decisive}) = \alpha_i(p)\ \Phi_i^+(\mathcal{W}, p) + (1-\alpha_i(p))\ \Phi_i^-(\mathcal{W}, p). \tag{3}$$

These three general measures, $\Phi, \Phi^+, \Phi^- : VR_N \times \mathfrak{P}_n \to R^n$, provide an assessment of the probability of playing a relevant role in three different senses, taking as inputs *the*

voting rule and the voting behavior (i.e., the distribution of probability over vote configurations). In each particular real-world voting situation the voters' preferences condition the probabilities of different vote configurations, and consequently the voters' power (in a positive/descriptive sense). In each particular case the probability distribution that better fits the situation must be approximated with the available data.

Only in a purely *normative* approach to power measurement these preferences should be ignored. A way of doing so is by assuming equally probable all vote configurations, the natural starting point in case of actual ignorance. As shown in Laruelle and Valenciano (2001a), denoting p^{Bz} to the distribution such that $p^{Bz}(S) = \frac{1}{2^n}$ for all $S \subseteq N$, it holds

$$\Phi_i^+(\mathcal{W}, p^{Bz}) = \Phi_i^-(\mathcal{W}, p^{Bz}) = \Phi_i(\mathcal{W}, p^{Bz}) = Bz_i(\mathcal{W}) = \sum_{\substack{S:i\in S \\ S\in\mathcal{W} \\ S\backslash i\notin\mathcal{W}}} \frac{1}{2^{n-1}} \,,$$

for every voting procedure $\mathcal{W}$, where $Bz_i(\mathcal{W})$ denotes i's Banzhaf index[3] in voting rule $\mathcal{W}$. This provides support to the Banzhaf index as a *normative* measure of voting power, adequate to assess the distribution of power originated *by the rule itself.*

As pointed out in Laruelle and Valenciano (2001a), different particularizations of formulae (1), (2) and (3) for different probability distributions over vote configurations yield the Shapley-Shubik index. For instance, the Shapley-Shubik index gives the (general) power (3) if the probability distribution over vote configurations is such that all the 'yes'-voters' configurations' sizes (from 0 to n) are equally probable; and all configurations of the same size are equally probable[4].

The Deegan and Packel (1978), Johnston (1978) and Holler and Packel (1983) indices however are not particular cases of any of the three general measures of voting power, even if a 'non normalized' version of Holler-Packel index 'almost fits' (3) as a particular case. This means that these indices do not give the probability that voters exert power in a precise sense.

Finally, as shown in (2001a) some game theoretic solution concepts, as probabilistic values (Weber, 1979), semivalues (Weber, 1979 and 1988) and weak semivalues (Calvo and Santos, 2000), when restricted to simple games[5] can be seen as particular cases of the three general measures for specific probability distributions. Namely, $\Phi_i^+(-, p)$ and $\Phi_i^-(-, p)$ become weak semivalues when for any two voters the probability of voting 'yes'

[3] Be aware that we refer to the 'non normalized' Banzhaf index (Owen, 1975), which is obtained by dividing the number of vote configurations where the voter is decisive by 2^{n-1}.

[4] For other probabilty distributions coincides with the positive or the negative power. See Laruelle and Valenciano (2001a) for more details.

[5] A voting rule can be formally described as a simple superadditive game by defining the 'characteristic function' v that assigns 1 to every winning configuration, and 0 to the others.

is the same, i.e., if $\alpha_i(p) = \alpha_j(p)$ for all i, j; while $\Phi_i(-,p)$, $\Phi_i^+(-,p)$ and $\Phi_i^-(-,p)$ become semivalues when the probability of a configuration of votes only depends on its number of 'yes' voters, i.e., if $p(S) = p(T)$ whenever $s = t$.

4 The better the seat, the more the power?

The paradoxes that we consider in this section refer to the conflict between the ranking of voters' power provided by a measure for a given voting rule and variations of the principle 'the better the seat, the more the power'. For some power measures, it may happen that a voter occupies a 'better' seat than another but has less power. There are several paradoxes of this type that result from different specifications of when a seat is considered as 'better' than another.

The first one concerns weighted majority rules, where it seems that it is better to have a large weight than a small one. Nevertheless not all power measures satisfy the property 'the larger the weight, the more the power'. Deegan and Packel (1982) show that their Deegan-Packel index does not satisfy it, and refer to the failure of satisfying this property as the 'paradox of weighted voting'. According to Felsenthal and Machover (1995), a valid measure of power should not display the paradox of weighted voting. They go even further, generalizing the property to any voting rule, arguing that any reasonable measure of power should satisfy 'the more desirable (as defined in section 2) the seat, the more the power'. The violation of this property is referred to as the 'dominance paradox', which can be formulated as follows for the power measures in voting situations introduced in the previous section:

Dominance paradox: A power measure Φ is said to display the dominance paradox if there exists some N-voting rule $\mathcal{W}$ and some $p \in \mathfrak{P}_N$, such that

$$\Phi_j(\mathcal{W}, p) < \Phi_i(\mathcal{W}, p) \text{ athough } j \succ_{\mathcal{W}} i.$$

A weaker form of the principle 'the more desirable seat the more power' is to require that a vetoer has at least as much power as any other voter. The violation of this property is referred to by Felsenthal and Machover as the 'preference for blocker paradox'.

