On some properties of the Hoede-Bakker index *

Agnieszka Rusinowska †(a), (b)
Harrie de Swart (a)

(a) Tilburg University, Department of Philosophy, P.O. Box 90153
5000 LE Tilburg, The Netherlands

(b) Warsaw School of Economics, Institute of Econometrics
Al. Niepodleglosci 162, 02-554 Warsaw, Poland

Abstract

The paper concerns an analysis of the decisional power index, the so called Hoede-Bakker index. This index takes the preferences of the players into account, as well as the social structure in which players may influence each other. We investigate the properties of the Hoede-Bakker index and the relations between this index and other (well-known) power indices. In the special case that all players are independent, i.e. no player influences any other player, the Hoede-Bakker index reduces to the absolute Banzhaf index. We also investigate whether this decisional power index displays some voting power paradoxes and whether it satisfies the postulates for power indices.

Key words: Hoede-Bakker index, power index, inclination vector, group decision, voting power paradoxes

1 Introduction

Many power indices have been introduced (Shapley and Shubik, 1954; Penrose, 1946; Banzhaf, 1965; Dubey and Shapley, 1979; Coleman, 1971; Johnston, 1978; Deegan and Packel, 1978; Holler, 1982; Holler and Packel, 1983; etc.). A review and comparison of these power indices have been made, for instance, in Laruelle (Laruelle, 2000). In the literature, some features of power indices, called voting power paradoxes, have been investigated, for instance, the paradox of new members and the paradox of large size (Brams, 1975; see also: Brams and Affuso, 1976), the paradox of redistribution (Fischer and Schotter, 1978), the

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quarrelling paradoxes (Kilgour, 1974). A theoretical analysis and extensive discussion of power indices and voting power paradoxes is given in Felsenthal and Machover (1998). There is also empirical research on paradoxes of power indices (see, for instance: Schotter, 1981). In particular, the occurrence of voting power paradoxes in real politics has been studied (Van Deemen and Rusinowska, 2001; Rusinowska, 2001). More recently, Felsenthal, Machover and Zwicker (Felsenthal and Machover, 1995, 1998; Felsenthal, Machover and Zwicker, 1998; see also: Laruelle, 2000) proposed some desirable properties of power indices, that they refer to as postulates, the non-fulfillment of which they considered as paradoxical.

The aim of this paper is to investigate properties of the, to the best of our knowledge, less well-known Hoede-Bakker index. Hoede and Bakker (1982) introduced the concept of decisional power. The essential point of the Hoede-Bakker index is the distinction between the inclination to say "yes" or "no" and the final decision apparent in a vote. As Hoede and Bakker noticed "In fact, it is the difference between inclination and final decision in which the exertion of power on an actor manifests itself".

Our paper is organized as follows. Section 2 contains a short survey of power indices, voting power paradoxes and some postulates. All remaining sections, that is, Sections 3-9, concern the Hoede-Bakker index. In Section 3, we describe the decisional power index. In Section 4, some examples are presented, showing in particular that this index depends on the group decision. Section 5 concerns a case with an even number of players. In Section 6, the decisional power index of a set of players is analysed. In Section 7, we deal with some further properties of the Hoede-Bakker index. In particular, we check whether this index satisfies some postulates and displays some paradoxes. In Section 8, the relations between the Hoede-Bakker index and other power indices are presented. Finally, Section 9 contains conclusions. The paper contains also two appendices. Appendix A contains examples of social networks for $n = 2, 3, 4, 5$. In Appendix B, we add tables presenting the results for figures from Appendix A. The last table in Appendix B summarizes the results.

2 On Power Indices and their Properties

2.1 Power Indices

A *(0,1)-game* is a pair (N, v), where $N = \{1, ..., n\}$ denotes the set of players and $v : 2^N \to \{0, 1\}$ is a function for which $v(\emptyset) = 0$. A *simple game* is a (0,1)-game such that $v(N) = 1$ and v is nondecreasing, i.e., $v(S) \leq v(T)$ whenever $S \subseteq T \subset N$. Any nonempty subset of N, $\emptyset \neq S \subseteq N$, is called a *coalition.* A coalition S is *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. Player $k \in S$ is a *swinger* in a winning coalition S, if his or her removal from this coalition makes it losing, that is, if $v(S) = 1$ and $v(S \setminus \{k\}) = 0$. We refer to a winning coalition in which all players are swingers as a *minimal winning coalition.* If one player, say player $k \in N$, forms the only minimal winning coalition, then k is called a

dictator. A *dummy* is a player who is a member of no minimal winning coalition. Hence, a dictator, if there is one, can be characterized as the sole player who is not a dummy.

One interesting class of simple games is the class of *weighted voting games*. We use the symbol $[q; w_1, w_2, ..., w_n]$ to represent a weighted voting game, where q is the *quota* needed for a coalition to win, and w_k is the number of votes of player k (k = 1, 2, ..., n). The quota q and the voting weights $w_1, w_2, ..., w_n$ are positive integers with $0 < q \leq \sum_{k=1}^{n} w_i$. Then, the expression $[q; w_1, w_2, ..., w_n]$ represents the simple game $v : 2^N \to \{0, 1\}$ defined for all $S \subseteq N$ by

$$v(S) = \begin{cases} 1 & if \quad \sum_{k \in S} w_k \geq q \\ 0 & \quad otherwise \end{cases} . \tag{1}$$

A *power index* is a function ϕ that associates with each simple game (N, v) a vector $\phi = (\phi_1, ..., \phi_n)$, where ϕ_k is interpreted as a measure of the influence that player k can exert on the outcome.

Many ideas how to evaluate the distribution of power among the players have appeared. We will give a short survey of some (well-known) power indices.

- The *Shapley-Shubik index* for the simple game (N, v) is the vector $Sh(v) = (Sh_1(v), ..., Sh_n(v))$, given by

$$Sh_k(v) = \frac{p_k}{n!} \tag{2}$$

  for each $k = 1, ..., n$, where p_k is the number of orders in which player k is *pivotal*. A player is said to be pivotal for an order if it turns the coalition of players preceding him in that order into a winning coalition.

- The *normalized Banzhaf index* for the simple game (N, v) is the vector $Bz(v) = (Bz_1(v), ..., Bz_n(v))$, where for each $k = 1, ..., n$

$$Bz_k(v) = \frac{\eta_k}{\sum_{j \in N} \eta_j} \tag{3}$$

  and η_k means the number of winning coalitions in which player k is a swinger.

- The *non-normalized Banzhaf index*, also called the *absolute Banzhaf index* is the vector $nnBz(v) = (nnBz_1(v), ..., nnBz_n(v))$, given by

$$nnBz_k(v) = \frac{\eta_k}{\beta_k} = \frac{\eta_k}{2^{n-1}} \tag{4}$$

  for each $k = 1, ..., n$, where β_k denotes the total number of coalitions containing player k.

- The *Deegan-Packel index* for the simple game (N, v) is the vector $DP(v) = (DP_1(v), ..., DP_n(v))$, defined by

$$DP_k(v) = \frac{1}{m} \sum_{\{S \in M : k \in S\}} \frac{1}{s} \tag{5}$$

for each $k = 1, ..., n$, where M means the set of all minimal winning coalitions, m is the total number of minimal winning coalitions, and s is the number of players in S.

- The *Holler-Packel index*, also called the *Holler index* or the *public good power index*, is the vector $HP(v) = (HP_1(v), ..., HP_n(v))$, where

$$HP_k(v) = \frac{m_k}{\sum_{j \in N} m_j} \tag{6}$$

for $k = 1, ..., n$, and m_k means the number of minimal winning coalitions containing player k.

2.2 Voting Power Paradoxes

In this subsection, we will recapitulate some counterintuitive features of power indices analysed in the literaure, called *paradoxes of power indices.*

- **Paradox of new members** (Brams, 1975; Brams and Affuso, 1976) -
It appears when a new party joins the assembly and at least one old party has greater voting power in this new situation than in the old one.
Let $V = [q; w_1, ..., w_n]$ and $V' = [q'; w_1, ..., w_n, w_{n+1}]$.
A power index ϕ displays the paradox of new members if

$$\text{for some } k \in N,\ \phi_k(V') > \phi_k(V).$$

- **Paradox of large size** (Brams, 1975) -
It occurs when the power index of a union of parties is less than the sum of the power indices of the separate parties of that union.
Let $P \subseteq N$ be the set of players who decided to unite. Let us call this union player U, where $w_U = \sum_{k \in P} w_k$.
A power index ϕ displays the paradox of large size if

$$\phi_U < \textstyle\sum_{k \in P} \phi_k.$$

- **Paradox of redistribution** (Fischer and Schotter, 1978) -
It appears when either a party's voting weight decreases and at the same time its power index increases, or when a party gains in terms of voting weight, but loses in voting power.
Let $V = [q; w_1, ..., w_n]$, $V' = [q; w'_1, ..., w'_n]$, and $\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} w'_k$.
A power index ϕ displays the redistribution paradox if

$$\text{for some } k,\ w'_k < w_k \text{ and } \phi_k(V') > \phi_k(V).$$

2.3 Some Postulates

In this subsection, we recapitulate some postulates concerning power indices, presented in particular in Felsenthal and Machover (1998) and Laruelle (2000). We add a postulate concerning a dummy.

- **Invariance postulate -**
  A voter's measure of power does not depend on his name.
- **Normalization postulate -** The voters' measures of power add up to 1.

$$\sum_{k \in N} \phi_k(v) = 1$$

- **Dummy postulate -** A player k is a dummy if and only if $\phi_k(v) = 0$.
- **Monotonicity postulate -**
  In a weighted voting body, a voter with a larger voting weight cannot be worse off than a voter with a smaller voting weight, i.e.,

$$\text{if } w_k > w_j, \text{ then } \phi_k(v) \geq \phi_j(v).$$

- **Bloc postulate -**
  If two voters always vote together so that they end up in forming a single voter, then the new voter has more power than each of the previous voters. Given (N, v), we consider $(N \setminus \{j\}, v')$ such that $v'(S) = v(S \cup \{j\})$ if $k \in S$, and $v'(S) = v(S)$ otherwise. The bloc postulate requires that:

$$\text{if } \phi_j(v) > 0, \text{ then } \phi_k(v') > \phi_k(v).$$

- **Donation postulate -**
  In a weighted voting body, a voter cannot gain power by distributing some of his voting weight to other voters.
  Let $V = [q; w_1, ..., w_n]$, $V' = [q; w'_1, ..., w'_n]$, and $\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} w'_k$.
  The donation postulate requires that:

$$\text{if } w'_j \geq w_j \text{ for each } j \neq k \text{ and } w'_k < w_k, \text{ then } \phi_k(V') \leq \phi_k(V).$$

3 Description of the Hoede-Bakker Index

The concept of decisional power was introduced by Hoede and Bakker (1982). It is based on the fact that in situations in which decisions have to be made, a distinction between the inclination to say yes or no and the actual decision may appear. A player may be, for instance, in favor of a certain decision (a certain point at issue), that is, his inclination is then "yes", but in fact he says "no" due to the influence of other player(s). Such a distinction between the inclination and the final decision illustrates the power of influencing players.

In this subsection, we recapitulate the definition of the decisional power index, the so called Hoede-Bakker index. We introduce a slightly different description than in Hoede and Bakker (1982). Let us consider the following situation. There are $n \geq 1$ players who have to make a decision about a certain point at issue (to accept or to reject a bill, a candidate, etc). By N we denote the set of all players (actors, voters), that is, $N = \{1, ..., n\}$. With respect to the point at issue, each player has an inclination either to say "yes" (denoted by "1") or "no" (denoted by "0"). Hence, for n players, we have 2^n possible *inclination vectors*, that is, n-vectors consisting of zeros and ones. An inclination vector will be denoted by i, the set of all n-vectors by I. For $n = 3$ we have 8 inclination vectors. For instance, $i = (1, 1, 0)$ means that player 1 and 2 have the inclination "yes", but actor 3 has the inclination "no".

Each inclination vector $i \in I$ is transformed into a *decision vector*, denoted by b. Formally, such a transformation may be represented by an operator
$B : I \rightarrow I$, that is,

$$b = Bi. \tag{7}$$

Decision vector b is n-vector consisting of zeros and ones ($b \in I$) and indicating the decisions made by all actors. For instance, $B(1, 1, 0) = (1, 0, 0)$ means that players 1 and 3 voted according to their inclination (player 1 said "yes", player 3 - "no"), and actor 2 decided for "no", although his inclination was "yes". The set of all decision vectors is denoted by $B(I)$.

The *group decision* $gd : B(I) \rightarrow \{1, -1\}$ is a function defined on the vectors b. It has the value $+1$ if the group decision is "yes" and the value -1 if the group decision is "no".

There are many possibilities to choose the operator B and the group decision gd. Hoede and Bakker (1982) proposed only two axioms which have to be satisfied by B and gd. Let us introduce first some definitions.

Definition 3.1 For each inclination vector $i = (i_1, ..., i_n)$, the *complement of* i is a vector $i^c = (i^c_1, ..., i^c_n)$ such that for each $k \in \{1, ..., n\}$

$$i^c_k = \begin{cases} 1 & if \quad i_k = 0 \\ 0 & if \quad i_k = 1 \end{cases}. \tag{8}$$

Definition 3.2

$$i \leq i' \iff \{k \in N \mid i_k = 1\} \subseteq \{k \in N \mid i'_k = 1\}. \tag{9}$$

According to Hoede and Bakker (1982), B and gd have to satisfy the following two axioms:

AXIOM (A-1):

$$\forall i \in I \ [gd(Bi^c) = -gd(Bi)] \tag{10}$$

AXIOM (A-2):

$$\forall i \in I \ \forall i' \in I \ [i \leq i' \Rightarrow gd(Bi) \leq gd(Bi')] \tag{11}$$

Axiom (A-1) says that changing all inclinations leads to the opposite group decision. Axiom (A-2) means that the group decision "yes" is not changed into "no" if the set of players with inclination "yes" is enlarged. Let us notice that we do not introduce separate axioms for B and gd. Axioms (A-1) and (A-2) concern both B and gd together.

