Ordinal equivalence of power indices

Josep Freixas[a] [1]

[a]Department of Applied Mathematics 3
Technical University of Catalonia

This paper will be presented at the Leverhulme Trust sponsored Voting Power in Practice Symposium held at the London School of Economics, 20 to 22 March 2011.

[1] Part of this work has been done jointly with D. Marciniak and M. Pons

- P-power: power as a payoff, a useful tool e.g. to divide a cake
  - Common property: efficiency $\sum_{i\in N} \psi_i = v(N)$
  - Theoretical foundations: axiomatic characterizations, i.e. lists of axioms that *uniquely* characterize the power index.
  - Some examples: Shapley–Shubik, Johnston, Holler, Deegan–Packel power indices, the nucleolus, the least-square core, etc.
- I-power: power as influence in a political committee
  - Theoretical foundations: Probabilistic approaches (mainly using probability distributions p over 2^N. The most relevant case is $p(S) = \frac{1}{2^n}$ for all $S \subseteq N$
  - Some examples: Banzhaf, Coleman to prevent, Coleman to initiate, Rae, König and Bräuninger power indices, etc.

- P-power: power as a payoff, a useful tool e.g. to divide a cake
  - Common property: efficiency $\sum_{i \in N} \psi_i = v(N)$
  - Theoretical foundations: axiomatic characterizations, i.e. lists of axioms that *uniquely* characterize the power index.
  - Some examples: Shapley–Shubik, Johnston, Holler, Deegan–Packel power indices, the nucleolus, the least-square core, etc.
- I-power: power as influence in a political committee
  - Theoretical foundations: Probabilistic approaches (mainly using probability distributions p over 2^N. The most relevant case is $p(S) = \frac{1}{2^n}$ for all $S \subseteq N$
  - Some examples: Banzhaf, Coleman to prevent, Coleman to initiate, Rae, König and Bräuninger power indices, etc.

Comments on power indices:

- When defining power indices we assume that all the information is encapsulated in the game (N, v); by only using n real numbers we intend to measure power. We cannot expect to find an ideal power index. Inverse problem.
- Every power index gives a total ranking of players, so that power for any two arbitrary voters is always comparable.
- Some power indices admit interpretations as both P-power and I-power. E.g. the Shapley–Shubik power index (typically seen as P-power) admits a probabilistic interpretation and the Banzhaf power index (typically seen as I-power) admits several axiomatizations that uniquely define it.

More comments on power indices:

- If we intend to measure influence, as is usual in politics, we will be interested in rankings rather than in “exact” values for power indices.
- Different power indices provide different rankings. However, if we restrict our attention to the main real–world voting systems what happens?

These comments/questions motivate the following study...

1. Power as a payoff and power as influence
2. **Ordinal equivalence of power indices**
3. Power indices respecting the desirability relation
4. Ordinal equivalence of Shapley–Shubik, Banzhaf and Johnston indices

Notion of ordinal equivalence of values

This problem goes back at least to Allingham (1975), who noticed that the Bz and SS are similar power indices in the class of weighted games.

Definition

Two values χ and ξ are ordinally equivalent for the cooperative game (N, v) if and only if the following equivalences hold for all $i \in N$ and $j \in N$:

$$\begin{aligned} \chi_i - \chi_j > 0 \quad &\text{iff} \quad \xi_i - \xi_j > 0 \\ \chi_i - \chi_j = 0 \quad &\text{iff} \quad \xi_i - \xi_j = 0, \end{aligned}$$

Completely ordinal equivalent power indices

- $Col_i^p[\mathcal{W}] \rightarrow \frac{\text{Number of winning coalitions in which } i \text{ is crucial}}{\text{Total number of winning coalitions}}$
- $Col_i^l[\mathcal{W}] \rightarrow \frac{\text{Number of losing coalitions in which } i \text{ is crucial}}{\text{Total number of losing coalitions}}$
- $Rae_i[\mathcal{W}] \rightarrow \frac{|S \subseteq N : i \in S \in \mathcal{W}| + |S \subseteq N : i \notin S \notin \mathcal{W}|}{2^N}$
- $KB_i[\mathcal{W}] \rightarrow \frac{\text{Number of winning coalitions that contain } i}{\text{Total number of winning coalitions}}$

Example

$$Rae_i(\mathcal{W}) = 0.5 + 0.5Bz_i(\mathcal{W})$$

$$\frac{Bz_i(\mathcal{W})}{\sum_{j\in N} Bz_j(\mathcal{W})} = \frac{Col_i^P(\mathcal{W})}{\sum_{j\in N} Col_j^P(\mathcal{W})} = \frac{Col_i^I(\mathcal{W})}{\sum_{j\in N} Col_j^I(\mathcal{W})}$$

Moreover, $c_i = 2w_i - w$ so that Bz and KB rank voters equally.