Preference for blocker paradox: A power measure Φ is said to display the preference for blocker paradox if there exists some N-decision-making rule $\mathcal{W}$ and some $p \in \mathfrak{P}_N$, such that

$$\Phi_j(\mathcal{W}, p) < \Phi_i(\mathcal{W}, p) \text{ although } j \text{ is a vetoer and } i \text{ is not.}$$

Now the question is: Is it reasonable to require or expect that 'the better the seat, the more the power' in any of the three forms? It could be a reasonable principle if the voters' power only depended on the rules, which is not the case. The probabilities of the vote configurations also matter. Therefore it may happen that a voter sitting in a more desirable seat has less power because the distribution of probability over the vote configurations more than compensates the voter in the worse seat. It is intuitively plausible that voter i, although sitting in a worse seat than voter j, may have a higher probability of being decisive than j if the configurations of votes where i is decisive have a very large probability of occurring compared to those where voter j is decisive.

Example: In the 4-person decision-making rule

$$\mathcal{W} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$$

seat 4 is more desirable than any other seat. Nevertheless, for the following probability distribution over vote configurations:

$$p(S) = \begin{cases} 1/2, & \text{if } S = \{1,2,3\} \text{ or } \{4\} \\ 0, & \text{otherwise,} \end{cases}$$

we obtain $\Phi_4^+(\mathcal{W},p) < \Phi_1^+(\mathcal{W},p)$, $\Phi_4^-(\mathcal{W},p) < \Phi_1^-(\mathcal{W},p)$ and $\Phi_4(\mathcal{W},p) < \Phi_1(\mathcal{W},p)$. This could be a stylized model for a four parties parliament, with three small left-wing parties (labelled 1, 2, and 3) and a large right-wing party (labelled 4). The large right-wing party has a smaller probability of exerting power than any small left-wing party because left-wing parties have similar (in the example identical) preferences, far different from the right-wing party's preferences.

Thus it is to be expected many occurrences of the dominance paradox for many distributions of probability over vote configurations. Notwithstanding, the dominance paradox never occurs for distributions of probability over vote configurations that exhibit a strong degree of symmetry. More precisely, if the probability of a vote configuration only depends on the number of its 'yes' voters, then a voter occupying a better seat will have at least as much power as the voter with a less desirable seat. In other words, semivalues (among them the Shapley-Shubik and the Banzhaf indices) do not display the paradox of dominance. Thus the following proposition sets a limit to the possibility of occurrence of the paradox of dominance (and therefore to the paradox of weighted voting or preference for blocker paradox).

Proposition 1 *For any decision rule $\mathcal{W}$ and any distribution of probability over vote*

configurations $p \in \mathfrak{P}_N$ such that $p(S) = p(T)$ whenever $s = t$, it holds

$$j \succ_{\mathcal{W}} i \Rightarrow \begin{cases} \Phi_j^+(\mathcal{W}, p) \geq \Phi_i^+(\mathcal{W}, p), \\ \Phi_j^-(\mathcal{W}, p) \geq \Phi_i^-(\mathcal{W}, p), \\ \Phi_j(\mathcal{W}, p) \geq \Phi_i(\mathcal{W}, p). \end{cases}$$

Proof. Let $j \succ_{\mathcal{W}} i$, that is, $S \cup i \in \mathcal{W} \Rightarrow S \cup j \in \mathcal{W}$, for any $S \subseteq N \setminus \{i, j\}$. Therefore $S \setminus i \notin \mathcal{W} \Rightarrow S \setminus j \notin \mathcal{W}$, for any S containing i and j. Then for any $p \in \mathfrak{P}_N$ we have

$$\begin{aligned} \Phi_i^+(\mathcal{W}, p) &= \sum_{\substack{S:S \ni i \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S)}{\alpha_i(p)} = \sum_{\substack{S:S \ni i,j \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S)}{\alpha_i(p)} + \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} \frac{p(S \cup i)}{\alpha_i(p)}, \\ \Phi_j^+(\mathcal{W}, p) &= \sum_{\substack{S:i,j \in S \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S)}{\alpha_j(p)} + \sum_{\substack{S:i,j \in S \\ S \in \mathcal{W} \\ S \setminus i \in \mathcal{W} \\ S \setminus j \notin \mathcal{W}}} \frac{p(S)}{\alpha_j(p)} + \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} \frac{p(S \cup j)}{\alpha_j(p)} + \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \notin \mathcal{W} \\ S \cup j \in \mathcal{W}}} \frac{p(S \cup j)}{\alpha_j(p)}. \end{aligned}$$

As $p(S \cup i) = p(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, and $\alpha_i(p) = \alpha_j(p)$, it yields $\Phi_i^+(\mathcal{W}, p) \leq \Phi_j^+(\mathcal{W}, p)$. Similarly for the negative power we have:

$$\begin{aligned} \Phi_i^-(\mathcal{W}, p) &= \sum_{\substack{S:i \notin S \\ S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} \frac{p(S)}{1 - \alpha_i(p)} = \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} \frac{p(S)}{1 - \alpha_i(p)} + \sum_{\substack{S:i,j \in S \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S \setminus i)}{1 - \alpha_i(p)} \\ \Phi_j^-(\mathcal{W}, p) &= \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \in \mathcal{W}}} \frac{p(S)}{1 - \alpha_j(p)} + \sum_{\substack{S:i,j \notin S \\ S \notin \mathcal{W} \\ S \cup i \notin \mathcal{W} \\ S \cup j \in \mathcal{W}}} \frac{p(S)}{1 - \alpha_j(p)} + \sum_{\substack{S:i,j \in S \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S \setminus j)}{1 - \alpha_j(p)} + \sum_{\substack{S:i,j \in S \\ S \in \mathcal{W} \\ S \setminus i \in \mathcal{W} \\ S \setminus j \notin \mathcal{W}}} \frac{p(S \setminus j)}{1 - \alpha_j(p)}. \end{aligned}$$

