# Voting Power in Weighted $(j, k)$ Games: A Limit Theorem and a Numerical Method

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The widely used instrument to analyse voting power is that of a simple voting game (SVG) which is binary as it offers each voter only to choose from 'yes' or 'no'. In real life decisions, however, options besides them clearly influence the outcome of a vote. The literature has recently restarted to take notice of voting games considering abstentions as an additional choice besides 'yes' and 'no'. Felsenthal and Machover (F&M) (2000) introduced a setup called a *ternary voting game* by adding abstention as a third option alongside 'yes' and 'no' (which extends an earlier work of Fishburn (1973, pp. 53-55)). As a further extension Freixas and Zwicker (2002) introduced so called weighted $(j, k)$ games in which a voter is endowed with $j$ many voting weights. The intuition is that the voter is now able to express different levels of approval ranging from complete enthousiasm to total opposition. The outcome set is also enlarged from the usual binary case to $k$ different levels of outcome. In their paper Freixas and Zwicker provide a combinatorical characterization of games that can be formulated as weighted $(j, k)$ games which is the property of grade trade robustness. This implies that all results with respect to these games also hold for games which are not weighted $(j, k)$ games in the first place but are grade trade robust and hence can be reformulated as such.

This paper introduces a broad notion of power in these games and provides a limit theorem, as well as a numerical method. The power concept is that of a voter being $r$-critical: a voter is $r$-critical if s/he can tip the balance between an outcome below or above an outcome level $r$. Hence, instead of introducing a single value that measures the decisiveness of a voter this concept provides a whole vector of decisiveness. With $k$ levels of output there are $k-1$ thresholds and hence the probability $\psi_a(r)$ of a voter being $r$-critical makes sense for $k-1$ different possible values of $r$. This concept covers all classical power indices in SVGs as well as the prevalent analogues in voting games with abstentions[1]. This generalization might not be the only reasonable one, however, we will show that this concept implies phenomena that are known from the SVG approach. The paper focuses on the *Penrose limit theorem* (PLT) which describes the phenomenon that under suitable conditions in weighted voting games the ratio of the powers of any two voters converges to the ratio of the voting weights.

[1] For a survey Lindner (2002).

Lindner & Machover (2002) have proven that it holds in SVGs with respect to the Shapley Shubik index under a non-atomic and replicative condition. This paper detects this effect in the more general approach of weighted $(j, k)$ games. As a by-product, it provides useful and easy to calculate approximations for $\psi_a(r)$. Further, the paper introduces a numerical method to compute $\psi_a(r)$ values which does not suffer from inapplicability. Due to huge number of possible scenarios - it grows exponentially with the number of voters - checking every possible scenario for possible $r$-criticalness of a voter is an impracticable task with small assemblies already. This paper provides a method in the spririt of the recursion of Mann & Shapley (1962) which does not suffer from these difficulties.

The paper is organized as follows. Section 2 introduces some notation and sets up the probabilistic tools that will serve us in the subsequent sections. Section 3 briefly reintroduces weighted $(j, k)$ games and picks out a subclass for which the PLT will be proven in Section 4. A numerical method for general $(j, k)$ games, which serves to verify the PLT exemplarily, is presented in Section 5.

# 1 Preliminaries

The main tool in this paper is from the field of probability theory and represents a general version of the central limit theorem.

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random variables, at least one of which has a non-degenerate distribution. Let the distribution of $X_i$ be denoted by $F_i$, its expectation by $EX_i = \mu_i$ and assume its variance $VarX_i = \sigma_i^2$ to be finite. Further put

$$s_n := Var \sum_{i \leq n} X_i = \sum_{i \leq n} \sigma_i^2$$

and

$$S_n := \frac{1}{s_n} \sum_{i \leq n} X_i - \mu_i.$$

**Theorem 1: (Lindeberg-Feller)**
In order that

$$\lim_{n \to \infty} \max_{i \leq n} \frac{\sigma_i}{s_n} = 0 \tag{1}$$

and

$$\lim_{n \to \infty} \sup_x |\Pr ob\{\frac{S_n}{s_n} < x\} - \Phi(x)| = 0 \tag{2}$$

it is necessary and sufficient that the following condition (the **Lindeberg condition)** be satisfied:

$$\lim_{n \to \infty} L_n(\varepsilon) = 0 \tag{3}$$

with

$$L_n(\varepsilon) \quad : \quad = s_n^{-1} \sum_{i \le n} E((X_i - \mu_i)^2; |X_i - \mu_i| \ge \varepsilon s_n) \tag{4}$$
$$= \quad s_n^{-1} \sum_{i \le n} \int_{\{|x-\mu_i| \ge \varepsilon \sqrt{s_n}\}} (x - \mu_i)^2 dF_i(x)$$

for every fixed $\varepsilon > 0$.

