

Computing Power Indices for Large Voting Games

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Voting power indices enable the analysis of the distribution of power in a legislature or voting body in which different members have different numbers of votes. Although this approach to the measurement of power has been known for a long time, its application to large games has been limited by the difficulty of computing these indices. This paper presents a new method for computing power indices that combines exact methods with an approximate method due to Owen. This method is of most utility in situations where the number of players is large and the voting weights are concentrated in the hands of a small number of members.

(Voting; Weighted Voting; Power Index; Weighted Majority Game; Empirical Game Theory)

1. Introduction

Many organizations have voting systems that are designed to give different amounts of influence to different members. For example, joint stock companies give each of the shareholders votes in proportion to their ownership of stock. Similarly, many international economic organizations (including the International Monetary Fund (IMF) and World Bank) give each member country a number of votes based on its financial contribution. Many political bodies (including the European Union Council of Ministers and the U.S. Presidential Electoral College) assign votes based on the populations of the represented countries or states.

In these kinds of voting systems, the distribution of actual voting power is often quite different from the nominal distribution of voting weights. For example, consider the original Council of the European Economic Community. Between 1958 and 1972, it had six member countries and used a system of qualified majority voting that allocated four votes each to France, West Germany, and Italy; two votes each to Belgium and The Netherlands; and one vote to Luxembourg. From these figures, one might think that the smaller countries had a disproportionately large

amount of power. For example, Luxembourg had one vote for 310,000 people, while West Germany had one vote for every 13,572,500. In fact, because the number of votes required for a decision was fixed at 12, it was impossible for Luxembourg's one vote to ever change the outcome of a vote.

Power indices have been developed to measure the ability of members to influence voting outcomes. The classic examples of power indices are those proposed by Shapley and Shubik (1954) and by Banzhaf (1965), both of which have been widely applied. Both indices measure how likely it is that a member can add his votes to those of a losing coalition so that it wins, but they differ in the way that such coalitions are counted. These power indices and others are surveyed by Straffin (1994), Felsenthal and Machover (1998), Lucas (1983), and Leech (2002a). These indices have recently received significant attention with the examination of voting structures of the European Union Council of Ministers and the IMF (see Felsenthal and Machover 2001; Leech 2002b, c). Applications to shareholder voting in British companies are described in Leech (1988, 2001).

This paper is concerned with the computation of the Shapley-Shubik and Banzhaf power indices when

there are a large but finite number of players. Large voting games are common. For example, the IMF had 182 members in 2002 and a typical publicly traded company has many thousands of shareholders. In these large voting games, power indices typically cannot be exactly computed and one must resort to simulation methods or the approximate methods like those of Owen (1972, 1975). However, the simulation methods may require a great deal of computation and the approximate methods may lead to large errors, particularly, when voting weights are concentrated in the hands of a few players (see §6 or Widgren 2000 for examples). This paper proposes a new method that combines direct methods with Owen's approximation, and allows one to control the trade-off between accuracy and computational effort. This method has been used to compute power indices for large voting bodies in Leech (2001, 2002a, b), including some with more than 400 players.

The Shapley-Shubik and Banzhaf power indices are defined in §2. Section 3 reviews methods for computing power indices and their limitations. Section 4 describes the approximate methods for large games due to Owen. Section 5 describes the new method. Two numerical examples are presented in §6 and §7 concludes.

2. Power Indices: Notation and Definitions

Consider a weighted voting game with n players represented by a set $N = \{1, \dots, n\}$ whose voting weights are w_1, \dots, w_n . The players are ordered by their weight representing their respective number of votes, so that $w_i \geq w_{i+1}$ for all i . Coalitions are subsets of players $T \subseteq N$ and the combined voting weight of such a coalition is $w(T) = \sum_{i \in T} w_i$. The decision rule for the voting game is defined in terms of a quota, q , such that a coalition T voting for a proposal wins if $w(T) \geq q$ and loses if $w(T) < q$. It is customary to impose the restriction $q > w(N)/2$ to ensure a unique decision in each vote.