Thus $\Phi_i^-(\mathcal{W}, p) \leq \Phi_j^-(\mathcal{W}, p)$. Finally, as $\Phi_i(\mathcal{W}, p) = \alpha_i(p)\Phi_i^+(\mathcal{W}, p) + (1 - \alpha_i(p))\Phi_i^-(\mathcal{W}, p)$, we also have $\Phi_i(\mathcal{W}, p) \leq \Phi_j(\mathcal{W}, p)$. ■

From the proof, it can be understood more deeply the two possible reasons that give rise to the 'dominance paradox'. First, it must be recalled that the positive and the negative measures of power are *conditional* probabilities. Sitting on a better seat does not imply that the conditional probability of exerting positive power is higher. If voter i has a lower probability of voting 'yes' than voter j has (i.e., $\alpha_i(p) < \alpha_j(p)$), then the probability that i exerts positive power given that i votes 'yes' may be higher that the probability that j exerts positive power given that j votes 'yes'. Second, in general voters i and j do not exert power in the same configurations of votes. Thus, even if all voters had the same probability of voting 'yes' (i.e., $\alpha_i(p) = \alpha_j(p)$, for all i, j, as for 'weak semivalues'), it might be that the vote configurations where j exerts power have a very low probability of occurrence, while the ones where i exerts power have a high probability, which may explain why a better seat does not guarantee a higher probability of exerting power.

Finally we have a result limiting the possibility of occurrence of the 'preference for blocker paradox'. In a voting rule with a veto seat, the vetoer has a more desirable seat than any other voter, and exerts positive power in *all* winning configurations. It is thus impossible for any other voter to exert positive power in a vote configuration where the vetoer does not. The second reason that gives rise to the 'dominance paradox' disappears. The only possibility for a voter to have a larger positive power than the vetoer is when she or he has very small probability of voting 'yes' relative to the vetoer. In other words, weak semivalues (Calvo and Santos, 2000), which appear when the probability of any voter voting 'yes' is the same, will not display the 'preference for blocker paradox'.

Proposition 2 *For any decision rule $\mathcal{W}$ and any distribution of probability over vote configurations $p \in \mathfrak{P}_N$ such that $\alpha_i(p) = \alpha_j(p)$ for any two voters, it holds*

$$j \text{ is a vetoer in } \mathcal{W} \;\Rightarrow\; \Phi_j^+(\mathcal{W}, p) \geq \Phi_i^+(\mathcal{W}, p) \text{ for all } i.$$

Proof. If j is a vetoer and i any other voter, $S \backslash i \notin \mathcal{W} \Rightarrow S \backslash j \notin \mathcal{W}$, for any S containing i and j. Then we have for any $p \in \mathfrak{P}_N$,

$$\begin{aligned}
\Phi_i^+(\mathcal{W}, p) &= \sum_{\substack{S: i \in S \\ S \in \mathcal{W} \\ S \backslash i \notin \mathcal{W}}} \frac{p(S)}{\alpha_i(p)} = \sum_{\substack{S: i,j \in S \\ S \in \mathcal{W} \\ S \backslash i \notin \mathcal{W}}} \frac{p(S)}{\alpha_i(p)} \\
\Phi_j^+(\mathcal{W}, p) &= \sum_{\substack{S: i,j \in S \\ S \in \mathcal{W} \\ S \backslash i \notin \mathcal{W}}} \frac{p(S)}{\alpha_j(p)} + \sum_{\substack{S: i,j \in S \\ S \in \mathcal{W} \\ S \backslash i \in \mathcal{W} \\ S \backslash j \notin \mathcal{W}}} \frac{p(S)}{\alpha_j(p)} + \sum_{\substack{S: i,j \notin S \\ S \notin \mathcal{W} \\ S \cup j \in \mathcal{W}}} \frac{p(S \cup j)}{\alpha_j(p)}
\end{aligned}$$

Then as $\alpha_i(p) = \alpha_j(p)$, this yields $\Phi_i^+(\mathcal{W}, p) \leq \Phi_j^+(\mathcal{W}, p)$. ∎

The following example shows how the negative power and the global power may display the preference for blocker paradox even if all voters have the same probability of voting 'yes'.

Example: In the 4-person decision-making rule $\mathcal{W} = \{\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}$, voters 1 and 2 have a veto. Suppose that the vote configurations have the following probabilities:

$$p(S) := \begin{cases} 9/32, & \text{if } S = \{1,2\} \text{ or } \{3,4\} \\ 1/32, & \text{otherwise.} \end{cases}$$

A simple calculation shows that $\Phi_1^-(\mathcal{W}, p) < \Phi_3^-(\mathcal{W}, p)$ and $\Phi_1(\mathcal{W}, p) < \Phi_3(\mathcal{W}, p)$. Note that all voters have the same probability of voting 'yes': $\alpha_i(p) = \frac{1}{2}$, for $i = 1, ..., 4$.

In sum the paradox of dominance is not that surprising, and if semivalues do not display it, it is because in this case the probability of a vote configuration only depends

on its number of 'yes'-voters, something not to be expected in real-world situations in general. Similarly, it can be understood why the positive measure of power do not display the preference for blocker paradox when all voters have the same probability of voting 'yes'.