For a proof see e.g. Petrov (1995), p.123-126.

We put

$$Q^{(n)} := \sum_{i \le n} w_i^2.$$

**Lemma 1**: For each $i$, let the independent random variable $X_i$ be given by

$$X_i = Cw_i,$$

where $C$ is a real-valued random variable with non-degenerate distribution on a compact set $[\alpha, \beta]$. Then $\{X_i\}_{i=1}^{\infty}$ satisfies the Lindeberg condition (3) iff

$$\lim_{n \to \infty} \frac{w_n}{\sqrt{Q^{(n)}}} = 0. \tag{5}$$

**Proof:** For each $i$ follows

$$EX_i = cw_i \tag{6}$$

$$VarX_i = d^2 w_i^2, \tag{7}$$

where $c$ and $d$ are reals independent of $i$, with $d > 0$ (since $C$ has a non-degenerate distribution). Hence

$$s_n = d\sqrt{Q^{(n)}}. \tag{8}$$

Now suppose the Lindeberg condition (3) is satisfied. Then by Theorem 1 we have (1), from which (5) follows at once in view of (7) and (8).

Conversely, suppose that (5) holds. We now show that

$$\lim_{n \to \infty} \max_{i \le n} \frac{w_i}{\sqrt{Q^{(n)}}} = 0. \tag{9}$$

For any $\varepsilon > 0$ fix $n'$ so large that $w_i/\sqrt{Q^{(i)}} < \varepsilon$ for all $i > n'$. Thus, for all $n > n'$ we have

$$\frac{w_i}{\sqrt{Q^{(n)}}} \le \frac{w_i}{\sqrt{Q^{(i)}}} < \varepsilon \qquad \text{for } i = n' + 1, ..., n.$$

Thus (9) holds. Now observe that for every $i$, the integral in (4) follows as

$$\int_{|x-cw_i|>\varepsilon d\sqrt{Q^{(n)}}} (x-cw_i)^2 dF_i(x). \tag{10}$$

But from $|x-cw_i| = |y-c|w_i$ for all $y \in [\alpha, \beta]$ and (9) it follows that, for any given $\varepsilon > 0$, if $n$ is sufficiently large, then

$$|y-c|w_i < \varepsilon d\sqrt{Q^{(n)}}$$

for all $y \in [\alpha, \beta]$ and all $i \leq n$. That implies the integral (10) vanishes for all $i \leq n$. Hence (3) holds.

**Remark 1**: Note that (5) does not apply to oceanic games since it implies

$$\max_{i \leq n} \hat{w}_i = \frac{w_i}{\sum_{i \leq n} w_i} \to 0 \qquad \text{for } n \to \infty. \tag{11}$$

However, condition (5) is stricter than (11).

# 2 Weighted (j,k) games

Let $N = \{1, ..., n\}$ denote the set of voters. According to Freixas and Zwicker[2] (2002) in a *weighted* $(j,k)$ *game* every voter is endowed with a weight vector $(w_i^1, ..., w_i^j)$. The set of options $\{w_i^1, ..., w_i^j\}$ can be interpreted as ranging from complete enthousiasm to total opposition. Further the $k$ different outcome levels are determined by $k-1$ real number quotas $Q_1 \geq Q_2 \geq ... \geq Q_{k-1}$ such that the outcome level is $r$ iff the total weight sum in the ballot box lies in the interval $[Q_r, Q_{r-1})$. Without loss of generality we assume that for every $r \in \{1, ..., k-1\}$ the ratio of the Quota $Q_r$ to the maximum of the weight sum in the ballot box $\sum_{i \leq n} w_i^1$ are values in $(0,1)$ and will operate with $q_r := Q_r / \sum_{i \leq n} w_i^1$.

In the following we will focus on a special class of weighted $(j,k)$ games to which will be refered to as *Multi-Partition Games* (MPG). Let there exist a real-valued vector $(c_1, ..., c_j)$ and $\{w_i\}_{i \in N}$ such that

$$(w_i^1, ..., w_i^j) = (c_1, ..., c_j)w_i. \tag{12}$$

Condition (12) says that we can think of the weight vector of every voter as a product of approval factors equal for every voter and an individual single weight that is characteristic for this voter. This means the voter faces a set $\{c_1, ..., c_j\}$ to weight his or her choice with his or her individual voting weight $w_i$. Hence each voter makes up his or her mind $c_i$ about the proposal and throws the overall product $c_i w_i$ into the ballot box. Although the voter might choose

[2] The following definition paraphrases their work. For their own formulation see Freixas and Zwicker (2002), Definition 2.2.

the highest level of approval $c_1$ the overall effect in the ballot box might be small due to a low $w_i$. In MPGs it makes sense to think of the voting weight $w_i$ as representative for voter $i$.