Definition 3.3 Given B and gd, the *decisional power index* (the *Hoede-Bakker index*) of a player $k \in N$ is given by

$$HB(k) = \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=1\}} gd(Bi). \tag{12}$$

Let us notice that when defining this power index of a player, we assume this player to have an inclination "yes" and consider the group decisions for all 2^{n-1} inclination vectors of the remaining players. If we assumed a player to have an inclination "no" (and then considered the group decisions for all 2^{n-1} inclination vectors of the remaining players), by virtue of axiom (A-1), the Hoede-Bakker index would be a non-positive number with absolute value equal to the one from Definition 3.3. Hence, without loss of generality, we assume the player has an inclination "yes".

4 Power Dependent on the Group Decision

Hoede and Bakker (1982) considered a social network modelled as a directed graph, and then, depending on the structure of the network, they defined the operator B and the group decision gd. Let us notice that, since B and gd have to satisfy only two axioms (A-1) and (A-2), defining B and gd does not have to be unique.

Example 4.1 We will analyse Figure 4(f) in Appendix A. Players 1, 3, 4 and 5 are not influenced, but player 2 is influenced by players 1, 3 and 4. We assume that if a player is not influenced, he decides according to his own inclination. If a player is influenced by only one voter, we can assume this influenced player follows his "boss". But what about a player who is influenced by more than one voter? What decision should he make if inclinations of influencing actors are opposite? Let us define B and gd for the social network from Figure 4(f). We will specify different *procedures* for defining B and gd.

Procedure 1

- Player k ($k \in \{1, 3, 4, 5\}$) always follows his own inclination.
- If at least two influencing players have the inclination "yes", player 2 decides for "yes", otherwise he says "no".
- If at least three players decide for "yes", the group decision is "yes" (that is, $gd = +1$), otherwise the group decision is "no" (that is, $gd = -1$).

Let us notice that in Procedure 1, player 2 never considers his own inclination. For instance, if players 2 and 4 have the inclination "yes", and players 1 and 3 have the inclination "no", player 2 still will choose for "no". Table 1 in Appendix B shows the group decision for Figure 4(f) with Procedure 1. Both axioms (A-1) and (A-2) are satisfied. By virtue of Table 1, we can calculate that:
$HB(1) = HB(3) = HB(4) = \frac{1}{16} \cdot (12 - 4) = \frac{1}{2}$,
$HB(2) = HB(5) = \frac{1}{16} \cdot (8 - 8) = 0$.

Example 4.2 Let us apply now a different procedure to Figure 4(f). One of the differences between the new procedure and the previous one is that now player 2 takes into account also his own inclinations. The procedure looks as follows:

Procedure 2

- Player k ($k \in \{1, 3, 4, 5\}$) always follows his own inclination.
- If all influencing players 1, 3 and 4 have the same inclination, player 2 follows their inclination, otherwise he decides according to his own inclination.
- If at least three players decide for "yes", the group decision is "yes" (that is, $gd = +1$), otherwise the group decision is "no" (that is, $gd = -1$).

Table 2 in Appendix B presents the group decision for Figure 4(f) with Procedure 2. As before, axioms (A-1) and (A-2) are satisfied. We receive that $HB(k) = \frac{1}{16} \cdot (11 - 5) = \frac{3}{8}$ for all k.
Let us notice that Procedure 2, taking into account player 2's inclinations in some cases, appears to be fruitful not only for player 2, but also for player 5 who did not change his behaviour. Players 1, 3 and 4 lost some of their power when applying Procedure 2.

Examples 4.1 and 4.2 illustrate that given a social network, you can define different $B's$ and $gd's$ giving different results. The power index does depend on how we define B and gd. The only requirement is that the axioms (A-1) and (A-2) have to be satisfied. This gives some additional possibilities, for instance, to consider a case of partial influence.

Example 4.3 Let us consider Figure 2(c). We have three players. Player 1 influences both players 2 and 3. We apply two procedures to this example. The first one is the following:

Procedure 3

- Player 1 always follows his own inclination.
- Player k ($k \in \{2, 3\}$) always follows the inclination of player 1.
- The group decision is "yes" if and only if all three players say "yes".

Formally, we can define this procedure by:

$$\forall i = (i_1, i_2, i_3)\ [Bi = (i_1, i_1, i_1)\ \wedge\ (gd(Bi) = 1 \iff Bi = (1, 1, 1))] \tag{13}$$

The results of the group decision are given in Table 3 (see Appendix B). Both B and gd satisfy axioms (A-1) and (A-2). From Table 3 we get:
$HB(1) = \frac{1}{4} \cdot (4 - 0) = 1$,
$HB(2) = HB(3) = \frac{1}{4} \cdot (2 - 2) = 0$.

Example 4.4 Let us apply now a different procedure. In a sense, we assume now player 1 to have "less influence" on the others.

Procedure 4

- Player 1 always follows his own inclination.
- If player 1 has the inclination "yes", then player 2 will follow this inclination, but player 3 will decide according to his own inclination. If player 1 has the inclination "no", then player 2 will decide according to his own inclination, and player 3 will follow player 1.
- The group decision is "yes" if and only if at least two players say "yes".

Procedure 4 can be written formally as:

$$\forall i = (i_1, i_2, i_3)\ [(i_1 = 1\ \Rightarrow\ Bi = (i_1, i_1, i_3))\ \wedge\ (i_1 = 0\ \Rightarrow\ Bi = (i_1, i_2, i_1))\ \wedge$$

$$(gd(Bi) = gd(b_1, b_2, b_3) = 1 \iff \sum_{k=1}^{3} b_k \geq 2)]. \tag{14}$$

The results of applying Procedure 4 to Figure 2(c) are shown in Table 4 (Appendix B). Let us notice that for the operator B defined in Procedure 4 we could not choose the group decision gd from Procedure 3, because then axiom (A-1) would not be satisfied. But B and gd defined together in Procedure 3 satisfy both axioms (A-1) and (A-2). From Table 4, we get the same results as before, that is, $HB(1) = 1$ and $HB(2) = HB(3) = 0$. In this case, either player 1 did not lose his power, or players 2 and 3 did not gain in power, although player 1's influence has been slightly limited.

5 Odd Versus Even Number of Players

Because it will be used frequently in what follows, we define a standard procedure which satisfies axioms (A-1) and (A-2) in case the number of players is odd.

Theorem 5.1 *Let $INF(k)$ denote the set of players influencing player k, $k \in N$, $0 \leq |INF(k)| \leq n-1$, $k \notin INF(k)$. Moreover, let*
$INF^{yes}(k,i) = \{m \in INF(k) \mid i_m = 1\}$,
$INF^{no}(k,i) = \{m \in INF(k) \mid i_m = 0\}$, *where $i = (i_1, ..., i_n)$ denotes the inclination vector. $Bi = (b_1, ..., b_n)$ means the decision vector for i.*
If n is **odd**, *then the operator B and group decision gd defined in the Standard Procedure below, satisfy both axioms (A-1) and (A-2):*

Procedure 5 (Standard Procedure)

For each $k \in N$:

1. *If $|INF(k)| = 0$, then for each $i = (i_1, ..., i_n)$, $b_k = i_k$.*
2. *If $|INF(k)| > 0$, then for each $i = (i_1, ..., i_n)$:*
   (a) *if $|INF^{yes}(k,i)| > |INF^{no}(k,i)|$, then $b_k = 1$,*
   (b) *if $|INF^{yes}(k,i)| < |INF^{no}(k,i)|$, then $b_k = 0$,*
   (c) *if $|INF^{yes}(k,i)| = |INF^{no}(k,i)|$, then $b_k = i_k$.*

For each $i = (i_1, ..., i_n)$:

$$gd(Bi) = \begin{cases} +1 & if \quad |\{k \in N \mid b_k = 1\}| \geq [\frac{n}{2}] + 1 \\ -1 & \qquad\qquad otherwise \end{cases}, \tag{15}$$

where $[x]$ means the greatest integer not greater than x.

Proof: Let $i = (i_1, ..., i_n)$ be arbitrary and let it contain x 1's and $n-x$ 0's, $0 \leq x \leq n$. We determine $Bi = (b_1, ..., b_n)$ and $gd(Bi)$. Let Bi contain y 1's and $n-y$ 0's. If $y \geq [\frac{n}{2}] + 1$, then $gd(Bi) = 1$, otherwise $gd(Bi) = -1$.
Let us consider now i^c. It contains $n-x$ 1's and x 0's. We determine $Bi^c = (b'_1, ..., b'_n)$. For each player k such that $|INF(k)| = 0$, $b'_k = i^c_k = -i_k = -b_k$. For each player k such that $|INF(k)| > 0$, also $b'_k = -b_k$. Then, $Bi^c = -(b_1, ..., b_n) = -Bi$. Hence, Bi^c contains y 0's and $n-y$ 1's.
Suppose that n is **odd**. Then, $[\frac{n}{2}]+1 = \frac{n+1}{2}$. If $y \geq \frac{n+1}{2}$, then $gd(Bi) = +1$ and $n - y \leq \frac{n-1}{2} < \frac{n+1}{2}$, what gives $gd(Bi^c) = -1$. If $y < \frac{n+1}{2}$, then $gd(Bi) = -1$ and $gd(Bi^c) = +1$.
If n is **even**, then we find a problem when $y = n-y$. Then, $[\frac{n}{2}] = \frac{n}{2}$, $y = \frac{n}{2} < \frac{n}{2} + 1$, and hence $gd(Bi) = -1$. But then, also $n - y = \frac{n}{2} < \frac{n}{2} + 1$ and $gd(Bi^c) = -1$, what means that axiom (A-1) is NOT satisfied. However, axiom (A-2) is satisfied.
For a given $i = (i_1, ..., i_n)$ having x 1's and $n-x$ 0's, we consider $i' = (i'_1, ..., i'_n)$ such that $\{k \in N \mid i_k = 1\} \subseteq \{k \in N \mid i'_k = 1\}$. Hence, in particular, i' has $x + \epsilon$ 1's and $n - x - \epsilon$ 0's, where $0 \leq \epsilon \leq n - x$. Let $Bi = (b_1, ..., b_n)$ and $Bi' = (b'_1, ..., b'_n)$. Then, $|\{k \in N \mid b'_k = 1\}| \geq |\{k \in N \mid b_k = 1\}|$. Hence, if $|\{k \in N \mid b_k = 1\}| \geq [\frac{n}{2}] + 1$, then $gd(Bi) = +1$ and $gd(Bi') = +1$. If $|\{k \in N \mid b_k = 1\}| < [\frac{n}{2}]+1$, then $gd(Bi) = -1$ and hence $gd(Bi') \geq gd(Bi)$. □

Remark 5.1 Let us notice that Procedure 1 applied to Figure 4(f) (see section 4) is, in fact, our Standard Procedure for $n = 5$ applied to Figure 4(f). Moreover, if in Procedure 3 applied to Figure 2(c) we change gd and we assume the group decision to be "yes" if and only if at least two players say "yes", we get the same result (see Table 3 in Appendix B), and moreover this modified Procedure 3 is again the Standard Procedure for $n = 3$ applied to Figure 2(c).

Remark 5.2 For n even, the operator B and the group decision gd described in the Standard Procedure, do satisfy axioms (A-1) and (A-2) if for each inclination vector i, the decision vector Bi has a different number of 1's and 0's. So, if for an even number of players there is no draw in Bi, we can still apply the Standard Procedure.

Hoede and Bakker (1982) considered mainly an odd number n of voters, but for n even they suggested the idea to "add an isolated dummy actor with prescribed voting behaviour, e.g. "yes" if that is the group decision in case of a draw". Let us consider some examples with an even number of players.

Example 5.1 We can apply our Standard Procedure to Figure 1(b). We have two players, and player 1 influences player 2. According to the Standard Procedure, player 1 will follow his own inclinations, and player 2 will always follow the inclination of his unique influencing voter - player 1. The results are described in Table 5. In this particular case, axioms (A-1) and (A-2) are satisfied. We get:
$HB(1) = \frac{1}{2} \cdot (2 - 0) = 1$, and $HB(2) = \frac{1}{2} \cdot (1 - 1) = 0$.

Example 5.2 Unfortunately, we cannot apply the Standard Procedure to Figure 1(a) in which there are two independent players, because then axiom (A-1) would not be satisfied. In this case, we get a draw twice, that is, when players have different inclinations. We follow the idea mentioned in Hoede and Bakker (1982) to add an isolated player, but now we assume him to say "no" in case of a draw. We get the following result:
$gd(B(1, 1)) = gd(1, 1) = +1$,
$gd(B(1, 0, 0)) = gd(1, 0, 0) = -1$,
$gd(B(0, 1, 0)) = gd(0, 1, 0) = -1$,
$gd(B(0, 0)) = gd(0, 0) = -1$,
and finally $HB(1) = HB(2) = \frac{1}{2} \cdot (1 - 1) = 0$.

Definition 5.1 By the *Drawn Standard Procedure* we mean the Standard Procedure adapted to the case of an even number of players as described in Example 5.2.

Example 5.3 We cannot apply the Standard Procedure either to Figure 3(a) or to Figure 3(c), because of a draw, but we can apply the Drawn Standard Procedure to these figures. Tables 6 and 7 show the group decision for Figures

3(a) and 3(c), respectively. In Figure 3(a), we have four independent voters. The result is similar to the one from Figure 1(a), that is,
$HB(k) = \frac{1}{8} \cdot (4 - 4) = 0$ for each $k \in \{1, 2, 3, 4\}$.

In Figure 3(c), there are also four players: players 1 and 2 influence player 4, but player 3 is independent (that is, he is not influenced and he does not influence anybody). By virtue of Table 7, we get:
$HB(1) = HB(2) = \frac{1}{8} \cdot (5 - 3) = \frac{1}{4}$,
$HB(3) = HB(4) = \frac{1}{8} \cdot (4 - 4) = 0$.

Example 5.4 As we already mentioned, we can apply the Standard Procedure for a social network with an even number of players, but only if no draw appears. In particular, we can apply it to Figures 3(b), 3(d) and 3(e). The results for these figures are given in Tables 8, 9 and 10, respectively.