Hence, these five power indices are ordinally equivalent.

Desirability relation

The desirability relation attempt to formalize the intuitive notion that underlies expressions such as the following:

- "i and j have equal power,"
- "i has at least as much power as j,"
- "i is preferable to j as a coalitional partner."

Definition

Desirability relation. Let $\mathcal{W} \in \mathcal{S}_N$, $i, j \in N$. Define

$$i \succsim_D j \quad \text{iff} \quad S \cup \{j\} \in \mathcal{W} \Rightarrow S \cup \{i\} \in \mathcal{W} \quad \text{for all} \quad S \subseteq N \backslash \{i, j\}.$$

- If $i \succsim_D$ is total then the game is called complete.
- Every weighted game is complete.

Some power indices do not respect the desirability relation

Given a power index ψ and a game $\mathcal{W}$:

$$i \succsim_D j \qquad \text{does not necesarilly imply} \qquad \psi_i \geq \psi_j.$$

Observation

The Deegan-Packel, Holler or Shift power indices fall, among others, within this group.

If we accept the desirability relation as an inescapable property for power indices perhaps we should discard these power indices...

Example

$[4; 2, 2, 1, 1, 1]$ a voter with weight 2 (i) is strictly more desirable than a voter with weight 1 (j), while both Holler and Deegan and Packel reverse this order.

$$i \succ_D j \text{ but } Hol_i < Hol_j \text{ and } DP_i < DP_j$$

Power indices which are not contradictive with the desirability relation

Given a power index ψ and a game $\mathcal{W}$:

$$i \succsim_D j \qquad \text{implies} \qquad \psi_i \geq \psi_j.$$

but

$$i \succ_D j \qquad \text{does not necessarily imply} \qquad \psi_i > \psi_j.$$

Observation

The nucleolus, the least core, non-regular semivalues fall, among others, within this group.

Example

$$\mathcal{W}^m = \{12, 134, 135, 145\}$$

Semivalues...

Semivalues [Weber, 1979] A characterization of semivalues in terms of weighting coefficients, provided by [Dubey *et. al.*, 1981].

$$\psi_i[v] = \sum_{S \subseteq N \setminus \{i\}} p_s[v(S \cup \{i\}) - v(S)]$$

for all $i \in N$ and all $v \in \mathcal{G}_N$, where $s = |S|$ and $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$, ($p_k \geq 0$)

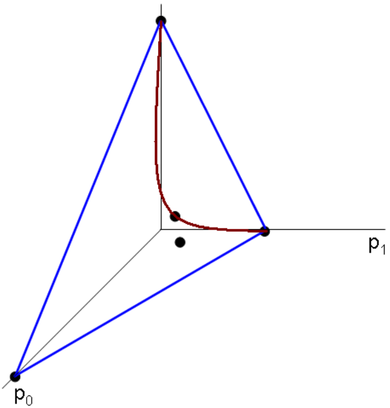
The Shapley value [Shapley, 1953] *SS* is defined by weighting coefficients $p_k = 1/n\binom{n-1}{k}$.

Banzhaf value [Owen, 1975] *Bz* is the only semivalue whose weighting coefficients are constant, i.e., $p_k = 1/2^{n-1}$ for all k.

Regular semivalues [Carreras and F., 1999] semivalue with $p_k > 0$ for all k.