For any positive integer $n$, we describe a MPG $\Gamma^{(n)}$ by the customary square-bracket notation

$$\Gamma^{(n)} := [qW^{(n)}; w_1, ..., w_n], \qquad q \in (0,1)^{k-1}. \tag{13}$$

The voters are labeled by the integers $1, ..., n$. The sum $W^{(n)}$ denotes the sum of all weights, i.e.

$$W^{(n)} = \sum_{i \leq n} w_i.$$

**Definition 1**: Let

$$N^{(0)} \subsetneq N^{(1)} \subsetneq N^{(2)} \subsetneq ...$$

be an infinite increasing chain of finite non-empty sets. For any two voters $a, b \in N = \cup_{n=0}^{\infty} N^{(n)}$ let their weights be given by the positive real numbers $w_a$ and $w_b$ and let $q$ be a real $\in (0,1)^{k-1}$. Let $\Gamma^{(n)}$ be the MPG whose assembly is $N^{(n)}$ and whose quota is $q$. We shall then say that $\{\Gamma^{(n)}\}_{n=0}^{\infty}$ is a $q$-*chain* of MPGs.

For each voter $i$, we represent the vote of $i$ as a random variable

$$X_i = C w_i \tag{14}$$

where the random variable $C$ takes the values

$$C = \begin{cases} c_1 \\ ... \\ c_j \end{cases}. \tag{15}$$

We do not yet have to specify the distributions of the $X_i$.

Let $Y^{(n)}$ be given by

$$Y^{(n)} = \sum_{i \leq n} X_i. \tag{16}$$

For any voter $a \in \{1, ..., n\}$ we put

$$Y_a^{(n)} = \sum_{i \neq a} X_i. \tag{17}$$

We say that voter $a$ is $r$-*critical* if the voters other than $a$ are so divided that $a$ can tip the balance between an outcome above level $r$ or below level $r$. Hence $a$ is $r$-critical iff the following two inequalities hold

$$\begin{array}{rcl} Y_a^{(n)} + c_1 w_a & \geq & q_r W^{(n)} \\ Y_a^{(n)} + c_j w_a & < & q_r W^{(n)}. \end{array}$$

Hence the probability of $a$ being $r$-critical is given by

$$\phi_a^{(n)}(r) = \text{Prob}\{q_r W^{(n)} - c_1 w_a \leq Y_a^{(n)} < q_r W^{(n)} - c_j w_a\}. \tag{18}$$

**Remark 2**: For $(2,2)$ games (SVGs) equation (18) provides with $c_1 = 1$ and $c_2 = 0$ and $p_1 = p_2 = 1/2$ the Banzhaf (Penrose) measure under the assumption that the voters vote independently. With the latter assumption and $p_1 = t$ and $t$ uniformly distributed over $[0,1]$ we get the Shapley-Shubik index[3].

We say that *Penrose's Limit Theorem (PLT) holds for the q-chain* $\{\Gamma^{(n)}\}_{n=0}^{\infty}$ *if for any* $a, b \in N$ *and* $r \in \{1, ..., k-1\}$

$$\frac{\phi_a^{(n)}(r)}{\phi_b^{(n)}(r)} \to \frac{w_a}{w_b}.$$

# 3 PLT in weighted (j,k) games

In this section we will assume that the $X_i$ in (14) are independent and each of them takes the values $c_1 w_i, ..., c_j w_i$ with probability $p_1, ..., p_j$ where at least two of the $p_l$ are positive and $\sum_{l \leq n} p_l = 1$.

From (14) and (17) follows

$$\mu_a^{(n)} := E(Y_a^{(n)}) = E(C)(W^{(n)} - w_a) \tag{19}$$

and

$$\sigma_a^{(n)} := \sqrt{Var(Y_a^{(n)})} = \sqrt{Var(C)(Q^{(n)} - w_a^2)}. \tag{20}$$

Further we define

$$l_a^{(n)}(r) := \frac{q_r W^{(n)} - w_a - \mu_a^{(n)}}{\sigma_a^{(n)}} \tag{21}$$

and

$$u_a^{(n)}(r) := \frac{q_r W^{(n)} - \mu_a^{(n)}}{\sigma_a^{(n)}}. \tag{22}$$

With this notation Theorem 1 provides

$$\phi_a^{(n)}(r) \approx \tilde{\phi}_a^{(n)}(r) := \frac{1}{\sqrt{2\pi}} \int_{l_a^{(n)}(r)}^{u_a^{(n)}(r)} \exp[-\frac{1}{2}r^2]dr. \tag{23}$$

where $|\phi_a^{(n)}(r) - \tilde{\phi}_a^{(n)}(r)| \to 0$ tends to zero as $n$ goes to infinity.