5 Transferring weight to gain power?

The paradox considered in this section concerns weighted majorities. The principle at stake is that a voter should not gain power by transferring part or all of her or his weight to another voter. Dreyer and Schotter (1980) consider a weighted majority where weights are redistributed, but keeping identical the total weight and the quota. They show that it may happen that a voter loses weight but increases her or his voting power according to some power indices. They refer to this phenomenon as the 'paradox of redistribution'. But as Felsenthal and Machover (1995) argue in the context of traditional power indices, the transfer of weight between two voters will affect the other voters. Therefore if there is more than one transfer of weight, the fact that a donor gains power is not paradoxical because it might be due to the transfers that have occurred among other voters. But if there is just one transfer between two voters, the power of the donor should not increase: 'it would be indeed paradoxical if a donor gained power purely as a result of his own donation: we do not really expect that in matters of power it is better to give than to receive. We surely ought to expect that donating weight may if anything cause a reduction in the donor's power.' (Felsenthal and Machover, 1998, p. 215). The violation of this principle is called the 'donation paradox'.

This paradox can be reformulated for the general class of power measures that we are considering here. In our approach the voting rule and the probability distribution over vote configurations are the two independent ingredients that jointly determine the voting power of the voters. In this case, a transfer of weight between two voters entails a change of voting rule. The question is then whether just one such transfer may increase the power of the donor given that the change of rule that does not modify the voters' voting behavior:

Donation paradox: A power measure Φ is said to display the donation paradox if there exist two weighted majority rules $\mathcal{W} = \{S \subseteq N : \sum_{i \in S} w_i > Q\}$ and $\mathcal{W}' = \{S \subseteq N : \sum_{i \in S} w'_i > Q'\}$, such that $Q = Q'$ and

$$w'_k = \begin{cases} w_i - \lambda, \text{ if } k = i \\ w_j + \lambda, \text{ if } k = j \\ w_k, \text{ if } k \neq i, j, \end{cases} \tag{4}$$

for some $0 < \lambda \leq w_i$, and some probability distribution $p \in \mathfrak{P}_N$, such that

$$\Phi_i(\mathcal{W}', p) > \Phi_i(\mathcal{W}, p).$$

The following result shows that neither of the three general measures of voting power under consideration exhibits this paradox.

Proposition 3 *Neither of the three general power measures given by (1), (2) and (3) displays the donation paradox.*

Proof. Let $\mathcal{W}$ and $\mathcal{W}'$ be two weighted majority rules, $\mathcal{W} = \{S \subseteq N : \sum_{i \in S} w_i > Q\}$ and $\mathcal{W}' = \{S \subseteq N : \sum_{i \in S} w_i' > Q'\}$, such that $Q = Q'$ and (4) for some $0 < \lambda \leq w_i$. In any weighted majority rule: $S \in \mathcal{W}_n \Leftrightarrow \omega(S) \geq Q$. On the other hand, $\omega'(S) = \omega(S)$ for all S s.t. $i, j \in S$, and $\omega'(S) = \omega(S) - \lambda$ for all S s.t. $i \in S$ and $j \notin S$. Then for any probability distribution over vote configurations $p \in \mathfrak{P}_N$, it holds:

$$\begin{aligned}
\Phi_i^+(\mathcal{W}, p) &= \sum_{\substack{S:i \in S \\ S \in \mathcal{W} \\ S \setminus i \notin \mathcal{W}}} \frac{p(S)}{\alpha_i(p)} = \sum_{\substack{S:i,j \in S \\ \omega(S) \geq Q \\ \omega(S) - w_i < Q}} \frac{p(S)}{\alpha_i(p)} + \sum_{\substack{S:i \in S, j \notin S \\ \omega(S) \geq Q \\ \omega(S) - w_i < Q}} \frac{p(S)}{\alpha_i(p)} \\
\Phi_i^+(\mathcal{W}, p) &= \sum_{\substack{S:i \in S \\ S \in \mathcal{W}' \\ S \setminus i \notin \mathcal{W}'}} \frac{p(S)}{\alpha_i(p)} = \sum_{\substack{S:i,j \in S \\ \omega'(S) \geq Q \\ \omega'(S) - w_i' < Q}} \frac{p(S)}{\alpha_i(p)} + \sum_{\substack{S:i \in S, j \notin S \\ \omega'(S) \geq Q \\ \omega'(S) - w_i' < Q}} \frac{p(S)}{\alpha_i(p)} \\
&= \sum_{\substack{S:i,j \in S \\ \omega(S) \geq Q \\ \omega(S) - w_i + \lambda < Q}} \frac{p(S)}{\alpha_i(p)} + \sum_{\substack{S:i \in S, j \notin S \\ \omega(S) - \lambda \geq Q \\ \omega(S) - w_i < Q}} \frac{p(S)}{\alpha_i(p)}
\end{aligned}$$

which entails $\Phi_i^+(\mathcal{W}', p) \leq \Phi_i^+(\mathcal{W}, p)$. The same inequality for Φ_i^- and Φ_i is derived similarly. ■

6 Joining to harm? Quarrelling to help?

The paradoxes considered in this section concern the effects in the voters' power of the formation of a 'bloc', or its opposite, that is, the effects of 'quarrelling'. Brams (1975) considers weighted rules where two voters decide to vote always together, forming a kind of indissoluble 'bloc'. The 'paradox of size' occurs when the power of the bloc is strictly smaller than the sum of the power of its components. Felsenthal and Machover (1995) generalize the idea of a bloc to any voting rule and argue that 'There are indeed very good common-sense arguments suggesting that the power of a bloc ought to be at least as great as the power of *the most powerful* of its component parts'. (Felsenthal and Machover, 1998,

p. 226). Otherwise the 'bloc paradox' emerges, that can be formulated in the traditional setting of 'power indices' as follows:

Bloc paradox: For any N-decision-making rule $\mathcal{W}$, and any two seats $i, j \in N$, where i is not a null seat[6], let $\mathcal{W}_B^{i,j}$ denote the N-decision-making rule such that:

$$\begin{aligned} S &\in \mathcal{W}_B^{i,j} \Leftrightarrow S \cup i \in \mathcal{W} \quad \text{for any } S \text{ containing } j, \\ S &\in \mathcal{W}_B^{i,j} \Leftrightarrow S \setminus i \in \mathcal{W} \quad \text{for any } S \text{ not containing } j. \end{aligned}$$

A power index ϕ suffers from the bloc paradox if for some N-voting rule $\mathcal{W}$ and some $i, j \in N$, as above,

$$\phi_j(\mathcal{W}_B^{i,j}) < \phi_j(\mathcal{W}).$$

The 'paradox of quarrelling members' is introduced as follows: 'We may suppose for example that two players are involved in a quarrel and refuse to join together to help forming a winning coalition. Although one might suspect that they could only succeed in hurting each other, it is a curious fact that the quarrel between two players may actually redound to their benefit by increasing both their individual and combined voting power. We call this phenomenon the paradox of quarreling members'. (Brams 1975, p. 181). In the traditional power indices' setting, the paradox can be stated as follows:

Paradox of quarreling members: For any N-decision-making rule $\mathcal{W}$, and $i, j \in N$, let $\mathcal{W}_Q^{i,j}$ denote the voting rule that results from deleting from the set of winning configurations those containing both voters i and j, that is

$$\mathcal{W}_Q^{i,j} = \mathcal{W} \setminus \{S \subseteq N : i, j \in S\}$$

A power index ϕ displays the paradox of quarrelling members if

$$\phi_i(\mathcal{W}_Q^{i,j}) > \Phi_i(\mathcal{W}) \quad \text{or} \quad \phi_j(\mathcal{W}_Q^{i,j}) > \Phi_j(\mathcal{W}).$$

Unlike the paradoxes considered so far, the reformulation of these paradoxes in the setting of the general measures under consideration is not obvious, because there are some difficulties concerning the adequacy of the formal statement of these paradoxes to the situations they refer to. Moreover, even the precise meaning of such situations requires some exam. First, note that neither the formation of a bloc, nor the quarrel of two voters, affects the voting rule, as implied by the above formulations[7]. If two voters change their

[6] A 'null seat' in a voting rule is a seat such that the result of a vote is never influenced by the vote cast from that seat. That is, i is a null seat in rule $\mathcal{W}$, if $S \in \mathcal{W} \Leftrightarrow S \setminus i \in \mathcal{W}$.

[7] Similar doubts were already raised by Straffin (1982) or Felsenthal and Machover (1998) concerning the quarrelling paradox.

voting behavior in some way so as to always vote together (or opposite), such a change concerns the voting behavior of the voters. Thus, if the starting point is a voting situation $(\mathcal{W}, p)$ we should keep unchanged the voting rule $\mathcal{W}$ and modify the second component of the voting situation, the voting behavior represented by p.

Now let us examine how the distribution of probability p should be modified in order to reflect adequately these changes of behavior. First, consider the case of a 'bloc': Which is the meaning of two voters, say voters i and j, switching to 'always vote together'? From this, one can infer that the probability of a vote configuration where these two voters vote differently becomes zero, while the other voters do not modify their voting behavior. But to complete a coherent description of the change of behavior, it must be specified which member of the bloc follows the other. Indeed, it does not make sense that *both* voters change their vote to vote as the other does! In case it is voter i who changes his or her behavior to vote permanently as voter j, we will say that 'i switches in favor of j'. This covers situations where voter i gives his or her seat or vote to voter j, or a voter with identical preferences to voter j replaces voter i, or even voter i decides to copy any vote cast by j.

Similarly, in the case of 'quarrelling' between i and j it is not sufficient to state that a vote configuration where both voters vote 'yes' or 'no' has a null probability after the quarrel, while the other voters do not modify their behavior. Again, it does not make any sense *both* voters switching to vote against each other! The specification of whose voting behavior changes is needed to give a coherent sense to such a quarrel. So, if it is voter i who changes his or her behavior we will say 'i switches against j' to mean that voter i decides to vote always opposite to voter j.

Thus we consider two similar and opposed changes *affecting only voter i's behavior,* from a previous situation described by probability distribution p. The change induced in the distribution of probability when i switches in favor of j are (1) the probability of any vote configuration where i and j vote opposite becomes zero, (2) the probability of a vote configuration S where i and j both vote 'yes' is increased by the previous probability of the vote configuration $S \setminus i$, and (3) the probability of a vote configuration S where i and j both vote 'no' is increased by the former probability of the vote configuration $S \cup i$. Denoting p_{ij}^{b} the probability distribution resulting from p by the bloc resulting from i switching in favor of j we have

$$p_{ij}^{b}(S) := \begin{cases} p(S) + p(S \setminus i), \text{ if } i, j \in S \\ p(S) + p(S \cup i), \text{ if } i, j \notin S \\ 0, \text{ otherwise.} \end{cases}$$