Semivalues, $n = 3$

$$p_0 + 2p_1 + p_2 = 1; \quad p_0 \geq 0, \quad p_1 \geq 0, \quad p_2 \geq 0$$



Observations

- Regular semivalues (Shapley and Banzhaf among others) and Johnston respect the desirability relation.
- All these indices preserve desirability order, so for complete games they rank voters equally.
- All hierarchies are possible for them excepting:

$$= \cdots => > \qquad \text{and} \qquad = \cdots => >>$$

Three classical power indices

Raw Banzhaf: $[\mathcal{C}_i = \{S \subseteq N \ : \ S \in \mathcal{W},\ S \setminus \{i\} \notin \mathcal{W}\}]$

$$c_i[\mathcal{W}] = \sum_{\substack{S\in\mathcal{W} \\ S\setminus\{i\}\notin\mathcal{W}}} 1 = \sum_{S\in\mathcal{C}_i} 1 = |\mathcal{C}_i| = \sum_{k=1}^{n} |\mathcal{C}_i(k)|; \quad Bz_i[\mathcal{W}] = \frac{c_i[\mathcal{W}]}{2^{n-1}}$$

Shapley–Shubik index: $[\mathcal{C}_i(k) = \{S \in \mathcal{C}_i \ : \ |S| = k\}]$

$$SS_i[\mathcal{W}] = \sum_{\substack{S\in\mathcal{W} \\ S\setminus\{i\}\notin\mathcal{W}}} \frac{(k-1)!(n-k)!}{n!} = \sum_{S\in\mathcal{C}_i} \frac{(k-1)!(n-k)!}{n!} = \sum_{k=1}^{n} |\mathcal{C}_i(k)| \frac{1}{\binom{n}{k} k}$$

Raw Johnston:

$$Jn_i[\mathcal{W}] = \sum_{\substack{S\in\mathcal{W} \\ S\setminus\{i\}\notin\mathcal{W}}} \frac{1}{\mathcal{X}(S)} = \sum_{S\in\mathcal{C}_i} \frac{1}{\mathcal{X}(S)} = \sum_{k=1}^{n} |\mathcal{C}'_i(k)| \frac{1}{k}$$

where $\mathcal{X}(S) = |\{j \in S \ : \ S \in \mathcal{C}_j\}|$ and $\mathcal{C}'_i(k) = \{S \in \mathcal{C}_i \ : \ \mathcal{X}(S) = k\}$.

Example

$$\mathcal{W}^m = \{\{1,2\},\{1,3,4\},\{2,3,4\}\} \qquad [4;2,2,1,1]$$

$$\mathcal{C}_1 = \{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$$

$$\mathcal{C}_3 = \{\{1,3,4\},\{2,3,4\}\}$$

marginal contributions

i\k	1	2	3	4
1, 2	0	1	3	0
3, 4	0	0	2	0

Table: $\mathcal{C}_i(k)$ numbers.

hierarchy

coefficients *SS* $\frac{1}{4}$ $\frac{1}{12}$ $\frac{1}{12}$ $\frac{1}{4}$

coefficients *Bz* $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$

ranking $1 \approx_d 2 \succ_d 3 \approx_d 4$

Example

$$\mathcal{W}^m = \{\{1,2\},\{1,3,4\},\{2,3,4\}\} \qquad [4;2,2,1,1]$$

$$\mathcal{C}_1^h = \{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$$

$$\mathcal{C}_3^h = \{\{1,3,4\},\{2,3,4\}\}$$

pure contributions

i\k	1	2	3	4
1, 2	0	3	1	0
3, 4	0	0	2	0

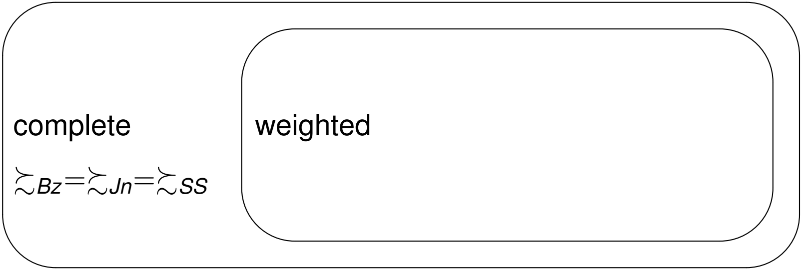
Table: $\mathcal{C}_i'(k)$ numbers.

hierarchy

coefficients Jn $\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$

ranking $1 \approx_d 2 \succ_d 3 \approx_d 4$

Ordinal equivalence for complete games



Shapley–Shubik, Johnston and Banzhaf indices preserve desirability relation.
Are all real–world binary voting systems modeled as complete games? Almost all of them...