In Figure 3(b), players 1, 2 and 3 influence player 4. According to Table 8, we get:
$HB(1) = HB(2) = HB(3) = \frac{1}{8} \cdot (6 - 2) = \frac{1}{2}$,
$HB(4) = \frac{1}{8} \cdot (4 - 4) = 0$.

In Figure 3(d), player 1 influences the others, that is, players 2, 3 and 4. In Figure 3(e), player 1 has "less" influence: he influences only players 2 and 3, and player 4 is independent. Nevertheless, the power index of player 1 is equal to 1 in both cases:
$HB(1) = \frac{1}{8} \cdot (8 - 0) = 1$, and
$HB(k) = \frac{1}{8} \cdot (4 - 4) = 0$ for $k \in \{2, 3, 4\}$.

For a social network with an arbitrary odd number of players, we can also apply a procedure slightly different from the Standard Procedure. The only difference is that in this new procedure, an influenced player k does not consider only $INF^{yes}(k, i)$ and $INF^{no}(k, i)$ without taking into account his own inclinations, but he compares the sets $INF^{yes}(k, i) \cup \{k\}$ and $INF^{no}(k, i) \cup \{k\}$. One can prove that:

Theorem 5.2 *Let $INF(k)$ denote the set of players influencing player k, $k \in N$, $0 \leq |INF(k)| \leq n - 1$, $k \notin INF(k)$. Moreover, let*
$INF2^{yes}(k, i) = \{m \in INF(k) \cup \{k\} \mid i_m = 1\}$,
$INF2^{no}(k, i) = \{m \in INF(k) \cup \{k\} \mid i_m = 0\}$, where $i = (i_1, ..., i_n)$ denotes the inclination vector. $Bi = (b_1, ..., b_n)$ means the decision vector for i.
If n is **odd**, *then the operator B and group decision gd, defined in Procedure 6, satisfy both axioms (A-1) and (A-2):*

Procedure 6

For each $k \in N$:

1. *If $|INF(k)| = 0$, then for each $i = (i_1, ..., i_n)$, $b_k = i_k$.*
2. *If $|INF(k)| > 0$, then for each $i = (i_1, ..., i_n)$:*

(a) if $|INF2^{yes}(k,i)| > |INF2^{no}(k,i)|$, then $b_k = 1$,

(b) if $|INF2^{yes}(k,i)| < |INF2^{no}(k,i)|$, then $b_k = 0$,

(c) if $|INF2^{yes}(k,i)| = |INF2^{no}(k,i)|$, then $b_k = i_k$.

For each $i = (i_1, ..., i_n)$:

$$gd(Bi) = \begin{cases} +1 & if \quad |\{k \in N \mid b_k = 1\}| \geq [\frac{n}{2}] + 1 \\ -1 & \quad otherwise \end{cases}, \tag{16}$$

where $[x]$ means the greatest integer not greater than x.

The proof of Theorem 5.2 is very similar to the proof of Theorem 5.1. The results of applying both Procedures 5 and 6 can be, of course, different. In fact, Procedure 2 applied to Figure 4(f) (see section 4) is Procedure 6 for $n = 5$ applied to Figure 4(f). As we already saw in section 4, we received different results when using Procedures 1 and 2.

One can show that the same results of applying B and gd may come from different social networks. Let us apply Procedure 6 presented in Theorem 5.2 to Figure 2(a) and 2(c). In fact, in both cases the results are precisely the same. They are presented in Table 11 (see Appendix B). Applying the Standard Procedure from Theorem 5.1 to Figure 2(a) gives, of course, the same results. We get then:
$HB(k) = \frac{1}{4} \cdot (3 - 1) = \frac{1}{2}$ for each $k \in \{1, 2, 3\}$.

6 Decisional Power of a Set of Players

Hoede and Bakker (1982) defined the decisional power index of a set S of players. In the definition, all actors in S are assumed to have the same inclination "yes", and we consider the group decisions for all $2^{n-|S|}$ inclination vectors of the others, where $|S|$ means, as usual, the number of players in S.

Definition 6.1 Given B and gd, the *decisional power index* (the *Hoede-Bakker index*) of a set of players $S \subseteq N$ is given by

$$HB(S) = \frac{1}{2^{n-|S|}} \cdot \sum_{\{i:\ i_k = 1\ for\ all\ k \in S\}} gd(Bi). \tag{17}$$

Hoede and Bakker (1982) showed that

Proposition 6.1 *For each set S of players, $0 \leq HB(S) \leq 1$.*

We like to show some other properties of the decisional power of a set of players. It is true that the power index of the whole set N of players is maximal, that is:

Proposition 6.2 *For each N, $HB(N) = 1$.*

Proof: From Definition 6.1 we have $HB(N) = gd(Bi^*)$, where $i^* = (1, ..., 1)$. By virtue of axioms (A-1) and (A-2), $gd(Bi^*) = +1$, and therefore $HB(N) = 1$. $\square$

Another property says that the larger the set of players, the greater (or equal) the Hoede-Bakker index.

Theorem 6.1 *For each $S \subset S'$, $HB(S) \leq HB(S')$.*

Proof: Assuming $S \subset S'$, there exists $S" \neq \emptyset$ such that $S' = S \cup S"$ and $S \cap S" = \emptyset$. Hence, $|S'| = |S| + |S"|$. Let us introduce the following notation:

$$S^+ = \{i \mid i_k = 1 \ for\ all\ k \in S\}. \tag{18}$$

Hence, of course,

$$(S \cup S")^+ = \{i \mid i_k = 1 \ for\ all\ k \in S \cup S"\}.$$

Let us notice that for each $i = (i_1, ..., i_n) \in S^+$, there exists $i^* \in (S \cup S")^+$ (that is, $i^*_k = 1$ for $k \in S \cup S"$) such that $i \leq i^*$ and $i^*_m = i_m$ for all $m \in N \setminus (S \cup S")$. Hence, by virtue of axiom (A-2), $gd(Bi) \leq gd(Bi^*)$. In the formula for $HB(S)$, we will replace each $i \in S^+$ by $i^* \in (S \cup S")^+$ having the same inclinations for each $m \in N \setminus (S \cup S")$. Since $|S^+| = 2^{|S"|} \cdot |(S \cup S")^+|$, we have:

$$HB(S) = \frac{1}{2^{n-|S|}} \cdot \sum_{\{i:\ i_k=1 \ for\ all\ k \in S\}} gd(Bi) \leq$$

$$\frac{1}{2^{n-|S|}} \cdot \sum_{\{i:\ i_k=1 \ for\ all\ k \in S \cup S"\}} 2^{|S"|} \cdot gd(Bi) =$$

$$\frac{2^{|S"|}}{2^{n-|S|}} \cdot \sum_{\{i:\ i_k=1 \ for\ all\ k \in S'\}} gd(Bi) = \frac{2^{|S"|}}{2^{n-|S|}} \cdot HB(S') \cdot 2^{n-|S'|} = HB(S').$$

$\square$

Remark 6.1 Let us notice that the property described in Theorem 6.1 is a (weak) version of the *BLOC POSTULATE* (see Section 2.3) for the decisional power of a set of players. A larger set of players S', where $S \subset S'$, has more or equal power than the set S and, by analogy, than the set $S' \setminus S$. As shown in Example 6.1, we can have $HB(S) = HB(S')$, even if $HB(S' \setminus S) > 0$. For this reason we speak about a weak version of the bloc postulate.

Hence, we can write:

Conclusion 6.1 *The Hoede-Bakker index of a set of players satisfies a weak version of the bloc postulate.*

Hoede and Bakker (1982) defined a dictator set as a set of players with the decisional power index equal to 1. We introduce an equivalent definition of a dictator set:

Definition 6.2 Given B and gd, a set of players $S \subseteq N$ is called a *dictator set* in a social network if for each $i \in I$ such that $i_k = 1$ for all $k \in S$, $gd(Bi) = +1$.

Remark 6.2 Notice that by virtue of axiom (A-1), the following holds:

$$\forall S \subseteq N \; [\forall i \in S^+ \; [gd(Bi) = +1] \iff \forall i \in S^- \; [gd(Bi) = -1]], \tag{19}$$

where S^+ is defined in formula (18), and

$$S^- = \{i \mid i_k = 0 \; for \; all \; k \in S\}. \tag{20}$$

Theorem 6.2 *Given B and gd for a social network,*

$$\forall S \subseteq N \; [HB(S) = 1 \iff S \; is \; a \; dictator \; set] \tag{21}$$

Proof: For given B, gd and an arbitrary $S \subseteq N$:
($\Rightarrow$) Let $HB(S) = 1$. Hence, $\sum_{\{i: \; i_k=1 \; for \; k \in S\}} gd(Bi) = 2^{n-|S|}$. Since we have $|\{i \mid i_k = 1 \; for \; k \in S\}| = 2^{n-|S|}$ and $gd(Bi) \in \{+1, -1\}$, it follows that for each $i \in I$ such that $i_k = 1$ for all $k \in S$, $gd(Bi) = 1$. Hence, by virtue of Remark 6.2, for each $i \in I$ such that $i_k = 0$ for all $k \in S$, $gd(Bi) = -1$. It means that S is a dictator set.
($\Leftarrow$) Let S be a dictator set. It means, in particular, that for each $i \in I$ such that $i_k = 1$ for all $k \in S$, $gd(Bi) = 1$. Hence, $HB(S) = 1$. □

The last question we like to discuss is the following:

$$Is \; there \; a \; relation \; between \; HB(S) \; and \; \sum_{k \in S} HB(k)?$$

In fact, the answer is:

Everything may happen!

Example 6.1 In order to see that each possible relation (from among $>$, $<$ and $=$) between $HB(S)$ and $\sum_{k \in S} HB(k)$ may appear, we will analyse Figure 4(e). In this social network, player 1 influences players 4 and 5, and additionally, player 2 influences player 4, and player 3 influences player 5. We will apply the Standard Procedure to this figure. The group decision is presented in Table 12 (see Apppendix B). By virtue of Table 12 we get:
$HB(1) = \frac{1}{16} \cdot (13 - 3) = \frac{5}{8}$,
$HB(2) = HB(3) = \frac{1}{16} \cdot (11 - 5) = \frac{3}{8}$,
$HB(4) = HB(5) = \frac{1}{16} \cdot (9 - 7) = \frac{1}{8}$.
By virtue of Definition 6.1, we get:
$HB(\{1, 5\}) = \frac{1}{8} \cdot (7 - 1) = \frac{3}{4} = HB(1) + HB(5)$,
$HB(\{1, 2, 3\}) = \frac{1}{4} \cdot (4 - 0) = 1 < \frac{11}{8} = HB(1) + HB(2) + HB(3)$,
$HB(\{2, 3, 4\}) = \frac{1}{4} \cdot (4 - 0) = 1 > \frac{7}{8} = HB(2) + HB(3) + HB(4)$.

Notice that $HB(N) = 1 = HB(\{2,3,4\})$, while $HB(\{1,5\}) = \frac{3}{4} > 0$.
See Remark 6.1.

In Example 6.1, we can recognize something like the *PARADOX OF LARGE SIZE* (see Section 2.2). The Hoede-Bakker index of a set of players may be less than the sum of the power indices of the individual players who constitute the set:
$HB(\{1,2,3\}) < HB(\{1\}) + HB(\{2\}) + HB(\{3\})$.

Conclusion 6.2 *The decisional power index of a set of players may display the paradox of large size.*

7 Some Properties of the Hoede-Bakker Index

In this section, we will analyse some further properties of the decisional power index of a player. We like to check whether the paradoxes mentioned in section 2.2 appear for the Hoede-Bakker index. Moreover, we will check whether the Hoede-Bakker index satisfies the postulates mentioned in section 2.3.

(A) INVARIANCE POSTULATE

The invariance postulate is satisfied, that is, the Hoede-Bakker index of a player does not depend on his name.

(B) NORMALIZATION POSTULATE

The normalization postulate is NOT satisfied by the Hoede-Bakker index. For instance, for Figure 4(f) with the Standard Procedure we have
$\sum_{k=1}^{5} HB(k) = \frac{3}{2}$.

Definition 7.1 Given B and gd, a player $k \in N$ is called a *dictator* in a social network if the group decision is always the same as the inclination of player k, that is, if for each $i \in I$ such that $i_k = 1$, $gd(Bi) = +1$.

Remark 7.1 Similarly to Remark 6.2 we note that it is enough to assume in Definition 7.1 that for each $i \in I$ such that $i_k = 1$, $gd(Bi) = +1$. The condition that for each $i \in I$ such that $i_k = 0$, $gd(Bi) = -1$, results from axiom (A-1).

Hoede and Bakker (1982) defined a dictator as a player with decisional power index equal to 1. We introduced an equivalent definition of a dictator, from which we can prove that:

Proposition 7.1 *Given B and gd for a social network,*

$$\forall k \in N \ [HB(k) = 1 \iff k \text{ is a dictator}]. \tag{22}$$

Proof: For given B, gd and an arbitrary $k \in N$:
($\Rightarrow$) Let $HB(k) = 1$. Hence, $\sum_{\{i:\ i_k=1\}} gd(Bi) = 2^{n-1}$. Since we have $|\{i \mid i_k = 1\}| = 2^{n-1}$ and $gd(Bi) \in \{+1, -1\}$, it follows that for each $i \in I$ such that $i_k = 1$, $gd(Bi) = +1$. Hence, by Remark 7.1 it follows that k is a dictator.
($\Leftarrow$) Let k be a dictator. It means, in particular, that for each $i \in I$ such that $i_k = 1$, $gd(Bi) = +1$. Hence, we have $HB(k) = 1$. □

Let us notice that a dictator does not have to influence all remaining players. We can show this by analysing Figures 3(d) and 3(e). In both cases, when applying the Standard Procedure, we have $HB(1) = 1$ and $HB(k) = 0$ for $k \in \{2, 3, 4\}$, what means that player 1 is a dictator (see also Tables 9 and 10). He has an influence on all remaining voters in the network presented in Figure 3(d), whereas he does not have an influence on player 4 in Figure 3(e).