A power index measures the ability of a player to determine the outcome of a vote, and is defined in terms of the number of times that a player can "swing" the decision by transferring his vote to a

coalition that is losing without him but wins with him. Formally, a swing for player i is defined as a pair of subsets $(T_i, T_i + \{i\})$ such that T_i is losing but $T_i + \{i\}$ is winning. In terms of voting weight, T_i is a swing if $q - w_i \leq w(T_i) < q$.

The power indices measure the likelihood that a player will swing the vote in a model in which, in some sense, coalitions are randomly formed. The Shapley-Shubik index is the probability that i swings (or is "pivotal" in the terminology of Shapley and Shubik) if all orderings of players are equally likely. Imagine listing all $n!$ possible orderings of N players and proceeding through one of these orderings sequentially, assuming that each player, in turn, votes in favor of the proposal. Player i is "pivotal" if, in this sequence, he casts the vote that puts the vote total at or over the required quota. The set of players preceding i in this sequence defines a set T_i , where i can cast a swing vote. Each such set T_i would appear $t!(n-t-1)!$ times in the list of all orderings, where t is the number of members of T_i and n is the number of players; $t!$ is the number of orderings of T_i and $(n-t-1)!$ is the number of orderings of $N - T_i - \{i\}$, the complement of T_i , excluding player i . The Shapley-Shubik index ϕ_i weights these swing sets T_i by the number of ways this ordering can appear, dividing by the total number of possible orderings

$$\phi_i = \sum_{T_i} \frac{t!(n-t-1)!}{n!} \quad (1)$$

If all orderings are equiprobable, ϕ_i is the probability that i casts the winning vote. Note that these probabilities must sum to one, as somebody must cast a winning vote in each ordering.

In contrast to the Shapley-Shubik index, the Banzhaf indices treat all coalitions as equally likely, players being arranged in no particular order. The *nonnormalized Banzhaf index* (or *Banzhaf swing probability*) for player i is the number of swings for that player divided by the total number of coalitions of other players. The number of swings is $\sum_{T_i} 1$ and the total number of coalitions, which do not include i is 2^{n-1} . The Banzhaf index is, thus,

$$\beta'_i = \frac{\sum_{T_i} 1}{2^{n-1}} \quad (2)$$

and can be interpreted as the probability that player i casts a swing vote, assuming all players are equally likely to vote for or against a proposal. These probabilities need not sum to one as there may be scenarios where no one player could change the decision as well as others, where multiple players could do so. The *normalized Banzhaf index* β_i normalizes these indices so they total one:

$$\beta_i = \beta'_i / \sum_i \beta'_i. \quad (3)$$

This can be interpreted as a relative measure of the players' voting powers.

3. Computing Power Indices

Several methods are available for computing the Shapley-Shubik indices, with simple modifications for the Banzhaf indices. The simplest approach is direct enumeration, which consists of applying the basic definitions of the indices by searching over all possible coalitions and counting swings using Equations (1) and (2). This method is straightforward but time consuming. Because the number of subsets of set N is 2^n , evaluating the Shapley-Shubik and nonnormalized Banzhaf indices for a player by using this method requires a computational effort on the order 2^n . Calculating the indices for all players has complexity of order $n2^n$. This method takes 5 seconds to compute the indices for games with 22 players and 85 seconds for games with 26 players.¹ The computation time approximately doubles with each additional player and, thus, using this system, it would require about 90 years to calculate indices for a game like the electoral college that involves 51 players.

Mann and Shapley (1960) proposed using Monte Carlo simulation methods for computing Shapley-Shubik indices and applied this method to analyze the electoral college. In this approach, one randomly generates voting outcomes and estimates the indices (as probabilities) by counting the number of times

the players swing or are pivotal. This method can be applied to games of any size, but, like all simulation methods, the results are subject to sample error and it may require many samples to determine the indices with the desired level of accuracy.

Mann and Shapley (1962) proposed an alternative exact method of calculating Shapley-Shubik indices that uses generating functions. This method requires computational time on the order of Cn^2 , where C is the number of possible vote totals (Bilbao et al. 2000). For games where every player has the same voting weight, there are $n + 1$ possible vote totals. For weighted voting games, where each player has a unique weight, the number of possible vote totals may be 2^n . This method may be practical for medium-sized games in which many players have similar voting weights. Mann and Shapley (1962) used this method to compute Shapley-Shubik indices for the electoral college. The method was extended to the Banzhaf indices by Brams and Affuso (1976) and is discussed in Lucas (1983), Lambert (1988), Bilbao et al. (2000), and Leech (2002d). The method has not yet been applied to games larger than the electoral college.