In the second case, when voter 'i switches against j', the resulting probability distribution p^q_{ij} from p, can be similarly derived in order to get:

$$p^q_{ij}(S) := \begin{cases} p(S) + p(S \cup i), \text{ if } j \in S \text{ and } i \notin S \\ p(S) + p(S \setminus i), \text{ if } j \notin S \text{ and } i \in S \\ 0, \text{ otherwise.} \end{cases}$$

At first sight it seems reasonable to expect that if voter i gives his or her vote to voter j this would not harm voter j's power. Similarly, if voter i switches to oppose j's vote permanently this would not benefit voter j's power. The violation of these properties gives rise to the following 'paradoxes' in terms of our power measures:

Bloc (i switching in favor of j) paradox: A power measure Φ is said to display the bloc (i switching in favor of j) paradox if there exists an N-decision-making rule $\mathcal{W}$, such that for some $i, j \in N$, and some $p \in \mathfrak{P}_N$,

$$\Phi_j(\mathcal{W}, p^b_{ij}) < \Phi_j(\mathcal{W}, p).$$

Quarrelling (i switching against j) paradox: A power measure Φ is said to display the quarrelling (i switching against j) paradox if there exists an N-decision-making rule $\mathcal{W}$, such that for some $i, j \in N$, and some $p \in \mathfrak{P}_N$,

$$\Phi_j(\mathcal{W}, p^q_{ij}) > \Phi_j(\mathcal{W}, p).$$

The following example shows that the measure of positive power (1) displays both paradoxes for certain probability distributions over the vote configurations. We left for the reader to work for himself or herself similar examples for the other two measures of power, and for occurrences of the quarrelling paradox.

Example: Consider the voting situation given by the 3-person majority rule $\mathcal{W} = \{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, and the following probability distribution over vote configurations:

$$p(S) = \begin{cases} 9/16, & \text{if } S = \{1,3\} \\ 1/16, & \text{otherwise.} \end{cases}$$

Assume voter 2 switches in favor of voter 1. Then

$$p^b_{21}(S) = \begin{cases} 5/8, \text{ if } S = \{1,2,3\} \\ 1/8, \text{ if } S = \emptyset,\ \{3\}, \text{ or } \{1,2\} \\ 0, \text{ otherwise.} \end{cases}$$

Voter 1 exerts power in configurations $\{1,2\}$ and $\{1,3\}$. With voter 2 joining voter 1, the vote configuration $\{1,2\}$ increases its probability from $1/16$ to $1/8$, while the vote

configuration $\{1,3\}$ decreases its probability from $9/16$ to 0. As a result, $\Phi_1^+(\mathcal{W}, p_{21}^b) < \Phi_1^+(\mathcal{W}, p)$.

If plausible at first sight, it is not surprising on deeper reflection that all the three power measures under consideration violate the 'principles' behind these 'paradoxes'. To fix ideas take the positive power and the first paradox (similar considerations apply to the other two measures). When voter i switches in favor of voter j, first note that voter j's probability of voting 'yes' is not modified, i.e., $\alpha_j(p) = \alpha_j(p_{ij}^b)$. But this change has two opposite effects on j's positive power. On the one hand, the probability of those winning configurations S containing i and j in which j is crucial increases in $p(S \setminus i)$. On the other hand, the probability of those winning configurations S containing j but not i in which j is crucial become 0. Thus if the first effect can increase j's positive power, the second can diminish it. The situation is similar for the quarrelling paradox.

Again once the paradox is explained it does not deserve the name of 'paradox' any more. Nevertheless the reader may feel some uneasiness to accept as trivial the paradox of violating what looked like two intuitive principles. In order to help conciliate the transparency of the explanation of the paradox with the frustration this reader may feel, it may be of some help to test against these paradoxes the concept of 'success' (Barry, 1980a and 1980b), or, better, its precise formulation and generalization proposed by us in Laruelle and Valenciano (2001b) based on the idea that a voter has 'success' whenever the decision coincides with the voter's vote, be her vote or not decisive for it. Again we can also distinguish between the 'positive' the 'negative' conditional variations of the concept.

Definition 2 *(Laruelle and Valenciano, 2001b) For a given N-decision-making procedure $\mathcal{W}$ and a distribution of probability $p \in \mathfrak{P}_N$ over the vote configurations,*

(i) voter i's measure of positive success in voting situation $(\mathcal{W}, p)$ is given by:

$$\Omega_i^+(\mathcal{W}, p) := P(\textit{the proposal is passed} \mid i \textit{ votes 'yes'}) = \sum_{\substack{S:S \ni i \\ S \in \mathcal{W}}} \frac{p(S)}{\alpha_i(p)}, \tag{5}$$

(ii) voter i's measure of negative success in voting situation $(\mathcal{W}, p)$ is given by:

$$\Omega_i^-(\mathcal{W}, p) := P(\textit{the proposal is rejected} \mid i \textit{ votes 'no'}) = \sum_{\substack{S:i \notin S \\ S \notin \mathcal{W}}} \frac{p(S)}{1 - \alpha_i(p)}, \tag{6}$$

(iii) voter i's measure of (general) success in voting situation $(\mathcal{W}, p)$ is given by:

$$\Omega_i(\mathcal{W}, p) := P(\textit{the decision coincides with } i\textit{'s vote}) = \sum_{\substack{S:S \ni i \\ S \in \mathcal{W}}} p(S) + \sum_{\substack{T:i \notin T \\ T \notin \mathcal{W}}} p(T). \tag{7}$$

These measures, that capture the basic idea behind 'success' in Barry and generalizes it (in its three variants) in probabilistic terms, can be formally seen as power measures based in the same inputs ($\mathcal{W}$ and p) and a different assessment of power. That is to say, power as 'success', where success means winning the vote, instead of power meaning winning the vote and being decisive for it. Then we have the following result which may reassure the uncomfortable reader convinced that there was something reasonable in the violated principles: joining cannot harm and opposing cannot help.. *success*.