One can also prove that:

Theorem 7.1 *Given B and gd for a social network,*

$$\textit{there is a dictator} \Rightarrow \sum_{k=1}^{n} HB(k) = 1. \tag{23}$$

Proof: Suppose that there exists a dictator, say $k \in N$, such that for each $i \in I$, if $i_k = 1$, then $gd(Bi) = +1$, and if $i_k = 0$, then $gd(Bi) = -1$. Let us consider another arbitrary player $m \in N$. We consider all inclination vectors i such that $i_m = 1$. There are 2^{n-1} such inclination vectors: from among them there are 2^{n-2} vectors with $i_k = 1$ and 2^{n-2} with $i_k = 0$. Moreover, for each i such that $i_m = 1$ and $i_k = 1$, there is i' such that $i'_m = 1$, $i'_k = 0$ and $i'_j = i_j$ for each $j \in N \setminus \{k, m\}$. Hence, $gd(Bi) = +1$ and $gd(Bi') = -1$, because $i_k = 1$, $i'_k = 0$ and k is a dictator. But this means that $\sum_{\{i:\ i_m=1\}} gd(Bi) = 2^{n-2} - 2^{n-2} = 0$, and therefore $\sum_{k=1}^{n} HB(k) = 1$. □

Whether the converse of Theorem 7.1 holds, is still an open problem.

$$(C)\ WHEN\ HB(k) = 0?$$

One can easily show that:

Proposition 7.2 *Given B and gd,*

$$\forall k \in N\ [HB(k) = 0 \iff$$
$$|\{i \in I \mid i_k = 1\ \wedge\ gd(Bi) = +1\}| = |\{i \in I \mid i_k = 1\ \wedge\ gd(Bi) = -1\}|] \tag{24}$$

Proof: For an arbitrary $k \in N$ we have:
$HB(k) = 0 \iff \frac{1}{2^{n-1}} \cdot \sum_{\{i:\ i_k=1\}} gd(Bi) = 0 \iff \sum_{\{i:\ i_k=1\}} gd(Bi) = 0 \iff$ $|\{i \in I \mid i_k = 1\ \wedge\ gd(Bi) = +1\}| = |\{i \in I \mid i_k = 1\ \wedge\ gd(Bi) = -1\}|$. □

In Section 2, we considered weighted voting games in which, in particular, players have their weights. When analysing the Hoede-Bakker index, we will

use different parameters, like the number of players influenced by a given voter, or the number of players influencing a given actor. For a given social network, and for player $k \in N$, we introduce the following notation:

$\alpha(k)$ - number of players influenced by player k,

$$\alpha(k) = |\{j \in N \mid k \in INF(j)\}|, \tag{25}$$

$\gamma(k)$ - number of players influencing player k,

$$\gamma(k) = |INF(k)| \tag{26}$$

Λ - number of influenced players in a social network,

$$\Lambda = |\{j \in N \mid \exists k \in N \setminus \{j\}\ [k \in INF(j)]\}|, \tag{27}$$

Γ - number of influencing players in a social network,

$$\Gamma = |\{j \in N \mid \exists k \in N \setminus \{j\}\ [j \in INF(k)]\}|. \tag{28}$$

One can ask the question whether there is a relation between the Hoede-Bakker index of a player k being equal to 0 and the statement $\alpha(k) = 0$ (there are no voters influenced by this player) or the statement $\gamma(k) > 0$ (there are some actors influencing the given player). One can show that:

Proposition 7.3 *NONE of the following implications is true:*

$$\forall k \in N\ [HB(k) = 0\ \Rightarrow\ \alpha(k) = 0],$$

$$\forall k \in N\ [\alpha(k) = 0\ \Rightarrow\ HB(k) = 0],$$

$$\forall k \in N\ [HB(k) = 0\ \Rightarrow\ \gamma(k) > 0],$$

$$\forall k \in N\ [\gamma(k) > 0\ \Rightarrow\ HB(k) = 0].$$

Proof: In order to show that the first and the third implication are not true, let us consider Figure 4(g). Player 1 influences players 3, 4 and 5, and additionally player 2 influences player 5. When applying the Standard Procedure to this figure we get that player 1 is a dictator (see also Table 13):
$HB(1) = 1$, $HB(k) = 0$ for $k \in \{2, 3, 4, 5\}$.
Hence, $HB(2) = 0$, $\alpha(2) = 1 > 0$ and $\gamma(2) = 0$.

In order to prove that the second and the fourth implication are not true, we consider Figure 4(e). The group decisions for Figure 4(e) with the Standard Procedure are shown in Table 12. We have $\alpha(4) = 0$, $\gamma(4) = 2 > 0$, and $HB(4) = \frac{1}{8} > 0$. □

It appears that the number of players influencing a given voter is NOT proportional to the lack of power of the influenced player. One can prove that:

Proposition 7.4 *Given a social network, B and gd, the Hoede-Bakker index does NOT have to be monotonic with respect to γ. It is NOT true that:*

$$\forall k, k' \in N \ [(\gamma(k) > \gamma(k') \ \wedge \ \alpha(k) = \alpha(k')) \ \Rightarrow \ HB(k) \leq HB(k')].$$

Proof: In order to show that $HB(k)$ is not monotonic with respect to $\gamma(k)$, we consider a counter-example. Let us consider Figure 4(d). There are two influencing voters: player 1 who influences actors 3 and 4, and player 2 influencing actors 4 and 5. The results for this social network with the Standard Procedure are presented in Table 14 (see Appendix B). We get:
$HB(1) = HB(2) = HB(4) = \frac{1}{16} \cdot (12 - 4) = \frac{1}{2}$,
$HB(3) = HB(5) = \frac{1}{16} \cdot (8 - 8) = 0$.
Hence, we receive:
$\gamma(4) = 2 > 1 = \gamma(3)$, $\alpha(3) = \alpha(4) = 0$, and $HB(4) = \frac{1}{2} > 0 = HB(3)$.
Hence, we showed that the implication mentioned is NOT true. □

It appears that the number of players influenced by a given voter does NOT have to illustrate the power of the influencing player if we apply, for instance, Procedure 6.

Proposition 7.5 *Given a social network, B and gd, the Hoede-Bakker index does NOT have to be monotonic with respect to α.*

Proof: Let us apply Procedure 6 described in Theorem 5.2 to Figure 4(j). In the social network presented in this figure, player 1 influences players 2 and 3, and additionally, players 3 and 5 influence player 4. The group decision is shown in Table 24. In fact, we get the same results (with changed names of players) as in Figure 4(h) with the Standard Procedure. We get:
$HB(1) = HB(2) = HB(4) = \frac{1}{4}$ and $HB(3) = HB(5) = \frac{1}{2}$.
Hence, we have:
$\alpha(1) = 2 > 1 = \alpha(5)$, $\gamma(1) = \gamma(5) = 0$, but $HB(1) = \frac{1}{4} < \frac{1}{2} = HB(5)$. □

If we consider TWO social networks with the same number of players, then the number of players influenced by a given voter does NOT have to illustrate the power of the influencing players. It is possible that a player who is influenced (and not influencing) in one social network has a greater power index in this network than in another network in which this player is influencing (but not influenced). One can prove that:

Proposition 7.6 *Given two social networks, B and gd, the Hoede-Bakker index does NOT have to be monotonic with respect to α. It is NOT true that:*

$$\forall k \in N \ [(\alpha'(k) > \alpha(k) \ \wedge \ \gamma'(k) \leq \gamma(k)) \ \Rightarrow \ HB'(k) \geq HB(k)],$$

where HB', α' and γ' refer to the second social network.

Proof: We can present several examples showing this fact.
(1) Let us consider Figures 4(b) and 4(d). We have:

Figure 4(b): $\alpha(1) = 1$, $\gamma(1) = 0$, $HB(1) = \frac{3}{4}$ (see Table 15)
Figure 4(d): $\alpha'(1) = 2$, $\gamma'(1) = 0$, $HB'(1) = \frac{1}{2}$ (see Table 14)

(2) If we consider Figures 4(b) and 4(e), then we get:

Figure 4(b): $\alpha(1) = 1$, $\gamma(1) = 0$, $HB(1) = \frac{3}{4}$,
Figure 4(e): $\alpha''(1) = 2$, $\gamma''(1) = 0$, $HB''(1) = \frac{5}{8}$

(3) Finally, we compare Figures 4(d) and 4(c). We have:

Figure 4(d): $\alpha'(4) = 0$, $\gamma'(4) = 2$, $HB'(4) = \frac{1}{2}$,
Figure 4(c): $\alpha'''(4) = 1$, $\gamma'''(4) = 0$, $HB'''(4) = \frac{3}{8}$. □

(E) PARADOX OF REDISTRIBUTION AND DONATION POSTULATE

In this subsection, we will check whether the redistribution paradox (see section 2.2) can appear for the Hoede-Bakker index, and whether the donation postulate (see section 2.3) is satisfied for this index. In the redistribution paradox we assume that $\sum_{k=1}^{n} w_k = \sum_{k=1}^{n} w'_k$, where w_1, ..., w_n and w'_1, ..., w'_n denote the weights of the players in the two simple games considered. When looking for the redistribution paradox for the Hoede-Bakker index, we will assume that $\sum_{k=1}^{n} \alpha(k) = \sum_{k=1}^{n} \alpha'(k)$, what is equivalent that $\sum_{k=1}^{n} \gamma(k) = \sum_{k=1}^{n} \gamma'(k)$. In fact, $\sum_{k=1}^{n} \alpha(k) = \sum_{k=1}^{n} \gamma(k)$ and $\sum_{k=1}^{n} \alpha'(k) = \sum_{k=1}^{n} \gamma'(k)$ denote the number of "arrows" in the first and the second social network, respectively.

Definition 7.2 Let $((\alpha(k))_{k=1}^{n}, (\gamma(k))_{k=1}^{n}, \Lambda, \Gamma)$ and $((\alpha'(k))_{k=1}^{n}, (\gamma'(k))_{k=1}^{n}, \Lambda', \Gamma')$ be the parameters of two n-player social networks A and B, respectively, where

$$\sum_{k=1}^{n} \alpha(k) = \sum_{k=1}^{n} \alpha'(k). \tag{29}$$

The Hoede-Bakker index displays the *paradox of redistribution* if

$$for\ some\ k,\ \alpha'(k) < \alpha(k),\ \gamma'(k) > \gamma(k),\ and\ HB'(k) > HB(k), \tag{30}$$

where HB and HB' denote the Hoede-Bakker index in social network A and B, respectively.

Example 7.1 Let us start with the comparison of Figures 4(c) and 4(d). In Figure 4(c), players 1, 3, 4 and 5 influence player 2. The group decision for Figure 4(c) with the Standard Procedure is given in Table 17. From Table 17 we get:
$HB(k) = \frac{1}{16} \cdot (11 - 5) = \frac{3}{8}$ for each $k \in \{1, 2, 3, 4, 5\}$.

Let us notice that player 2, influenced by four players, has the same power as all voters influencing him. In Figure 4(c), we have $\Lambda = 1$, $\Gamma = 4$, and hence $\Lambda + \Gamma = 5$,

$$\sum_{k=1}^{5} \alpha(k) = \alpha(1) + \alpha(3) + \alpha(4) + \alpha(5) = 4 = \sum_{k=1}^{5} \gamma(k) = \gamma(2),$$

$\alpha(4) = 1$, $\gamma(4) = 0$, and $HB(4) = \frac{3}{8}$.
The group decision for Figure 4(d) with the Standard Procedure is presented in Table 14. In Figure 4(d), we have:
$\Lambda' = 3$, $\Gamma' = 2$, hence additionally $\Lambda' + \Gamma' = 5 = \Lambda + \Gamma$,

$$\sum_{k=1}^{5} \alpha'(k) = \alpha'(1) + \alpha'(2) = 4 = \sum_{k=1}^{5} \alpha(k),$$

and equivalently,

$$\sum_{k=1}^{5} \gamma'(k) = \gamma'(3) + \gamma'(4) + \gamma'(5) = 4 = \sum_{k=1}^{5} \gamma(k),$$

$\alpha'(4) = 0 < 1 = \alpha(4)$, $\gamma'(4) = 2 > 0 = \gamma(4)$, but $HB'(4) = \frac{1}{2} > \frac{3}{8} = HB(4)$.

By virtue of Example 7.1 we can conclude that:

Conclusion 7.1 *The Hoede-Bakker index may display the paradox of redistribution.*

Example 7.2 We like to present another example in which the redistribution paradox occurs. Let us compare Figures 4(g) and 4(c). The group decision for Figure 4(g) with the Standard Procedure is given in Table 13. In Figure 4(g), we have $\Lambda'' = 3$, $\Gamma'' = 2$, and hence $\Lambda'' + \Gamma'' = 5 = \Lambda + \Gamma$,

$$\sum_{k=1}^{5} \alpha''(k) = \alpha''(1) + \alpha''(2) = 4 = \sum_{k=1}^{5} \alpha(k),$$

and equivalently,

$$\sum_{k=1}^{5} \gamma''(k) = \gamma''(3) + \gamma'(4) + \gamma''(5) = 4 = \sum_{k=1}^{5} \gamma(k),$$

$\alpha''(2) = 1 > 0 = \alpha(2)$, $\gamma''(2) = 0 < 4 = \gamma(2)$, but $HB''(2) = 0 < \frac{3}{8} = HB(2)$.

Let us redefine now the donation postulate for the Hoede-Bakker index.

Definition 7.3 Let $((\alpha(k))_{k=1}^{n}, (\gamma(k))_{k=1}^{n}, \Lambda, \Gamma)$ and $((\alpha'(k))_{k=1}^{n}, (\gamma'(k))_{k=1}^{n}, \Lambda', \Gamma')$ be the parameters of two n-player social networks A and B, respectively, where

$$\sum_{k=1}^{n} \alpha(k) = \sum_{k=1}^{n} \alpha'(k). \tag{31}$$

The donation postulate for the Hoede-Bakker index requires that

$$\begin{aligned} &if\ \alpha'(j) \geq \alpha(j)\ and\ \gamma'(j) \leq \gamma(j)\ for\ each\ j \neq k,\ and \\ &\alpha'(k) < \alpha(k)\ and\ \gamma'(k) > \gamma(k),\ then\ HB'(k) \leq HB(k), \end{aligned} \tag{32}$$

where HB and HB' denote the Hoede-Bakker index in social network A and B, respectively.