Owen (1972, 1975) describes methods for computing the power indices exactly based on the multilinear extension (MLE) of a game. These exact calculations have the same complexity issues as direct enumeration and, hence, can be used only for small games. The method does, however, provide the basis of an approximation method that has been successfully used with large games. The method proposed in this paper is based on Owen's approximation, which is described in detail in the next section.

4. Owen's Approximate Method

Expression (1) for the Shapley-Shubik index can be rewritten by noting that the term inside the summation is a beta function:

$$B(t+1, n-t) = \frac{t!(n-t-1)!}{n!} = \int_0^1 x^t (1-x)^{n-t-1} dx. \quad (4)$$

The integrand on the right-hand side of (4), $x^t (1-x)^{n-t-1}$, can be interpreted as the probability that the coalition T_i votes for the proposal and $N - T_i - \{i\}$

¹ These calculations were done using the author's FORTRAN program *ipexact* running on a 400 MHz Sparc II processor under the Solaris 7 operating system. This program, and all other programs mentioned in this paper, are available from the author's website: www.warwick.ac.uk/~ecaae/.

votes against the proposal, where x is the probability of any player voting for a proposal. Summing this expression across all swings gives the probability of a swing for i . Let us call this probability $f_i(x)$:

$$f_i(x) = \sum_{T_i} x^t (1-x)^{n-t-1}. \quad (5)$$

Integrating x out of (5) gives the Shapley-Shubik index, because substituting (4) into (1) gives

$$\begin{aligned} \phi_i &= \sum_{T_i} \int_0^1 x^t (1-x)^{n-t-1} dx \\ &= \int_0^1 \left[\sum_{T_i} x^t (1-x)^{n-t-1} \right] dx = \int_0^1 f_i(x) dx. \end{aligned} \quad (6)$$

Thus, we can evaluate the Shapley-Shubik index by integrating $f_i(x)$. The Banzhaf index is given by $f_i(0.5)$ because (5) reduces to the definition of the Banzhaf swing probability in this case.

Assuming each player votes the same way as i with probability x , independently of the others, we can define a random variable $v_i(x)$ that counts the number of votes cast by others on the same side as i . Its first two moments are

$$\mu_i(x) = E(v_i(x)) = xw(N - \{i\}) = xw(N) - xw_i$$

and

$$\begin{aligned} \sigma_i(x)^2 &= \text{Var}(v_i(x)) = x(1-x)h(N - \{i\}) \\ &= x(1-x)h(N) - x(1-x)w_i^2, \end{aligned}$$

where $h(T) = \sum_{i \in T} w_i^2$ is the sum of squared weights.

In large games with many small weights and no large weights, $v_i(x)$ will be approximately normally distributed and the desired swing probability

$$f_i(x) = \Pr[q - w_i \leq v_i(x) < q] \quad (7)$$

can be obtained approximately using the normal distribution function, $\Phi(\cdot)$ by evaluating the expression

$$f_i(x) \approx \Phi\left(\frac{q - \mu_i(x)}{\sigma_i(x)}\right) - \Phi\left(\frac{q - \mu_i(x) - w_i}{\sigma_i(x)}\right). \quad (8)$$

The Banzhaf index is approximated by setting $x = 0.5$ in (8) and the Shapley-Shubik index is approximated by numerically integrating out x in (8).

The complexity of this method is linear in the number of players. The calculations for the Shapley-Shubik and nonnormalized Banzhaf indices for a player depend on the number of players only through the statistics $w(N)$ and $h(N)$, which need only be calculated once because they are common to all players. The normalized Banzhaf indices require all indices to be simultaneously found to determine the normalizing constant. This simultaneous calculation still has linear complexity because the statistics $w(N)$ and $h(N)$ are common for all players.