Proposition 4 *Neither of the three general measures of success given by (5), (6) and (7) displays the bloc (i switching in favor of j) paradox, nor the quarrelling (i switching against j) paradox.*

We leave the easy proof to the reader.

7 Bicameral paradox?

Felsenthal, Machover and Zwicker (1998) consider a bicameral system, where a bill requires the approval of two distinct chambers to be passed. A bicameral system can be modelled as follows. Let N_1 and N_2 denote the seats in either chamber ($N_1 \cap N_2 = \emptyset$), and let $\mathcal{W}_{N_1}$ and $\mathcal{W}_{N_2}$ denote the decision-making rules used by each chamber. Then a bicameral procedure based on these rules is defined by the N-decision-making rule $\mathcal{W}_N$, with $N = N_1 \cup N_2$, where

$$\mathcal{W}_N = \{S \subseteq N : S \cap N_1 \in \mathcal{W}_{N_1} \text{ and } S \cap N_2 \in \mathcal{W}_{N_2}\}.$$

They argue that it would be unreasonable that the order of power between two voters would be reversed from one chamber to the bicameral system: if one voter has more power in one chamber than another voter then she or he should also be more powerful in the bicameral system. If this is not so, the index would display the 'weak bicameral paradox'. More precisely, in traditional power indices' terms:

Weak bicameral paradox: A power index ϕ displays the weak bicameral paradox if for some bicameral system $\mathcal{W}_N$ based on $\mathcal{W}_{N_1}$ and $\mathcal{W}_{N_2}$, the following property is not satisfied for any pair of voters i and j from the first chamber:

$$\phi_i(\mathcal{W}_{N_1}) < \phi_j(\mathcal{W}_{N_1}) \Leftrightarrow \phi_i(\mathcal{W}_N) < \phi_j(\mathcal{W}_N).$$

Again we have a formulation whose translation into the more general terms of the power measures we are dealing with is not straightforward. Again traditional power indices take the voting rules as the only explicit input, while now we need a probability distribution describing voters' voting behavior.

In our model we need the two independent inputs, voting rule and voting behavior, for the two chambers and for the bicameral system. But mind we are speaking of a single set of voters, N, and in our model the voting behavior is represented by a probability distribution p_N over all N-vote configurations. Now N_1 and N_2 sets of voters are *subsets* of N. Therefore the behavior of these subsets of voters (that can be described by some probability distributions p_{N_1} and p_{N_2}) can be derived from p_N. Namely, the probability $p_{N_1}(S)$ of an N_1-configuration $S \subseteq N_1$ in which voters with labels $i \in S$ vote 'yes' while voters with labels $j \in N_1 \backslash S$ vote 'no', should be equal to the probability $p_N(R)$ of an N-configuration $R \subseteq N$ such that $R \cap N_1 = S$. The same can be said for the voters in the second chamber. Then the requirement of consistency between the voting behavior of voters in N, N_1, and N_2, can be stated as follows:

$$
\begin{aligned}
p_{N_1}(S) &= \sum_{\substack{R \subset N \\ R \cap N_1 = S}} p_N(R) = \sum_{T \subset N_2} p_N(S \cup T) \quad \text{for any } S \subset N_1, \\
p_{N_2}(S) &= \sum_{\substack{R \subset N \\ R \cap N_2 = S}} p_N(R) = \sum_{T \subset N_1} p_N(S \cup T) \quad \text{for any } S \subset N_2.
\end{aligned}
$$

In particular this entails that the probability that a voter from the first chamber votes in favor of the proposal is the same in the first chamber and in the bicameral system $\alpha_i(p_{N_1}) = \alpha_i(p_N)$, for all $i \in N_1$. Similarly $\alpha_i(p_{N_2}) = \alpha_i(p_N)$, for all $i \in N_2$. Observe that p_{N_1} and p_{N_2} are fully determined by p_N, while the voting rule $\mathcal{W}_N$ is fully determined by $\mathcal{W}_{N_1}$ and $\mathcal{W}_{N_2}$. Then the 'weak bicameral paradox' can be reformulated for our general measures of power.

Weak bicameral paradox: A power measure Φ displays the weak bicameral paradox if for some bicameral system $\mathcal{W}_N$ based on $\mathcal{W}_{N_1}$ and $\mathcal{W}_{N_2}$, and some $p_N \in \mathfrak{P}_N$, the following property is not satisfied for any pair of voters i and j from the first chamber:

$$\Phi_i(\mathcal{W}_{N_1}, p_{N_1}) < \Phi_j(\mathcal{W}_{N_1}, p_{N_1}) \Leftrightarrow \Phi_i(\mathcal{W}_N, p_N) < \Phi_j(\mathcal{W}_N, p_N). \tag{8}$$

Is that paradoxical the violation of this property for the measures considered, as is the case for the three of them? Not really. These positive/descriptive measures of power do not depend exclusively on the rules, but also on the probabilities over vote configurations. Therefore it might be that voter i has a lower probability of exerting power than voter j in a chamber, but a larger one in the whole bicameral system, because in the first chamber the vote configurations where voter i is decisive have relatively low probabilities of occurring with respect to those where voter j is decisive, while the reverse happens in the whole bicameral system. As an extreme example, consider a bicameral system in

which decisions are made by simple majority in both chambers. Imagine that in the first chamber all voters independently toss a coin to vote 'yes' or 'no', while in the second chamber all voters blindly vote as a particular voter from the first chamber. Then while in the first chamber all voters will have the identical positive power, in the bicameral system the voter whose vote is always followed by the members of the second chamber will have a larger power than any other from the first chamber.