Example 7.3 In order to show that the donation postulate described in Definition 7.3, is not satisfied by the Hoede-Bakker index, we will compare Figures 4(i) and 4(d). In Figure 4(i), player 1 influences players 3 and 5, and additionally players 2 and 4 influence each others. The group decisions for Figure 4(i) with the Standard Procedure are given in Table 18. Since the group decision is always the same like player 1's inclination, we have:
$HB(1) = 1$ and $HB(k) = 0$ for each $k \in \{2, 3, 4, 5\}$.
In Figure 4(i), we have:
$\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 0$, $\alpha(4) = 1$, $\alpha(5) = 0$, and
$\gamma(1) = 0$, $\gamma(2) = 1$, $\gamma(3) = 1$, $\gamma(4) = 1$, $\gamma(5) = 1$, and $HB(4) = 0$.
Let us consider now Figure 4(d). We have:
$\alpha'(1) = 2$, $\alpha'(2) = 2$, $\alpha'(3) = 0$, $\alpha'(4) = 0$, $\alpha'(5) = 0$, and
$\gamma'(1) = 0$, $\gamma'(2) = 0$, $\gamma'(3) = 1$, $\gamma'(4) = 2$, $\gamma'(5) = 1$, and $HB'(4) = \frac{1}{2}$.
Hence, we get:

$$\sum_{k=1}^{5} \alpha(k) = \sum_{k=1}^{5} \alpha'(k) = 4,$$

and of course,

$$\sum_{k=1}^{5} \gamma(k) = \sum_{k=1}^{5} \gamma'(k) = 4,$$

$$\forall j \neq 4\ [\alpha'(j) \geq \alpha(j)\ \wedge\ \gamma'(j) \leq \gamma(j)],$$

and moreover, $\alpha'(4) < \alpha(4)$, $\gamma'(4) > \gamma(4)$, and $HB'(4) > HB(4)$.

Hence, by virtue of Example 7.3 we can conclude that:

Conclusion 7.2 *The Hoede-Bakker index does not satisfy the donation postulate.*

(F) PARADOX OF LARGE SIZE AND BLOC POSTULATE

As was shown in Section 6 (see Conclusion 6.2), the decisional power index of a set of players may display the paradox of large size. Now, we like to check whether the Hoede-Bakker index of an individual player may display this paradox as well. We assume that if some voters create an union, then the new player-union will influence all voters who were influenced by the players forming the union, and moreover, the voter-union will be influenced by all voters who influenced at least one member of the union. Since a player cannot influence himself, if a voter influenced by another one will form the union with his "boss", this influence will be, in a sense, lost. We can redefine the paradox of large size for the Hoede-Bakker index of an individual player in the following way:

Definition 7.4 Let $((\alpha(k))_{k=1}^n, (\gamma(k))_{k=1}^n, \Lambda, \Gamma)$ be the parameters of n-player social network A. Let $P \subseteq N$ be the set of players who form a single player-union U, resulting in a new social network B with $N \setminus P \cup \{U\}$ as the set of players. We assume that:

$$\forall k \in P \; \forall j \in N \setminus P \; [k \in INF(j) \; \Rightarrow U \in INF(j)], \tag{33}$$

$$\forall k \in P \; \forall j \in N \setminus P \; [j \in INF(k) \; \Rightarrow j \in INF(U)]. \tag{34}$$

The Hoede-Bakker index displays the *paradox of large size* if

$$HB'(U) < \sum_{k \in P} HB(k), \tag{35}$$

where HB and HB' denote the Hoede-Bakker index in social network A and B, respectively.

In this subsection, we would like also to check whether the bloc postulate, the stronger property of a power index, is satisfied by the Hoede-Bakker index of a separate player. According to this postulate, if two players form a single voter, then the new voter has more power than each of the previous players. As was shown in Section 6 (see Conclusion 6.1), a weak version of the bloc postulate is satisfied by the Hoede-Bakker index of a set of players. We introduce the following definition of the bloc postulate for the Hoede-Bakker index of an individual player:

Definition 7.5 Let $((\alpha(k))_{k=1}^n, (\gamma(k))_{k=1}^n, \Lambda, \Gamma)$ be the parameters of n-player social network A. Let $P \subseteq N$ be the set of players who form a single player-union U, resulting in a new social network B with $N \setminus P \cup \{U\}$ as the set of players. We assume that:

$$\forall k \in P \; \forall j \in N \setminus P \; [k \in INF(j) \; \Rightarrow U \in INF(j)], \tag{36}$$

$$\forall k \in P \; \forall j \in N \setminus P \; [j \in INF(k) \; \Rightarrow j \in INF(U)], \tag{37}$$

$$\forall k \in P \; [HB(k) > 0]. \tag{38}$$

The bloc postulate for the Hoede-Bakker index requires that

$$\forall k \in P\ [HB'(U) > HB(k)], \tag{39}$$

where HB and HB' mean the Hoede-Bakker index in social network A and B, respectively.

Example 7.4 Let us consider Figure 4(e). The group decisions for this figure with the Standard Procedure are given in Table 12. The Hoede-Bakker indices are as follows:
$HB(1) = \frac{5}{8}$, $HB(2) = HB(3) = \frac{3}{8}$, and $HB(4) = HB(5) = \frac{1}{8}$.
Let us suppose now that player 1, 2 and 3 decide to unite, resulting in only three players: player 1+2+3, player 4 and player 5. Since in Figure 4(e), player 4 is influenced by voters 1 and 2, and player 5 is influenced by voters 1 and 3, the new player 1+2+3 will influence actors 4 and 5. Hence, we get Figure 2(c) with 1+2+3 instead of 1, 4 instead of 2 and 5 instead of 3. Hence:
$HB'(1+2+3) = 1$, $HB'(4) = HB'(5) = 0$.
We can recognize an occurrence of the paradox of large size, because:
$HB'(1+2+3) = 1 < \frac{11}{8} = HB(1) + HB(2) + HB(3)$.

Let us assume now that in Figure 4(e), players 4 and 5 form a new voter, called player 4+5. Then, players 1, 2 and 3 influence the new voter 4+5. Hence, we get Figure 3(b) with 4+5 instead of 4 (see Table 8), and therefore:
$HB''(1) = HB''(2) = HB''(3) = \frac{1}{2}$ and $HB''(4+5) = 0$.
In this case, the paradox of large size does appear again, because:
$HB''(4+5) = 0 < \frac{1}{4} = HB(4) + HB(5)$,
but additionally, the bloc postulate is not satisfied, because:
$HB''(4+5) = 0 < \frac{1}{8} = HB(k)$ for $k \in \{4, 5\}$.

We start again with Figure 4(e), but now we assume that players 2, 3 and 4 decide to form one player 2+3+4. In this case, player 2 loses his influence on player 4, but player 4 should gain, because there is only one voter influencing him (that is, player 1) and moreover, thanks to the union with player 3, he will influence voter 5. The new situation is described by Figure 2(e) (see Table 21) with 2+3+4 instead of 2 and 5 instead of 3. Hence, by virtue of Table 21 we get:
$HB'''(1) = \frac{1}{4} \cdot (4 - 0) = 1$,
$HB'''(2+3+4) = HB'''(5) = \frac{1}{4} \cdot (2 - 2) = 0$,
what means that the bloc postulate is not satisfied and that all players forming the new voter 2+3+4 lost in power, even player 4:
$HB'''(2+3+4) = 0 < \frac{3}{8} = HB(2)$,
$HB'''(2+3+4) = 0 < \frac{3}{8} = HB(3)$, and
$HB'''(2+3+4) = 0 < \frac{1}{8} = HB(4)$.

By virtue of Example 7.4, we can conclude that:

Conclusion 7.3 *The Hoede-Bakker index may display the paradox of large size.*

Conclusion 7.4 *The Hoede-Bakker index does not satisfy the bloc postulate.*

Of course, Conclusion 7.3 follows from Conclusion 7.4, because if the bloc postulate is not satisfied, then the new single voter-union has smaller (or equal) power than at least one previous voter in the union. Hence, the index of the voter-union is smaller than the sum of the indices of the players forming the union.

Example 7.5 We like to show another example of the occurrence of the paradox of large size and the violation of the bloc postulate (for three voters forming a new player). Let us consider Figure 4(b). In this case, we have (see Table 15): $HB(1) = \frac{3}{4}$, $HB(2) = 0$, and $HB(k) = \frac{1}{4}$ for $k \in \{3, 4, 5\}$.
Let us assume now that the three independent players (players 3, 4 and 5) form a new single voter, called player 3+4+5. Then, we get Figure 2(d) in which player 1 influences player 2, and the third player is independent. Hence, we have (see Table 20):
$HB'(1) = \frac{1}{4} \cdot (4 - 0) = 1$,
$HB'(2) = HB'(3 + 4 + 5) = \frac{1}{4} \cdot (2 - 2) = 0$.
Hence, we recognize an occurrence of the paradox of large size, that is,
$HB'(3 + 4 + 5) = 0 < \frac{3}{4} = HB(3) + HB(4) + HB(5)$,
and moreover, the bloc postulate for three players is not satisfied in this case, because:
$HB'(3 + 4 + 5) = 0 < \frac{1}{4} = HB(k)$ for $k \in \{3, 4, 5\}$.

(G) PARADOX OF NEW MEMBERS

In this subsection, we will present several examples showing that the Hoede-Bakker index may display the paradox of new members. Let us first redefine the paradox of new members for the Hoede-Bakker index.

Definition 7.6 Let $((\alpha(k))_{k=1}^{n}, (\gamma(k))_{k=1}^{n}, \Lambda, \Gamma)$ and $((\alpha'(k))_{k=1}^{n+n'}, (\gamma'(k))_{k=1}^{n+n'}, \Lambda', \Gamma')$ be the parameters of two n-player social networks A and B, respectively, where B is obtained from A by adding $n' \geq 1$ players in such a way that:

$$\forall_{k=1}^{n} \; [\alpha'(k) \leq \alpha(k) \;\wedge\; \gamma'(k) \geq \gamma(k)]. \tag{40}$$

The Hoede-Bakker index displays the *paradox of new members if*

$$for\ some\ k \in \{1, ..., n\},\ HB'(k) > HB(k), \tag{41}$$

where HB and HB' denote the Hoede-Bakker index in social network A and B, respectively.

Example 7.6 Let us start with Figure 1(b). We have:
$\alpha(1) = 1$, $\gamma(1) = 0$, $\alpha(2) = 0$, $\gamma(2) = 1$,
$HB(1) = 1$ and $HB(2) = 0$.
Now we assume that a third voter, player 3, appears and he has an influence on player 2. Then, we get Figure 2(b). The group decisions for Figure 2(b) are given in Table 19. We have:
$\alpha'(1) = 1$, $\gamma'(1) = 0$, $\alpha'(2) = 0$, $\gamma'(2) = 2$,
$HB'(k) = \frac{1}{4} \cdot (3 - 1) = \frac{1}{2}$ for each player $k \in \{1, 2, 3\}$.
Hence, $HB(2) = 0 < \frac{1}{2} = HB'(2)$,
what means that player 2 gained in power after voter 3 (with influence on player 2) appeared.

Hence, we can write that:

Conclusion 7.5 *The Hoede-Bakker index may display the paradox of new members.*

Example 7.7 We will show two examples (both starting with Figure 2(d)) in which the paradox of new members occurs. Since the Hoede-Bakker index satisfies the invariance postulate, we are allowed to change the name of player 3 in Figure 2(d). Let us call him player 5. Hence, we have:
$\alpha(1) = 1$, $\gamma(1) = 0$, $\alpha(2) = 0$, $\gamma(2) = 1$, $\alpha(5) = 0$, $\gamma(5) = 0$,
$HB(1) = 1$ and $HB(2) = HB(5) = 0$.
Let us suppose now that two new players appear: player 3 influencing voter 2 and player 4 as an independent player. Hence, we get Figure 4(h) (see Table 16):
$\alpha'(1) = 1$, $\gamma'(1) = 0$, $\alpha'(2) = 0$, $\gamma'(2) = 2$, $\alpha'(5) = 0$, $\gamma'(5) = 0$,
$HB'(1) = HB'(3) = \frac{1}{2}$ and $HB'(2) = HB'(4) = HB'(5) = \frac{1}{4}$.
It means that two old players, voter 2 and voter 5, gained in power after players 3 and 4 entered, because:
$HB'(k) = \frac{1}{4} > 0 = HB(k)$ for $k \in \{2, 5\}$.

If for Figure 2(d) with players 1, 2 and 5, we add two independent players (voter 3 and voter 4), then we get Figure 4(b) (see Table 15). We have then:
$\alpha''(1) = 1$, $\gamma''(1) = 0$, $\alpha''(2) = 0$, $\gamma''(2) = 1$, $\alpha''(5) = 0$, $\gamma''(5) = 0$,
$HB''(1) = \frac{3}{4}$, $HB''(2) = 0$ and $HB''(k) = \frac{1}{4}$ for $k \in \{3, 4, 5\}$,
what means that we face again the paradox of new members, because:
$HB''(5) = \frac{1}{4} > 0 = HB(5)$.

8 Relations with other Power Indices

As already noticed by Hoede and Bakker (1982), the decisional power index differs from the other power indices. In the definition of the familiar power

indices like the Shapley-Shubik index, the normalized and absolute Banzhaf indices, the Deegan-Packel index and the Holler-Packel index, no structure is assumed and no influence between players is considered. Moreover, there is no reference to the inclination and the actual decision of a player. Nevertheless, we can find some relations between the Hoede-Bakker index and the other power indices.
First of all, there is a relation between the Hoede-Bakker index and the absolute Banzhaf index. In a sense, the absolute Banzhaf index is a special case of the Hoede-Bakker index. More precisely, if we apply the Standard Procedure to a network with an odd number of independent players, and we define a simple game in which a coalition is winning if and only if it contains more than half of the players, then the values of the Hoede-Bakker index and the absolute Banzhaf index are equal.