5. A Combined Method for Large Games

For games where n is too large for the direct methods to be feasible and with votes concentrated in the hands of a few, we can combine the essential features of both approaches. In this approach, the players are divided into two subsets: (1) Major players with the largest weights $M = \{1, 2, \dots, m\}$, and (2) minor players $N - M$. This combined method treats the major players using enumeration as in the direct approach, but treats minor players using Owen's approximate technique. Larger values of m will improve accuracy but increase computation time.

Consider the computation of the indices for a major player, $i \in M$. To determine the swing probability, we will search across all subsets S of M that do not include player i . As before, let $f_i(x)$ be the swing probability given a probability x that each member votes for the proposal. This probability can be written as

$$f_i(x) = \Pr(\text{swing for } i) = \sum_{S \subseteq M - \{i\}} p(S, x) g_i(S, x),$$

where $p(S, x) = x^s (1-x)^{m-s-1}$ is the probability of subset S of $M - \{i\}$ voting for the proposal (s is the number of elements in S) and $g_i(S, x)$ is the probability of a swing for player i given that set S has voted for the proposal. To find $g_i(S, x)$, define the random variable $v_i(x) = \sum_{j \in N-M} v_j$, where v_j represents the random number of votes cast by the minor players on the same side as i . The mean and variance of v_i are

$$E(v_i(x)) = xw(N - M) = \mu_i(x)$$

and

$$\text{Var}(v_i(x)) = x(1-x)h(N-M) = \sigma_i(x)^2.$$

Using these moments and the normal approximation to the distribution of $v_i(x)$, we can approximate the required probability as before:

$$\begin{aligned} g_i(S, x) &= \Pr[q - w(S) - w_i \leq v_i(x) < q - w(S)] \\ &= \Phi\left(\frac{q - w(S) - \mu_i(x)}{\sigma_i(x)}\right) \\ &\quad - \Phi\left(\frac{q - w(S) - w_i - \mu_i(x)}{\sigma_i(x)}\right). \end{aligned} \quad (9)$$

Therefore, we can write

$$f_i(x) = \sum_{S \subseteq M - \{i\}} x^s(1-x)^{m-s-1} g_i(S, x). \quad (10)$$

The approximate Shapley-Shubik index is then

$$\begin{aligned} \phi_i &= \int_0^1 f_i(x) dx = \int_0^1 \left[\sum_{S \subseteq M - \{i\}} x^s(1-x)^{m-s-1} g_i(S, x) \right] dx \\ &= \sum_{S \subseteq M - \{i\}} \int_0^1 x^s(1-x)^{m-s-1} g_i(S, x) dx, \end{aligned} \quad (11)$$

which can be found by searching across all subsets of $M - \{i\}$, integrating out x by numerical integration for each subset then summing. The Banzhaf index β'_i is obtained from (10) by setting $x = 0.5$ instead of integrating it out, then summing to give $\beta'_i = f_i(0.5)$.

To calculate the indices for a minor player rather than a major player, we must consider all subsets of M (rather than just the subsets of $M - \{i\}$), and we must not include the contribution of the minor player i in the means and variances used in (9), using $\mu_i(x) = xw(N - M - \{i\})$ and $\sigma_i(x)^2 = x(1-x)h(N - M - \{i\})$ when calculating $g_i(S, x)$.

Examining these calculations, it is clear that one can compute indices for both major and minor players in a single search across the subsets of M . For each player i , the calculations in (9) must be done for each subset S of M , resulting in a complexity of the order 2^m ; the means and variances required for (9) do not depend on S and can be calculated only once. To calculate indices for all players, one only needs to search across subsets once, but must calculate indices for each player for each subset, resulting in a complexity on the order of $n2^m$.

6. Two Examples

This section describes the results of using the combined method in two examples. The first is an analysis of the electoral college and the second is the IMF Board of Governors.²

The U.S. Presidential Electoral College. Table 1 shows the votes in the electoral college for each of its 51 members (50 states and the District of Columbia) based on the 1990 Census data and used in the 1992, 1996, and 2000 presidential elections. The weights vary from California's 54 votes and New York's 33 down to eight members with 3 votes each. The total voting weight is 538 and the quota is 270. In this example, all participants have integer weights and exact indices can be computed by the method of generating functions; these values (obtained using programs written by the author, *ipgenf* and *ssgenf*) are shown in the table. Table 1 also shows relative approximation errors for different values of m between $m = 0$ and $m = 20$. Here, we see that the errors from the Owen MLE approximation method ($m = 0$) are substantial. In all cases, the errors are reduced as m increases and, when $m \geq 10$, the relative errors are negligible.