[3]This result can be found in Straffin (1982).

Now, for two voters $a$ and $b$ it follows for the ratio

$$\frac{\phi_a^{(n)}(r)}{\phi_b^{(n)}(r)} = \frac{\tilde{\phi}_a^{(n)}(r)}{\tilde{\phi}_b^{(n)}(r)} + \varepsilon_{a,b}^{(n)},$$

where $\varepsilon_{a,b}^{(n)}$ is the approximation error.

**Theorem 2**: Let $p_1, ..., p_j$ be given such that at least two of the $p_l$ are positive and $\sum_{l \leq n} p_l = 1$.
If the sequence $\{w_n\}_{n=0}^{\infty}$ satisfies (5) and $\varepsilon_{a,b}^n \to 0$ for any two voters $a, b \in N$ then PLT holds for the corresponding $q$-chain $\{\Gamma^{(n)}\}_{n=0}^{\infty}$ for any $q \in (0,1)^{k-1}$.

**Proof**: We have to show that

$$\frac{\tilde{\phi}_a^{(n)}(r)}{\tilde{\phi}_b^{(n)}(r)} \to \frac{w_a}{w_b}.$$

The mean value theorem says that there exist values $m_a^{(n)} \in [l_a^{(n)}(r), u_a^{(n)}(r)]$ and $m_b^{(n)} \in [l_b^{(n)}(r), u_b^{(n)}(r)]$ such that

$$\frac{\tilde{\phi}_a^{(n)}(r)}{\tilde{\phi}_b^{(n)}(r)} = \frac{u_a^{(n)}(r) - l_a^{(n)}(r)}{u_b^{(n)}(r) - l_b^{(n)}(r)} \exp[-\frac{1}{2}(m_a^{(n)})^2 + \frac{1}{2}(m_b^{(n)})^2].$$

With (21) and (22) follows

$$\frac{\tilde{\phi}_a^{(n)}(r)}{\tilde{\phi}_b^{(n)}(r)} = \frac{w_a/\sigma_a^{(n)}}{w_b/\sigma_b^{(n)}} \exp[-\frac{1}{2}(m_a^{(n)})^2 + \frac{1}{2}(m_b^{(n)})^2].$$

From (20) it is easy to see that

$$\frac{w_a/\sigma_a^{(n)}}{w_b/\sigma_b^{(n)}} \to \frac{w_a}{w_b}.$$

Since the lengths of the intervals $[l_a^{(n)}(r), u_a^{(n)}(r)]$ and $[l_b^{(n)}(r), u_b^{(n)}(r)]$ tend to zero as well as the difference of the upper bounds $u_a^{(n)}(r) - u_b^{(n)}(r)$ it follows that

$$\exp[-\frac{1}{2}(m_a^{(n)})^2 + \frac{1}{2}(m_b^{(n)})^2] \to 1.$$

*q.e.d.*

# 4 Computing Power Indices

The approximation (23) provides a useful rule of thumb for a voter $a$ being $r$-critical. If we want to compute the exact value, however, already a small number of voters provides serious problems with respect to computation time. The reason is due to the fact that if every voter is endowed with $j$ voting weights then there are $j^{n-1}$ possible scenarios to check whether voter $a$ is $r$-critical which implies exponential growth of the computational extent. This section provides a method to evade this impracticability. The main idea is to operate with the coefficients of so called generating functions which represents an extension of the recursion of Mann & Shapley[4] for SVGs.

The following briefly sketches the recursion of Mann & Shapley and extends it to games with several levels of approval afterwards. For notational convenience, the latter will be formulated for the special case of $j = 3$ (which can be interpreted as voting games with abstentions). For general $j$, the method works essentially similar albeit notationally more messy.

Assume in a simple weighted voting game with quota $Q$ and total maximal weight sum $W$. Let $c_{mv}$ be the number of ways in which the $v$ players, other than $a$, can have a sum of votes equal to $m$, where $0 \leq v \leq n-1, 0 \leq m \leq W - w_a$.