Theorem 8.1 *Consider a social network with a set $N = \{1, ..., n\}$ (n - odd) of independent players, i.e.,*

$$\forall i \in I \ [Bi = (b_1, ..., b_n) = i]; \tag{42}$$

$$gd(Bi) = \begin{cases} +1 & if \quad |\{k \in N \mid b_k = 1\}| \geq \frac{n+1}{2} \\ -1 & \qquad otherwise \end{cases} . \tag{43}$$

Next consider a simple game (N, v) such that for each $S \subseteq N$

$$v(S) = \begin{cases} +1 & if \quad |S| \geq \frac{n+1}{2} \\ -1 & \quad otherwise \end{cases} . \tag{44}$$

Then

$$\forall k \in N \ [HB(k) = nnBz_k(v) = \frac{1}{2^{n-1}} \cdot \binom{n-1}{\frac{n-1}{2}}], \tag{45}$$

where $HB(k)$ and $nnBz_k(v)$ denote the Hoede-Bakker index and the absolute Banzhaf index of player k, respectively.

Proof: Consider a social network in which each player is independent, i.e. for all $i \in I$, $Bi = i$. We look for $HB(k)$ for an arbitrary $k \in N$. For $x \in \{0, 1, ..., n-1\}$ we define:
$I_x^+ = \{i \in I : \ i_k = 1 \ \wedge \ |\{j \in N \setminus \{k\} : \ i_j = 1\}| = x\}$,
$I_x^- = \{i \in I : \ i_k = 1 \ \wedge \ |\{j \in N \setminus \{k\} : \ i_j = 0\}| = x\}$.
Hence, we have:

$$\sum_{\{i: \ i_k=1\}} gd(Bi) = \sum_{\{i: \ i_k=1\}} gd(i) = \sum_{x=\frac{n-1}{2}}^{n-1} |I_x^+| - \sum_{x=\frac{n+1}{2}}^{n-1} |I_x^-| =$$

$$\sum_{x=\frac{n-1}{2}}^{n-1} \binom{n-1}{x} - \sum_{x=\frac{n+1}{2}}^{n-1} \binom{n-1}{x} = \binom{n-1}{\frac{n-1}{2}}$$

Hence,

$$HB(k) = \frac{1}{2^{n-1}} \cdot \binom{n-1}{\frac{n-1}{2}}.$$

From formula (4), for each $k \in N$, $nnBz_k(v) = \frac{\eta_k}{2^{n-1}}$, where η_k denotes the number of winning coalitions in which player k is a swinger. For the simple game analysed, player k is a swinger for each coalition S such that $k \in S$ and $|S| = \frac{n+1}{2}$. Since there are $\binom{n-1}{\frac{n-1}{2}}$ such coalitions,

$$nnBz_k(v) = \frac{1}{2^{n-1}} \cdot \binom{n-1}{\frac{n-1}{2}}.$$

□

Example 8.1 If we apply our formula of $HB(k)$ to Figure 2(a), where we have three independent players, and to Figure 4(a) with five independent players, we get:
Figure 2(a): $HB(k) = \frac{1}{4} \cdot \binom{2}{1} = \frac{1}{2}$ for $k \in \{1, 2, 3\}$,
Figure 4(a): $HB(k) = \frac{1}{16} \cdot \binom{4}{2} = \frac{3}{8}$ for $k \in \{1, 2, 3, 4, 5\}$.
Hence, we receive, of course, the same results as before, when using the definition of the Hoede-Bakker index.

As was shown in Section 7, the Hoede-Bakker index is not normalized. Nevertheless, from Theorem 7.1 we know that if there is a dictator in a social network, then $\sum_{k=1}^{n} HB(k) = 1$. Hence, there is also a certain relation between the Hoede-Bakker index and the power indices satisfying the normalization postulate, like the Shapley-Shubik index, the normalized Banzhaf index, the Deegan-Packel index and the Holler-Packel index. One can prove that

Theorem 8.2 *Suppose $N = \{1, ..., n\}$ is a set of players in a social network and k^* is a dictator in this network. Let (N, v) be a simple game in which player $k^* \in N$ is a dictator too. Then:*

$$HB(k^*) = \phi_{k^*}(v) = 1, \tag{46}$$

$$HB(k) = \phi_k(v) = 0 \ for \ each \ k \in N \setminus \{k^*\}, \tag{47}$$

for each $\phi \in \{Sh, Bz, DP, HP\}$.

Proof: Since k^* is a dictator in a social network, $HB(k^*) = 1$ and $HB(k) = 0$ for each $k \in N \setminus \{k^*\}$. Player k^* as a dictator in a simple game, is the sole player who is not a dummy. One can prove that for $\phi \in \{Sh, Bz, nnBz, DP, HP\}$, $\phi_k = 0$ if and only if k is a dummy. Hence, $\phi_k(v) = 0$ for each $k \in N \setminus \{k^*\}$. Since $\phi \in \{Sh, Bz, DP, HP\}$ satisfies the normalization postulate, $\phi_{k^*}(v) = 1$.
□

9 Conclusions

In this paper, we analysed the decisional power index (the Hoede-Bakker index). Given a social network, different operators B and group decision operators gd can be constructed that satisfy the two axioms of Hoede and Bakker for B and gd, but may yield different outcomes. This confirms that the decisional power of an individual is dependent on the operators B and gd. We proposed a Standard Procedure for B and gd which satisfies all adopted axioms for a network with an arbitrary odd number of players. Also, we discussed the case of an even number of players and we considered the decisional power index of a set of players. Using examples, we showed that the Hoede-Bakker index displays a number of voting power paradoxes: the paradox of redistribution, the paradox of large size and the paradox of new members. In addition, we showed that the Hoede-Bakker index does not satisfy some of the postulates for power indices: the monotonicity, the donation and the bloc postulate. However, the paradoxes appear mainly if players influenced are involved. The Hoede-Bakker index of a "strong" player, that is, a player who influences the others without being influenced by the others, seems to be less sensitive to the paradoxes. Hence, the ability to influence the others seems to indicate real power. Finally, in the paper we analysed the relations between the decisional power index and other power indices. It turns out that the absolute Banzhaf index is precisely the Hoede-Bakker index in case all players in the network are independent.

References

[1] Banzhaf, J.(1965). Weighted voting doesn't work: a mathematical analysis, *Rutgers Law Review* 19: 317-343.

[2] Brams, S.J. (1975). *Game Theory and Politics*, New York, Free Press.

[3] Brams, S.J. and Affuso, P. (1976). Power and size: a new paradox, *Theory and Decision* 7: 29-56.

[4] Deegan, J. and Packel, E.W. (1978). A new index of power for simple n-person games, *International Journal of Game Theory* 7: 113-123.

[5] Dubey, P. and Shapley, L.S. (1979). Mathematical properties of the Banzhaf power index. *Mathematics of Operations Research* 4: 99-131.

[6] Felsenthal, D.S. and Machover, M. (1995). Postulates and paradoxes of relative voting power - a critical reappraisal. *Theory and Decision* 38: 195-229

[7] Felsenthal, D.S. and Machover, M. (1998). *The measurement of voting power: theory and practice, problems and paradoxes*, London: Edward Elgar Publishers.

[8] Felsenthal, D.S., Machover, M. and Zwicker, W.S. (1998). The bicameral postulates and indices of a priori voting power. *Theory and Decision* 44: 83-116

[9] Fischer, D. and Schotter, A. (1978). The inevitability of the paradox of redistribution in the allocation of voting weights. *Public Choice* 33: 49-67.

[10] Hoede, C. and Bakker, R. (1982). A theory of decisional power. *Journal of Mathematical Sociology* 8: 309-322.

[11] Holler, M.J. (1982). Forming coalitions and measuring voting power. *Political Studies* 30: 262-271.

[12] Holler, M.J. and Packel E.W. (1983). Power, luck and the right index. *Journal of Economics* 43 (1983) 21-29.

[13] Kilgour, D.M. (1974). A Shapley value for cooperative games with quarreling, in: A. Rapoport (ed.) *Game theory as a theory of conflict resolution*, 193-206. Boston: Reidel.

[14] Laruelle, A. (2000). On the choice of a power index. Universidad de Alicante, Alicante, Spain.

[15] Penrose, L.S. (1946). The elementary statistics of majority voting. *Journal of the Royal Statistical Society* 109: 53-57.

[16] Rusinowska, A. (2001). Paradox of redistribution in Polish politics. Marie Curie Fellowships Annals (MCFA), volume II (in press).

[17] Schotter, A. (1981). The paradox of redistribution: some theoretical and empirical results. In Holler, M. J. (ed.) *Power, voting, and voting power.* Wurzburg-Wien: Physica-Verlag.

[18] Shapley, L.S. (1953). A value for n-person games. *Annals of Mathematics Studies* 28: 307-317.

[19] Shapley, L.S. and Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review* 48: 787-792.

[20] Van Deemen, A.M.A. and Rusinowska A. (2001). Paradoxes of voting power in Dutch politics. *Public Choice*, to appear.

Appendix A

Figure 1: Examples of social network for $n = 2$

Figure 2: Examples of social network for $n = 3$

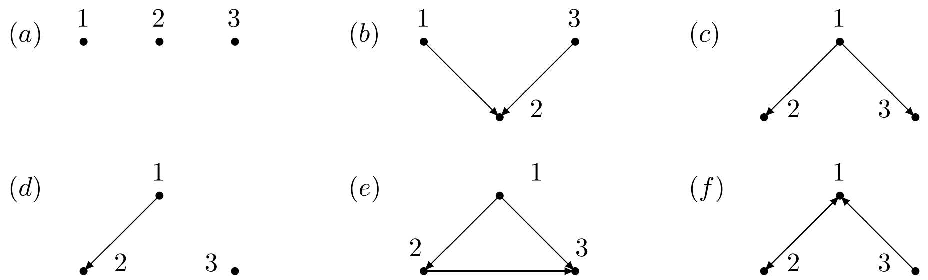


Figure 3: Examples of social network for $n = 4$

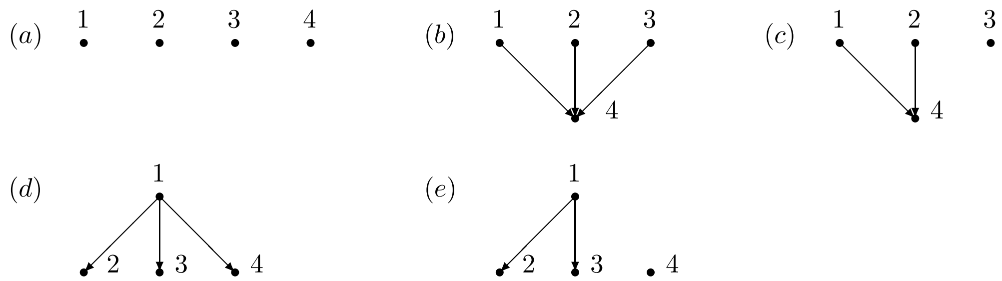
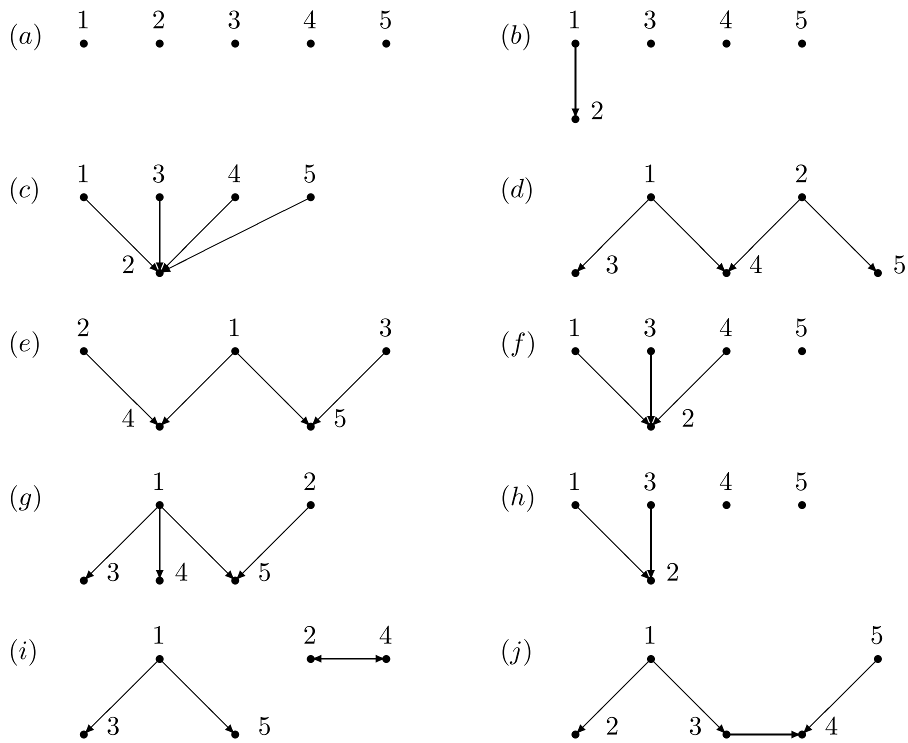