To get a similar level of accuracy with Monte Carlo simulation, we would need a large number of trials. For example, with a true Shapley-Shubik index of $\phi = 0.02$, a 95% confidence interval would be approximately $\pm 2(\phi(1-\phi)/k)^{0.5}$ in width, where k is the number of trials. To achieve a relative accuracy of $\pm 1\%$ (or absolute accuracy of $\pm 0.01 \times 0.02$) with this ϕ and confidence level, it would require consideration of approximately 1,962,000 trial scenarios. In contrast, the combined method with $m = 10$ requires considering a total of $2^{10} = 1,024$ scenarios. With $m = 5$, only $2^5 = 32$ scenarios are considered. Using the author's program *mmle* on the system described in Footnote 1, these computations required less than 1 second for $m = 0$ and 5, 4 seconds for $m = 10$, 15 seconds for $m = 95$, and 3,048 seconds for $m = 20$.

² The implementation here uses a subroutine that finds every subset of a set exactly once, given in Nijenhuis and Wilf (1983), a numerical integration routine due to Patterson (1968) for the Shapley-Shubik index and double precision arithmetic.

Table 1 The Approximation Errors for the Electoral College

States	Shapley-Shubik index										Normalized Banzhaf index										Nonnormalized Banzhaf index									
	Vote	Exact value	Relative error (%)					Exact value	Relative error (%)					Exact value	Relative error (%)					Exact value	Relative error (%)									
			m = 0	m = 5	m = 10	m = 15	m = 20		m = 0	m = 5	m = 10	m = 15	m = 20		m = 0	m = 5	m = 10	m = 15	m = 20											
CA	54	0.10812541	0.24	0.08	0.03	0.02	0.01	0.11142757	-2.10	0.14	0.03	0.01	0.00	0.46645413	1.42	0.26	0.05	0.02	0.00	0.46645413	1.42	0.26	0.05	0.02	-0.00					
NY	33	0.06296833	0.41	0.00	0.00	0.00	0.00	0.06201049	0.56	-0.02	-0.00	-0.00	-0.00	0.25958612	4.19	0.10	0.02	0.00	-0.00	0.25958612	4.19	0.10	0.02	0.00	-0.00					
TX	32	0.06093272	0.41	0.00	0.00	0.00	0.00	0.06001062	0.54	-0.01	-0.00	-0.00	-0.00	0.25121433	4.16	0.11	0.02	0.00	-0.00	0.25121433	4.16	0.11	0.02	0.00	-0.00					
FL	25	0.04693093	0.41	0.00	0.00	0.00	0.00	0.04631072	0.40	-0.01	-0.00	-0.00	-0.00	0.1938643	4.02	0.11	0.02	0.00	-0.00	0.1938643	4.02	0.11	0.02	0.00	-0.00					
PA	23	0.04300633	0.41	0.00	0.00	0.00	0.00	0.04248381	0.37	-0.01	-0.00	-0.00	-0.00	0.1778442	3.98	0.11	0.02	0.00	-0.00	0.1778442	3.98	0.11	0.02	0.00	-0.00					
IL	22	0.04105613	0.41	0.00	0.00	0.00	0.00	0.04058256	0.35	-0.05	-0.00	-0.00	-0.00	0.16988528	3.96	0.08	0.02	0.00	-0.00	0.16988528	3.96	0.08	0.02	0.00	-0.00					