- weighted games: $[q_1; w_1, ..., w_n]$,
- voting by count and account $[q_1; w_1, ..., w_n] \cap [q_2; 1, ..., 1]$,
- triple or fourfold... weighted ordered intersections. E.g. the European Economic Union (after 25 and 27 member enlargements)
  $[q_1; w_1, ..., w_n] \cap [q_2; v_1, ..., v_n] \cap [q_3; u_1, ..., u_n]$ (with weights in non–increasing order)

Few exceptions...

- The United States Federal System voting system is not complete. E.g. a senator and a member of the representative chamber are not comparable by $\succsim_D$. However, this system is semicomplete.

Second Relation

Given a simple game $\mathcal{W}$ in N, for each $i \in N$ and $1 \leq k \leq n$,

$$\mathcal{C}_i = \{S \in \mathcal{W} : S \setminus \{i\} \notin \mathcal{W}\} \quad \text{and} \quad \mathcal{C}_i(k) = \{S \in \mathcal{C}_i : |S| = k\}.$$

$\mathcal{C}_i$ is the set of winning coalitions S for which i is crucial, while $\mathcal{C}_i(k)$ is the subset of such coalitions having cardinality k.

Definition

The weak desirability relation. Let $\mathcal{W}$ be a simple game in N and consider

$$i \succsim_d j \quad \text{iff} \quad |C_i(k)| \geq |C_j(k)| \quad \text{for each} \quad k = 1, 2, \ldots, n.$$

A simple game $\mathcal{W}$ on a finite set of players N is weakly complete whenever $\succsim_d$, the weak desirability relation, is total.

An example of a weakly complete game but not complete

Example

$$\mathcal{W}^m = \{\{1,2,3\},\{1,4,5\}\}$$

i\k	1	2	3	4	5	sum
1	0	0	2	4	1	7
2, 3	0	0	1	2	0	3
4, 5	0	0	1	2	0	3
weights *SS*	$\frac{1}{5}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{5}$	
weights *Bz*	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	

Table: $\mathcal{C}_i(k)$ numbers.

$$1 \succ_d 2 \approx_d 3 \approx_d 4 \approx_d 5$$

Properties:

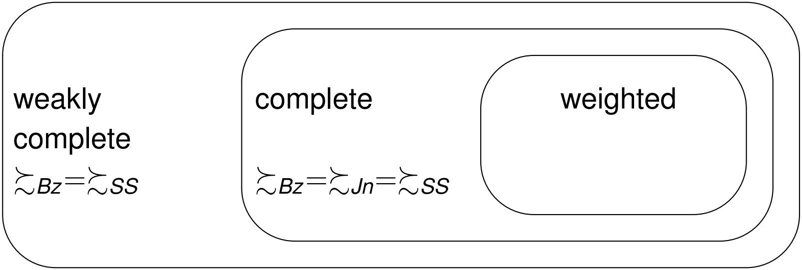
$\succsim_D$ is a subpreordering of $\succsim_d$. Hence, $\mathcal{W}$ complete implies $\mathcal{W}$ weakly complete.

Power indices: The Shapley–Shubik and Banzhaf power indices preserve the weak desirability relation:

$$\begin{array}{ccccc} i \succ_d j & \Leftrightarrow & SS_i > SS_j & \Leftrightarrow & Bz_i > Bz_j \\ i \approx_d j & \Leftrightarrow & Bz_i = Bz_j & \Leftrightarrow & Bz_i = Bz_j \end{array}$$

but not necessarily is fulfilled for the Johnston index. However, the Johnston and Banzhaf indices preserve another relation for which $\succsim_D$ is a sub–preordering.

Ordinal equivalence for weakly complete games



Third Relation

For each $i \in N$ and $1 \leq h \leq n$, $\mathcal{C}_i^h$ denotes the set formed by all coalitions in $\mathcal{C}_i$ having exactly h crucial players (player i and exactly $h-1$ additional players).

Definition

The feeble desirability relation. Let $\mathcal{W}$ be a simple game in N and define

$$i \succsim_{d'} j \quad \text{iff} \quad \sum_{h=1}^{k} |\mathcal{C}_i^h| \geq \sum_{h=1}^{k} |\mathcal{C}_j^h| \quad \text{for all} \quad k = 1, 2, \ldots, n.$$

A simple game $\mathcal{W}$ on N is feebly complete whenever $\succsim_{d'}$ on N, is total.

Neither $\succsim_d$ is a sub–preordering of $\succsim_{d'}$ nor $\succsim_{d'}$ is a sub–preordering of $\succsim_d$.