Figure 4: Examples of social network for $n = 5$



Appendix B

Table 1: Group decision for Figure 4(f) with Procedure 1

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,0)	+1	(0,0,0,0,1)	(0,0,0,0,1)	−1
(1,1,1,0,1)	(1,1,1,0,1)	+1	(0,0,0,1,0)	(0,0,0,1,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,1,1,1,1)	+1	(0,1,0,0,0)	(0,0,0,0,0)	−1
(0,1,1,1,1)	(0,1,1,1,1)	+1	(1,0,0,0,0)	(1,0,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,0,0)	+1	(0,0,0,1,1)	(0,0,0,1,1)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,1,1,1,0)	+1	(0,1,0,0,1)	(0,0,0,0,1)	−1
(0,1,1,1,0)	(0,1,1,1,0)	+1	(1,0,0,0,1)	(1,0,0,0,1)	−1
(1,1,0,0,1)	(1,0,0,0,1)	−1	(0,0,1,1,0)	(0,1,1,1,0)	+1
(1,0,1,0,1)	(1,1,1,0,1)	+1	(0,1,0,1,0)	(0,0,0,1,0)	−1
(0,1,1,0,1)	(0,0,1,0,1)	−1	(1,0,0,1,0)	(1,1,0,1,0)	+1
(1,0,0,1,1)	(1,1,0,1,1)	+1	(0,1,1,0,0)	(0,0,1,0,0)	−1
(0,1,0,1,1)	(0,0,0,1,1)	−1	(1,0,1,0,0)	(1,1,1,0,0)	+1
(0,0,1,1,1)	(0,1,1,1,1)	+1	(1,1,0,0,0)	(1,0,0,0,0)	−1

Table 2: Group decision for Figure 4(f) with Procedure 2

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,0)	+1	(0,0,0,0,1)	(0,0,0,0,1)	−1
(1,1,1,0,1)	(1,1,1,0,1)	+1	(0,0,0,1,0)	(0,0,0,1,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,1,1,1,1)	+1	(0,1,0,0,0)	(0,0,0,0,0)	−1
(0,1,1,1,1)	(0,1,1,1,1)	+1	(1,0,0,0,0)	(1,0,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,0,0)	+1	(0,0,0,1,1)	(0,0,0,1,1)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,1,1,1,0)	+1	(0,1,0,0,1)	(0,0,0,0,1)	−1
(0,1,1,1,0)	(0,1,1,1,0)	+1	(1,0,0,0,1)	(1,0,0,0,1)	−1
(1,1,0,0,1)	(1,1,0,0,1)	+1	(0,0,1,1,0)	(0,0,1,1,0)	−1
(1,0,1,0,1)	(1,0,1,0,1)	+1	(0,1,0,1,0)	(0,1,0,1,0)	−1
(0,1,1,0,1)	(0,1,1,0,1)	+1	(1,0,0,1,0)	(1,0,0,1,0)	−1
(1,0,0,1,1)	(1,0,0,1,1)	+1	(0,1,1,0,0)	(0,1,1,0,0)	−1
(0,1,0,1,1)	(0,1,0,1,1)	+1	(1,0,1,0,0)	(1,0,1,0,0)	−1
(0,0,1,1,1)	(0,0,1,1,1)	+1	(1,1,0,0,0)	(1,1,0,0,0)	−1

Table 3: Group decision for Figure 2(c) with Procedure 3

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	-1
(1,1,0)	(1,1,1)	+1	(0,0,1)	(0,0,0)	-1
(1,0,1)	(1,1,1)	+1	(0,1,0)	(0,0,0)	-1
(0,1,1)	(0,0,0)	-1	(1,0,0)	(1,1,1)	+1

Table 4: Group decision for Figure 2(c) with Procedure 4

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	-1
(1,1,0)	(1,1,0)	+1	(0,0,1)	(0,0,0)	-1
(1,0,1)	(1,1,1)	+1	(0,1,0)	(0,1,0)	-1
(0,1,1)	(0,1,0)	-1	(1,0,0)	(1,1,0)	+1

Table 5: Group decision for Figure 1(b) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1)	(1,1)	+1	(0,0)	(0,0)	-1
(1,0)	(1,1)	+1	(0,1)	(0,0)	-1

Table 6: Group decision for Figure 3(a) with Drawn Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1)	(1,1,1,1)	+1	(0,0,0,0)	(0,0,0,0)	-1
(1,1,1,0)	(1,1,1,0)	+1	(0,0,0,1)	(0,0,0,1)	-1
(1,1,0,1)	(1,1,0,1)	+1	(0,0,1,0)	(0,0,1,0)	-1
(1,0,1,1)	(1,0,1,1)	+1	(0,1,0,0)	(0,1,0,0)	-1
(0,1,1,1)	(0,1,1,1)	+1	(1,0,0,0)	(1,0,0,0)	-1
(1,1,0,0,0)	(1,1,0,0,0)	-1	(0,0,1,1,0)	(0,0,1,1,0)	-1
(1,0,1,0,0)	(1,0,1,0,0)	-1	(0,1,0,1,0)	(0,1,0,1,0)	-1
(1,0,0,1,0)	(1,0,0,1,0)	-1	(0,1,1,0,0)	(0,1,1,0,0)	-1

Table 7: Group decision for Figure 3(c) with Drawn Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1)	(1,1,1,1)	$+1$	(0,0,0,0)	(0,0,0,0)	-1
(1,1,1,0)	(1,1,1,1)	$+1$	(0,0,0,1)	(0,0,0,0)	-1
(1,1,0,1)	(1,1,0,1)	$+1$	(0,0,1,0)	(0,0,1,0)	-1
(1,0,1,1)	(1,0,1,1)	$+1$	(0,1,0,0)	(0,1,0,0)	-1
(0,1,1,1)	(0,1,1,1)	$+1$	(1,0,0,0)	(1,0,0,0)	-1
(1,1,0,0)	(1,1,0,1)	$+1$	(0,0,1,1)	(0,0,1,0)	-1
(1,0,1,0,0)	(1,0,1,0,0)	-1	(0,1,0,1,0)	(0,1,0,1,0)	-1
(1,0,0,1,0)	(1,0,0,1,0)	-1	(0,1,1,0,0)	(0,1,1,0,0)	-1

Table 8: Group decision for Figure 3(b) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1)	(1,1,1,1)	$+1$	(0,0,0,0)	(0,0,0,0)	-1
(1,1,1,0)	(1,1,1,1)	$+1$	(0,0,0,1)	(0,0,0,0)	-1
(1,1,0,1)	(1,1,0,1)	$+1$	(0,0,1,0)	(0,0,1,0)	-1
(1,0,1,1)	(1,0,1,1)	$+1$	(0,1,0,0)	(0,1,0,0)	-1
(0,1,1,1)	(0,1,1,1)	$+1$	(1,0,0,0)	(1,0,0,0)	-1
(1,1,0,0)	(1,1,0,1)	$+1$	(0,0,1,1)	(0,0,1,0)	-1
(1,0,1,0)	(1,0,1,1)	$+1$	(0,1,0,1)	(0,1,0,0)	-1
(1,0,0,1)	(1,0,0,0)	-1	(0,1,1,0)	(0,1,1,1)	$+1$

Table 9: Group decision for Figure 3(d) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1)	(1,1,1,1)	$+1$	(0,0,0,0)	(0,0,0,0)	-1
(1,1,1,0)	(1,1,1,1)	$+1$	(0,0,0,1)	(0,0,0,0)	-1
(1,1,0,1)	(1,1,1,1)	$+1$	(0,0,1,0)	(0,0,0,0)	-1
(1,0,1,1)	(1,1,1,1)	$+1$	(0,1,0,0)	(0,0,0,0)	-1
(0,1,1,1)	(0,0,0,0)	-1	(1,0,0,0)	(1,1,1,1)	$+1$
(1,1,0,0)	(1,1,1,1)	$+1$	(0,0,1,1)	(0,0,0,0)	-1
(1,0,1,0)	(1,1,1,1)	$+1$	(0,1,0,1)	(0,0,0,0)	-1
(1,0,0,1)	(1,1,1,1)	$+1$	(0,1,1,0)	(0,0,0,0)	-1

Table 10: Group decision for Figure 3(e) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1)	(1,1,1,1)	+1	(0,0,0,0)	(0,0,0,0)	−1
(1,1,1,0)	(1,1,1,0)	+1	(0,0,0,1)	(0,0,0,1)	−1
(1,1,0,1)	(1,1,1,1)	+1	(0,0,1,0)	(0,0,0,0)	−1
(1,0,1,1)	(1,1,1,1)	+1	(0,1,0,0)	(0,0,0,0)	−1
(0,1,1,1)	(0,0,0,1)	−1	(1,0,0,0)	(1,1,1,0)	+1
(1,1,0,0)	(1,1,1,0)	+1	(0,0,1,1)	(0,0,0,1)	−1
(1,0,1,0)	(1,1,1,0)	+1	(0,1,0,1)	(0,0,0,1)	−1
(1,0,0,1)	(1,1,1,1)	+1	(0,1,1,0)	(0,0,0,0)	−1

Table 11: Group decision for Figure 2(a) with Procedure 5/Procedure 6 = Group decision for Figure 2(c) with Procedure 6

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	−1
(1,1,0)	(1,1,0)	+1	(0,0,1)	(0,0,1)	−1
(1,0,1)	(1,0,1)	+1	(0,1,0)	(0,1,0)	−1
(0,1,1)	(0,1,1)	+1	(1,0,0)	(1,0,0)	−1

Table 12: Group decision for Figure 4(e) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,1)	+1	(0,0,0,0,1)	(0,0,0,0,0)	−1
(1,1,1,0,1)	(1,1,1,1,1)	+1	(0,0,0,1,0)	(0,0,0,0,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,0,1,1,1)	+1	(0,1,0,0,0)	(0,1,0,0,0)	−1
(0,1,1,1,1)	(0,1,1,1,1)	+1	(1,0,0,0,0)	(1,0,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,1,1)	+1	(0,0,0,1,1)	(0,0,0,0,0)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,0,1,1,1)	+1	(0,1,0,0,1)	(0,1,0,0,0)	−1
(0,1,1,1,0)	(0,1,1,1,0)	+1	(1,0,0,0,1)	(1,0,0,0,1)	−1
(1,1,0,0,1)	(1,1,0,1,1)	+1	(0,0,1,1,0)	(0,0,1,0,0)	−1
(1,0,1,0,1)	(1,0,1,0,1)	+1	(0,1,0,1,0)	(0,1,0,1,0)	−1
(0,1,1,0,1)	(0,1,1,0,1)	+1	(1,0,0,1,0)	(1,0,0,1,0)	−1
(1,0,0,1,1)	(1,0,0,1,1)	+1	(0,1,1,0,0)	(0,1,1,0,0)	−1
(0,1,0,1,1)	(0,1,0,1,0)	−1	(1,0,1,0,0)	(1,0,1,0,1)	+1
(0,0,1,1,1)	(0,0,1,0,1)	−1	(1,1,0,0,0)	(1,1,0,1,0)	+1

Table 13: Group decision for Figure 4(g) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,1)	+1	(0,0,0,0,1)	(0,0,0,0,0)	−1
(1,1,1,0,1)	(1,1,1,1,1)	+1	(0,0,0,1,0)	(0,0,0,0,0)	−1
(1,1,0,1,1)	(1,1,1,1,1)	+1	(0,0,1,0,0)	(0,0,0,0,0)	−1
(1,0,1,1,1)	(1,0,1,1,1)	+1	(0,1,0,0,0)	(0,1,0,0,0)	−1
(0,1,1,1,1)	(0,1,0,0,1)	−1	(1,0,0,0,0)	(1,0,1,1,0)	+1
(1,1,1,0,0)	(1,1,1,1,1)	+1	(0,0,0,1,1)	(0,0,0,0,0)	−1
(1,1,0,1,0)	(1,1,1,1,1)	+1	(0,0,1,0,1)	(0,0,0,0,0)	−1
(1,0,1,1,0)	(1,0,1,1,0)	+1	(0,1,0,0,1)	(0,1,0,0,1)	−1
(0,1,1,1,0)	(0,1,0,0,0)	−1	(1,0,0,0,1)	(1,0,1,1,1)	+1
(1,1,0,0,1)	(1,1,1,1,1)	+1	(0,0,1,1,0)	(0,0,0,0,0)	−1
(1,0,1,0,1)	(1,0,1,1,1)	+1	(0,1,0,1,0)	(0,1,0,0,0)	−1
(0,1,1,0,1)	(0,1,0,0,1)	−1	(1,0,0,1,0)	(1,0,1,1,0)	+1
(1,0,0,1,1)	(1,0,1,1,1)	+1	(0,1,1,0,0)	(0,1,0,0,0)	−1
(0,1,0,1,1)	(0,1,0,0,1)	−1	(1,0,1,0,0)	(1,0,1,1,0)	+1
(0,0,1,1,1)	(0,0,0,0,0)	−1	(1,1,0,0,0)	(1,1,1,1,1)	+1

Table 14: Group decision for Figure 4(d) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,1)	+1	(0,0,0,0,1)	(0,0,0,0,0)	−1
(1,1,1,0,1)	(1,1,1,1,1)	+1	(0,0,0,1,0)	(0,0,0,0,0)	−1
(1,1,0,1,1)	(1,1,1,1,1)	+1	(0,0,1,0,0)	(0,0,0,0,0)	−1
(1,0,1,1,1)	(1,0,1,1,0)	+1	(0,1,0,0,0)	(0,1,0,0,1)	−1
(0,1,1,1,1)	(0,1,0,1,1)	+1	(1,0,0,0,0)	(1,0,1,0,0)	−1
(1,1,1,0,0)	(1,1,1,1,1)	+1	(0,0,0,1,1)	(0,0,0,0,0)	−1
(1,1,0,1,0)	(1,1,1,1,1)	+1	(0,0,1,0,1)	(0,0,0,0,0)	−1
(1,0,1,1,0)	(1,0,1,1,0)	+1	(0,1,0,0,1)	(0,1,0,0,1)	−1
(0,1,1,1,0)	(0,1,0,1,1)	+1	(1,0,0,0,1)	(1,0,1,0,0)	−1
(1,1,0,0,1)	(1,1,1,1,1)	+1	(0,0,1,1,0)	(0,0,0,0,0)	−1
(1,0,1,0,1)	(1,0,1,0,0)	−1	(0,1,0,1,0)	(0,1,0,1,1)	+1
(0,1,1,0,1)	(0,1,0,0,1)	−1	(1,0,0,1,0)	(1,0,1,1,0)	+1
(1,0,0,1,1)	(1,0,1,1,0)	+1	(0,1,1,0,0)	(0,1,0,0,1)	−1
(0,1,0,1,1)	(0,1,0,1,1)	+1	(1,0,1,0,0)	(1,0,1,0,0)	−1
(0,0,1,1,1)	(0,0,0,0,0)	−1	(1,1,0,0,0)	(1,1,1,1,1)	+1