Notwithstanding, it is possible to set a clear limit to the occurrence of the weak bicameral paradox. Consider a voting situation consisting of a bicameral system in which the voting behavior of voters in one chamber is *independent* form that of voters in the other one. In this case we have:

$$p_N(R) = p_{N_1}(R \cap N_1)\ p_{N_2}(R \cap N_2) \quad \text{for all } R \subseteq N. \tag{9}$$

In these situations we have the following result:

Proposition 5 *For any bicameral system in which the voting behavior of voters in one chamber is independent form that of the voters in the other one, the three power measures given by (1), (2) and (3) satisfy the weak bicameral property (8).*

Proof. Let $\mathcal{W}_N$ be a bicameral system based in $\mathcal{W}_{N_1}$ and $\mathcal{W}_{N_2}$. And let $p_N \in \mathfrak{P}_N$ satisfying (9). In this case for any voter i in N_1, $\Phi_i^+(\mathcal{W}_N, p_N)$ is the conditional probability (given that i votes 'yes') of being decisive in the chamber to which the voter belongs multiplied by the probability of a winning vote configuration occurring in the other chamber. That is, as

$$P(\text{Chamber 2 accepts the proposal}) = \sum_{T \in \mathcal{W}_{N_2}} p_{N_2}(T),$$

we have

$$\Phi_i^+(\mathcal{W}_N, p_N) = \Phi_i^+(\mathcal{W}_{N_1}, p_{N_1}) \sum_{T \in \mathcal{W}_{N_2}} p_{N_2}(T).$$

Similarly

$$\Phi_i^-(\mathcal{W}_N, p_N) = \Phi_i^-(\mathcal{W}_{N_1}, p_{N_1}) \sum_{T \in \mathcal{W}_{N_2}} p_{N_2}(T).$$

¿From which it follows that

$$\Phi_i(\mathcal{W}_N, p_N) = \Phi_i(\mathcal{W}_{N_1}, p_{N_1}) \sum_{T \in \mathcal{W}_{N_2}} p_{N_2}(T).$$

Then, unless the probability of acceptance in the second chamber is zero, (8) holds for Φ_i^+, Φ_i^- and Φ_i. ■

Thus this simple result provides a wide class of examples of bicameral situations in which the weak bicameral paradox does not occur. Note that the case in which each voter independently votes 'yes' with a certain probability α_i is included. In particular, the Banzhaf index will not display the paradox, because the corresponding distribution of probability is the special case of independence with $\alpha_i = 1/2$ for any voter. But in real-world bicameral situations, the voting behavior in both chambers is not independent: some correlation between the vote of the two chambers is to be expected, and occurrence of the weak bicameral paradox is not surprising.

8 Conclusion

We have tested 'against each other' some of the best known voting power paradoxes and the three general measures of voting power introduced in Laruelle and Valenciano (2001a). As a result of this reciprocal test these power measures come out reinforced, in the sense that their coherence challenges these so-called paradoxes. Also as a result a better understanding of these paradoxes is gained.

We have shown that neither of the three general measures display the only paradox based on a really compelling postulate in a positive/descriptive sense: the donation paradox. In the other cases, in which the 'paradox' may occur for certain voting behaviors, the situation can be explained in clear and simple terms consistent with real-world experience, so that the paradoxes dissipate as such. This is the case of the dominance paradox, the bloc paradox, the quarrelling paradox and the bicameral paradox. This test confirms the coherence a concept of power based on the two ingredients that enter any voting situation: the voting rule and the voters' behavior.

This reciprocal test has also shed some light on the nature of some of these paradoxes. For this test to be possible a previous reformulation of these paradoxes has been necessary in terms of power measures that are based on both the voting rule and the voters' voting behavior. This reformulation has disclosed some internal difficulties in the traditional terms' formulation of some of these paradoxes, at least when voting behavior is included in the model via a probability distribution over vote configurations. This is the case of the bloc and quarrelling paradoxes, as well as the bicameral paradox. Another interesting result is the 'limit' obtained for the possibility of occurrence of some paradoxes, as the dominance paradox, which never occurs for any of the three measures when they are semivalues, and preference for blocker paradox, which never occurs for the positive power when it is a weak semivalue. This shows how only a bit of symmetry is enough to avoid some paradoxes. Similarly the independence of the chambers guarantees that the weak bicameral paradox does not occur. This 'limiting results' set a limit in fact to the selective

value of these paradoxes for normative purposes.

¿From the positive/descriptive point of view associated with this general approach to voting power measurement based on the voting rule and the voting behavior, the irrelevance of most paradoxes seems the most clear outcome, beyond the 'deeper insight into the true nature of voting power' (Felsenthal and Machover (1998, p. 276)) their discussion helps to gain. The precedent discussion (section 6 exemplifies it in a specially clear way) may serve as a warning about how unsafe turns out to be an aprioristic paradox-based evaluation of power measures. Before hurrying to raise expectations about how they should behave, it is only wiser a previous deep understanding and consistent formulation of whatever one is talking about. Only basic misunderstandings may account for speaking about 'paradoxes'. In other words, paradoxes tell more about our prejudices about power measures than about the concept of power measurement.

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