Then the classical power measures for a voter $a$ are given by

$$\phi_a = \sum_{v=0}^{n-1} fac(v,n) \sum_{m=Q-w_a}^{Q-1} c_{mv} \tag{24}$$

where the factor $fac(v,n)$ is given by

$$fac(v,n) = \frac{v!(n-1-v)!}{n!} \tag{25}$$

for the Shapley Shubik index and

$$fac(v,n) = 2\hat{}(n-1) \tag{26}$$

for the Banzhaf measure. Due to notational convenience $c_{mv}$ and $fac(v,n)$ are not indexed by $a$.

The crucial problem is of course to compute the $c_{mv}$. Cantor's suggestion was to use the generating function

$$f(x,y) = \prod_{i \neq a} (1 + x^{w_i} y). \tag{27}$$

This is a polynomial in $x$ and $y$ and the coefficient of $x^m y^v$ is precisely $c_{mv}$. The problem reduces then to multiplying out the polynomial and determining the coefficients. If we do it taking one factor at a time, then we get a sequence of coefficients $C^{(i)}$. This leads to the following recursion

[4]The key idea of the recursion Mann&Shapley was due to David Cantor who suggested it to the former following a lecture at Princeton University on October 1960.

$$c_{mv}^{(i)} = c_{mv}^{(i-1)} + c_{m-w_i,v-1}^{(i-1)} \tag{28}$$

where the last term is understood to be 0 if either subscript is negative. The initialization $C^{(0)}$ is 0 except for $c_{00}^{(0)} = 1$.

If player $a$ is left out, then $C^{(n-1)}$ is the matrix with elements equal to $c_{mv}$.

For weighted voting games with $j = 3$ let the weight vector of voter $v$ be given by $(w_v^1, w_v^2, w_v^3)$. Consider the following generating function

$$f(x, y, z, t) = \prod_{v \neq a} (1 + x^{w_v^1} y + z^{w_v^2} t). \tag{29}$$

This is a polynomial in $x, y, z$ and $t$ where a term looks e.g. like

$$x^{w_2^1 + w_5^1 + w_6^1} y^3 z^{w_3^2 + w_4^2} t^2. \tag{30}$$

We interprete the exponent of $x$ as the weight sum of the 'yes' voters, the exponent of $y$ as the number of 'yes' voters, the exponent of $z$ as the weight sum of abstainers and the exponent of $t$ as the number of abstainers. In the above example term (30) this means that the players 2,5 and 6 vote 'yes' and the players 3 and 4 abstain.

Let the coefficient of $x^m y^v z^s t^l$ be given by $c_{mv}^{sl}$.

The recursion to obtain $c_{mv}^{sl}$ is then given by

$$(c_{mv}^{sl})^{(i)} = (c_{mv}^{sl})^{(i-1)} + (c_{m-w_i^1,v-1}^{sl})^{(i-1)} + (c_{mv}^{s-w_i^2,l-1})^{(i-1)} \tag{31}$$

where the terms are understood to be 0 if either subscript is negative. The initialization $C^{(0)}$ is 0 except for $c_{00}^{(0)} = 1$.

For (18) follows

$$\phi_a(r) = \sum_{v,l} fac(v, l, n) \sum_{m=0}^{W} \sum_{s=Q(r)-w_j(a)-m}^{Q(r)-w_1(a)-1-m} c_{mv}^{sl}. \tag{32}$$

The threshold $Q(r)$ (that has to be achieved by the overall sum for the outcome to lie above level $r$) is given by

$$Q(r) = \lceil q_r W \rceil \tag{33}$$

i.e. $Q(r)$ is the smallest integer above $q_r W$.

The factor $fac(v, l, n)$ depends on the specific measure, e.g. applying the Bernoullian principle of insufficient reason provides

$$fac(v, l, n) = 3\hat{}(n-1). \tag{34}$$

Or if the probability to abstain is given by $\alpha$, and to vote 'yes' and 'no' $(1-\alpha)/2$ in each case, then this turns out as

$$fac(v,l,n) = \alpha^l (1-\alpha)^{n-1-l}/4 \tag{35}$$

**Numerical extent:** Since $v, l \in \{0, ..., n-1\}$ and $m, s \in \{0, ..., W\}$ we have to determine $(n-1)^3|W|^2$ many different $c_{mv}^{sl}$ to the maximum for a player $a$ (the number if iterations for each $c_{mv}^{sl}$ is given by $n-1$). Since we have $n$ players, the overall extent is given by

$$n(n-1)^3|W|^2$$

and hence not exponentially growing in $n$.

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