Table 15: Group decision for Figure 4(b) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,0)	+1	(0,0,0,0,1)	(0,0,0,0,1)	−1
(1,1,1,0,1)	(1,1,1,0,1)	+1	(0,0,0,1,0)	(0,0,0,1,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,1,1,1,1)	+1	(0,1,0,0,0)	(0,0,0,0,0)	−1
(0,1,1,1,1)	(0,0,1,1,1)	+1	(1,0,0,0,0)	(1,1,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,0,0)	+1	(0,0,0,1,1)	(0,0,0,1,1)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,1,1,1,0)	+1	(0,1,0,0,1)	(0,0,0,0,1)	−1
(0,1,1,1,0)	(0,0,1,1,0)	−1	(1,0,0,0,1)	(1,1,0,0,1)	+1
(1,1,0,0,1)	(1,1,0,0,1)	+1	(0,0,1,1,0)	(0,0,1,1,0)	−1
(1,0,1,0,1)	(1,1,1,0,1)	+1	(0,1,0,1,0)	(0,0,0,1,0)	−1
(0,1,1,0,1)	(0,0,1,0,1)	−1	(1,0,0,1,0)	(1,1,0,1,0)	+1
(1,0,0,1,1)	(1,1,0,1,1)	+1	(0,1,1,0,0)	(0,0,1,0,0)	−1
(0,1,0,1,1)	(0,0,0,1,1)	−1	(1,0,1,0,0)	(1,1,1,0,0)	+1
(0,0,1,1,1)	(0,0,1,1,1)	+1	(1,1,0,0,0)	(1,1,0,0,0)	−1

Table 16: Group decision for Figure 4(h) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,0)	+1	(0,0,0,0,1)	(0,0,0,0,1)	−1
(1,1,1,0,1)	(1,1,1,0,1)	+1	(0,0,0,1,0)	(0,0,0,1,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,1,1,1,1)	+1	(0,1,0,0,0)	(0,0,0,0,0)	−1
(0,1,1,1,1)	(0,1,1,1,1)	+1	(1,0,0,0,0)	(1,0,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,0,0)	+1	(0,0,0,1,1)	(0,0,0,1,1)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,1,1,1,0)	+1	(0,1,0,0,1)	(0,0,0,0,1)	−1
(0,1,1,1,0)	(0,1,1,1,0)	+1	(1,0,0,0,1)	(1,0,0,0,1)	−1
(1,1,0,0,1)	(1,1,0,0,1)	+1	(0,0,1,1,0)	(0,0,1,1,0)	−1
(1,0,1,0,1)	(1,1,1,0,1)	+1	(0,1,0,1,0)	(0,0,0,1,0)	−1
(0,1,1,0,1)	(0,1,1,0,1)	+1	(1,0,0,1,0)	(1,0,0,1,0)	−1
(1,0,0,1,1)	(1,0,0,1,1)	+1	(0,1,1,0,0)	(0,1,1,0,0)	−1
(0,1,0,1,1)	(0,0,0,1,1)	−1	(1,0,1,0,0)	(1,1,1,0,0)	+1
(0,0,1,1,1)	(0,0,1,1,1)	+1	(1,1,0,0,0)	(1,1,0,0,0)	−1

Table 17: Group decision for Figure 4(c) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,0)	+1	(0,0,0,0,1)	(0,0,0,0,1)	−1
(1,1,1,0,1)	(1,1,1,0,1)	+1	(0,0,0,1,0)	(0,0,0,1,0)	−1
(1,1,0,1,1)	(1,1,0,1,1)	+1	(0,0,1,0,0)	(0,0,1,0,0)	−1
(1,0,1,1,1)	(1,1,1,1,1)	+1	(0,1,0,0,0)	(0,0,0,0,0)	−1
(0,1,1,1,1)	(0,1,1,1,1)	+1	(1,0,0,0,0)	(1,0,0,0,0)	−1
(1,1,1,0,0)	(1,1,1,0,0)	+1	(0,0,0,1,1)	(0,0,0,1,1)	−1
(1,1,0,1,0)	(1,1,0,1,0)	+1	(0,0,1,0,1)	(0,0,1,0,1)	−1
(1,0,1,1,0)	(1,1,1,1,0)	+1	(0,1,0,0,1)	(0,0,0,0,1)	−1
(0,1,1,1,0)	(0,1,1,1,0)	+1	(1,0,0,0,1)	(1,0,0,0,1)	−1
(1,1,0,0,1)	(1,1,0,0,1)	+1	(0,0,1,1,0)	(0,0,1,1,0)	−1
(1,0,1,0,1)	(1,1,1,0,1)	+1	(0,1,0,1,0)	(0,0,0,1,0)	−1
(0,1,1,0,1)	(0,1,1,0,1)	+1	(1,0,0,1,0)	(1,0,0,1,0)	−1
(1,0,0,1,1)	(1,1,0,1,1)	+1	(0,1,1,0,0)	(0,0,1,0,0)	−1
(0,1,0,1,1)	(0,1,0,1,1)	+1	(1,0,1,0,0)	(1,0,1,0,0)	−1
(0,0,1,1,1)	(0,1,1,1,1)	+1	(1,1,0,0,0)	(1,0,0,0,0)	−1

Table 18: Group decision for Figure 4(i) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	+1	(0,0,0,0,0)	(0,0,0,0,0)	−1
(1,1,1,1,0)	(1,1,1,1,1)	+1	(0,0,0,0,1)	(0,0,0,0,0)	−1
(1,1,1,0,1)	(1,0,1,1,1)	+1	(0,0,0,1,0)	(0,1,0,0,0)	−1
(1,1,0,1,1)	(1,1,1,1,1)	+1	(0,0,1,0,0)	(0,0,0,0,0)	−1
(1,0,1,1,1)	(1,1,1,0,1)	+1	(0,1,0,0,0)	(0,0,0,1,0)	−1
(0,1,1,1,1)	(0,1,0,1,0)	−1	(1,0,0,0,0)	(1,0,1,0,1)	+1
(1,1,1,0,0)	(1,0,1,1,1)	+1	(0,0,0,1,1)	(0,1,0,0,0)	−1
(1,1,0,1,0)	(1,1,1,1,1)	+1	(0,0,1,0,1)	(0,0,0,0,0)	−1
(1,0,1,1,0)	(1,1,1,0,1)	+1	(0,1,0,0,1)	(0,0,0,1,0)	−1
(0,1,1,1,0)	(0,1,0,1,0)	−1	(1,0,0,0,1)	(1,0,1,0,1)	+1
(1,1,0,0,1)	(1,0,1,1,1)	+1	(0,0,1,1,0)	(0,1,0,0,0)	−1
(1,0,1,0,1)	(1,0,1,0,1)	+1	(0,1,0,1,0)	(0,1,0,1,0)	−1
(0,1,1,0,1)	(0,0,0,1,0)	−1	(1,0,0,1,0)	(1,1,1,0,1)	+1
(1,0,0,1,1)	(1,1,1,0,1)	+1	(0,1,1,0,0)	(0,0,0,1,0)	−1
(0,1,0,1,1)	(0,1,0,1,0)	−1	(1,0,1,0,0)	(1,0,1,0,1)	+1
(0,0,1,1,1)	(0,1,0,0,0)	−1	(1,1,0,0,0)	(1,0,1,1,1)	+1

Table 19: Group decision for Figure 2(b) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	−1
(1,1,0)	(1,1,1)	+1	(0,0,1)	(0,0,0)	−1
(1,0,1)	(1,0,1)	+1	(0,1,0)	(0,1,0)	−1
(0,1,1)	(0,1,1)	+1	(1,0,0)	(1,0,0)	−1

Table 20: Group decision for Figure 2(d) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	−1
(1,1,0)	(1,1,0)	+1	(0,0,1)	(0,0,1)	−1
(1,0,1)	(1,1,1)	+1	(0,1,0)	(0,0,0)	−1
(0,1,1)	(0,0,1)	−1	(1,0,0)	(1,1,0)	+1

Table 21: Group decision for Figure 2(e) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	−1
(1,1,0)	(1,1,1)	+1	(0,0,1)	(0,0,0)	−1
(1,0,1)	(1,1,1)	+1	(0,1,0)	(0,0,0)	−1
(0,1,1)	(0,0,1)	−1	(1,0,0)	(1,1,0)	+1

Table 22: Group decision for Figure 2(f) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1)	(1,1,1)	+1	(0,0,0)	(0,0,0)	−1
(1,1,0)	(1,1,0)	+1	(0,0,1)	(0,0,1)	−1
(1,0,1)	(1,1,1)	+1	(0,1,0)	(0,0,0)	−1
(0,1,1)	(1,0,1)	+1	(1,0,0)	(0,1,0)	−1

Table 23: Group decision for Figure 4(a) with Standard Procedure

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	$+1$	(0,0,0,0,0)	(0,0,0,0,0)	-1
(1,1,1,1,0)	(1,1,1,1,0)	$+1$	(0,0,0,0,1)	(0,0,0,0,1)	-1
(1,1,1,0,1)	(1,1,1,0,1)	$+1$	(0,0,0,1,0)	(0,0,0,1,0)	-1
(1,1,0,1,1)	(1,1,0,1,1)	$+1$	(0,0,1,0,0)	(0,0,1,0,0)	-1
(1,0,1,1,1)	(1,0,1,1,1)	$+1$	(0,1,0,0,0)	(0,1,0,0,0)	-1
(0,1,1,1,1)	(0,1,1,1,1)	$+1$	(1,0,0,0,0)	(1,0,0,0,0)	-1
(1,1,1,0,0)	(1,1,1,0,0)	$+1$	(0,0,0,1,1)	(0,0,0,1,1)	-1
(1,1,0,1,0)	(1,1,0,1,0)	$+1$	(0,0,1,0,1)	(0,0,1,0,1)	-1
(1,0,1,1,0)	(1,0,1,1,0)	$+1$	(0,1,0,0,1)	(0,1,0,0,1)	-1
(0,1,1,1,0)	(0,1,1,1,0)	$+1$	(1,0,0,0,1)	(1,0,0,0,1)	-1
(1,1,0,0,1)	(1,1,0,0,1)	$+1$	(0,0,1,1,0)	(0,0,1,1,0)	-1
(1,0,1,0,1)	(1,0,1,0,1)	$+1$	(0,1,0,1,0)	(0,1,0,1,0)	-1
(0,1,1,0,1)	(0,1,1,0,1)	$+1$	(1,0,0,1,0)	(1,0,0,1,0)	-1
(1,0,0,1,1)	(1,0,0,1,1)	$+1$	(0,1,1,0,0)	(0,1,1,0,0)	-1
(0,1,0,1,1)	(0,1,0,1,1)	$+1$	(1,0,1,0,0)	(1,0,1,0,0)	-1
(0,0,1,1,1)	(0,0,1,1,1)	$+1$	(1,1,0,0,0)	(1,1,0,0,0)	-1

Table 24: Group decision for Figure 4(j) with Procedure 6

inclination i	Bi	$gd(Bi)$	inclination i	Bi	$gd(Bi)$
(1,1,1,1,1)	(1,1,1,1,1)	$+1$	(0,0,0,0,0)	(0,0,0,0,0)	-1
(1,1,1,1,0)	(1,1,1,1,0)	$+1$	(0,0,0,0,1)	(0,0,0,0,1)	-1
(1,1,1,0,1)	(1,1,1,1,1)	$+1$	(0,0,0,1,0)	(0,0,0,0,0)	-1
(1,1,0,1,1)	(1,1,0,1,1)	$+1$	(0,0,1,0,0)	(0,0,1,0,0)	-1
(1,0,1,1,1)	(1,0,1,1,1)	$+1$	(0,1,0,0,0)	(0,1,0,0,0)	-1
(0,1,1,1,1)	(0,1,1,1,1)	$+1$	(1,0,0,0,0)	(1,0,0,0,0)	-1
(1,1,1,0,0)	(1,1,1,0,0)	$+1$	(0,0,0,1,1)	(0,0,0,1,1)	-1
(1,1,0,1,0)	(1,1,0,0,0)	-1	(0,0,1,0,1)	(0,0,1,1,1)	$+1$
(1,0,1,1,0)	(1,0,1,1,0)	$+1$	(0,1,0,0,1)	(0,1,0,0,1)	-1
(0,1,1,1,0)	(0,1,1,1,0)	$+1$	(1,0,0,0,1)	(1,0,0,0,1)	-1
(1,1,0,0,1)	(1,1,0,0,1)	$+1$	(0,0,1,1,0)	(0,0,1,1,0)	-1
(1,0,1,0,1)	(1,0,1,1,1)	$+1$	(0,1,0,1,0)	(0,1,0,0,0)	-1
(0,1,1,0,1)	(0,1,1,1,1)	$+1$	(1,0,0,1,0)	(1,0,0,0,0)	-1
(1,0,0,1,1)	(1,0,0,1,1)	$+1$	(0,1,1,0,0)	(0,1,1,0,0)	-1
(0,1,0,1,1)	(0,1,0,1,1)	$+1$	(1,0,1,0,0)	(1,0,1,0,0)	-1
(0,0,1,1,1)	(0,0,1,1,1)	$+1$	(1,1,0,0,0)	(1,1,0,0,0)	-1

Table 25: The Hoede-Bakker index with Standard Procedure

Figure	$HB(1)$	$HB(2)$	$HB(3)$	$HB(4)$	$HB(5)$
1(b)	1	0	–	–	–
2(a)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	–	–
2(b)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	–	–
2(c)	1	0	0	–	–
2(d)	1	0	0	–	–
2(e)	1	0	0	–	–
2(f)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	–	–
3(b)	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	–
3(d)	1	0	0	0	–
3(e)	1	0	0	0	–
4(a)	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
4(b)	$\frac{3}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
4(c)	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
4(d)	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
4(e)	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
4(f)	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0
4(g)	1	0	0	0	0
4(h)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
4(i)	1	0	0	0	0