Fourth Relation

Definition

The moderate desirability relation. Let $\mathcal{W}$ on N and define:

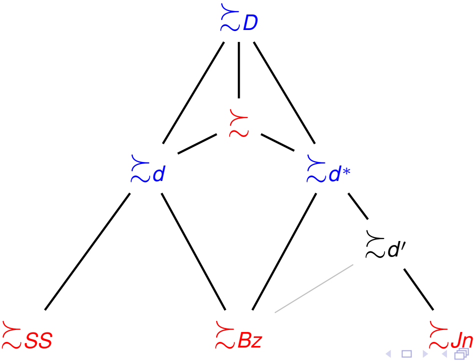
$$\begin{array}{lcll} i \succ_{d^*} j & \text{iff} & i \succsim_{d'} j \quad \text{and} & |\mathcal{C}_i| > |\mathcal{C}_j| \\ i \approx_{d^*} j & \text{iff} & i \approx_{d'} j & \end{array}$$

A simple game $\mathcal{W}$ on N is moderately complete whenever $\succsim_{d^*}$, the moderate desirability relation on N, is total.

Taking into account that $|\mathcal{C}_i| = \sum_{h=1}^{n} |\mathcal{C}_i^h|$, the moderate desirability relation $\succsim_{d^*}$ is a sub–preordering of the feebly desirability relation $\succsim_{d'}$. The converse is not true.

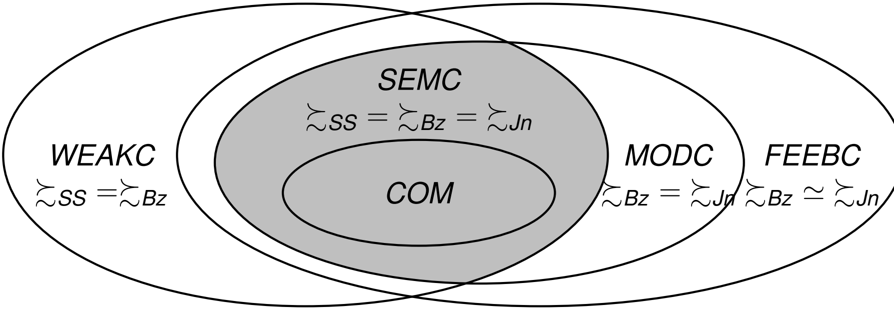
Chain of Preorderings

We show the inter–relations among the preorderings considered till now. These inter–relations can be visualized in the following picture, in which black lines indicate sub–preorderings and the grey line indicates almost–sub–preordering.



SS, Bz and Jn ordinal equivalence

Within semicomplete games we have the ordinal equivalence of Shapley and Shubik, Banzhaf and Johnston indices.



The OE of SS and Bz holds even outside weakly complete games

model	coalitions	number of coalitions
(3,0,0)	$\{1,2,3\}$	1
(0,2,2)	$\{4,5,6,7\}, \ldots, \{4,5,9,10\}$	10
(2,0,3)	$\{1,2,6,7,8\}, \ldots, \{2,3,8,9,10\}$	30
(2,1,2)	$\{1,2,4,6,7\}, \ldots, \{2,3,5,9,10\}$	60
(1,0,5)	$\{1,6,7,8,9,10\}, \ldots, \{3,6,7,8,9,10\}$	3

Table: List of the 5 models representing the 104 minimal winning coalitions.

i\k	0	1	2	3	4	5	6	7	8	9	sum
$1,2,3$	0	0	1	7	81	56	22	0	0	0	167
$4,5$	0	0	0	10	70	35	16	0	0	0	131
$6,7,8,9,10$	0	0	0	6	54	15	6	0	0	0	81

Table: $\mathcal{C}_i(k)$ numbers.

Conclusion:

- We have identified a large class of games for which the Banzhaf, Shapley–Shubik and Johnston power indices are ordinally equivalent and all the hierarchies for simple games are achievable within this class. Moreover, this class contains a lot of real–life binary voting systems.
- We have identified a large class of games for which the Banzhaf and Shapley–Shubik power indices are ordinally equivalent.
- From a qualitative (but not quantitative) point of view the Banzhaf and Shapley–Shubik indices are almost indistinguishable. This (partially) closes the importance of choosing one or the other measure if the purpose is to evaluate power as influence.
- Extensions to cooperative games are not difficult.

Thanks for your attention