OH	21	0.03911385	0.41	0.00	0.00	0.00	0.00	0.03686889	0.33	-0.04	-0.00	-0.00	-0.00	0.16195807	3.94	0.08	0.02	0.00	-0.00	0.16195807	3.94	0.08	0.02	0.00	-0.00					
MI	18	0.0333336	0.41	0.00	0.00	0.00	0.00	0.03304912	0.28	-0.03	-0.00	-0.00	-0.00	0.13834904	3.89	0.09	0.02	0.00	-0.00	0.13834904	3.89	0.09	0.02	0.00	-0.00					
NJ	15	0.02762124	0.40	0.00	0.00	0.00	0.00	0.02746282	0.24	-0.02	-0.00	-0.00	-0.00	0.1149639	3.84	0.10	0.02	0.00	-0.00	0.1149639	3.84	0.10	0.02	0.00	-0.00					
NC	14	0.02573179	0.40	0.00	0.00	0.00	0.00	0.02561084	0.22	-0.02	-0.00	-0.00	-0.00	0.1072112	3.83	0.10	0.02	0.00	-0.00	0.1072112	3.83	0.10	0.02	0.00	-0.00					
GA, VA	13	0.02384954	0.40	0.00	0.00	0.00	0.00	0.02376332	0.21	-0.02	-0.01	-0.00	-0.00	0.09947716	3.82	0.11	0.01	0.00	-0.00	0.09947716	3.82	0.11	0.01	0.00	-0.00					
IN, MA	12	0.0219744	0.40	0.00	0.00	0.00	0.00	0.0219199	0.20	-0.02	-0.01	-0.00	-0.00	0.09176031	3.81	0.11	0.02	0.00	-0.00	0.09176031	3.81	0.11	0.02	0.00	-0.00					
MO, TN, WA, WI	11	0.0201063	0.40	0.00	0.00	0.00	0.00	0.02008025	0.19	-0.01	-0.01	-0.00	-0.00	0.08405922	3.79	0.11	0.02	0.00	-0.00	0.08405922	3.79	0.11	0.02	0.00	-0.00					
MD, MN	10	0.01824514	0.39	0.00	0.00	0.00	0.00	0.01824402	0.18	-0.01	-0.00	-0.00	-0.00	0.07637249	3.78	0.11	0.02	0.00	-0.00	0.07637249	3.78	0.11	0.02	0.00	-0.00					
AL, LA	9	0.01639087	0.39	0.00	0.00	0.00	0.00	0.0164109	0.17	-0.01	-0.00	-0.00	-0.00	0.06869873	3.77	0.11	0.02	0.00	-0.00	0.06869873	3.77	0.11	0.02	0.00	-0.00					
AZ, CO, CT, KY, OK, SC	8	0.01454339	0.39	0.00	0.00	0.00	0.00	0.01458055	0.16	-0.01	-0.00	-0.00	-0.00	0.06103658	3.76	0.11	0.02	0.00	-0.00	0.06103658	3.76	0.11	0.02	0.00	-0.00					
IA, MS, OR	7	0.01270265	0.39	0.00	0.00	0.00	0.00	0.01275266	0.15	-0.01	-0.00	-0.00	-0.00	0.05338472	3.76	0.11	0.02	0.00	-0.00	0.05338472	3.76	0.11	0.02	0.00	-0.00					
AR, KS	6	0.01086856	0.39	0.00	0.00	0.00	0.00	0.0109269	0.14	-0.01	-0.00	-0.00	-0.00	0.04574181	3.75	0.11	0.02	0.00	-0.00	0.04574181	3.75	0.11	0.02	0.00	-0.00					
NE, NM, UT, WV	5	0.00904106	0.38	0.00	0.00	0.00	0.00	0.00910297	0.14	-0.01	-0.00	-0.00	-0.00	0.03810655	3.74	0.11	0.02	0.00	-0.00	0.03810655	3.74	0.11	0.02	0.00	-0.00					
HI, ID, ME, NH, NV, RI	4	0.00722008	0.38	0.00	0.00	0.00	0.00	0.00728056	0.13	-0.01	-0.00	-0.00	-0.00	0.03047764	3.74	0.11	0.02	0.00	-0.00	0.03047764	3.74	0.11	0.02	0.00	-0.00					
AK, DC, DE, MT, ND, SD, VT, WY	3	0.00540555	0.38	0.00	0.00	0.00	0.00	0.00545937	0.13	-0.01	-0.00	-0.00	-0.00	0.0228538	3.74	0.11	0.02	0.00	-0.00	0.0228538	3.74	0.11	0.02	0.00	-0.00					
Sum		0.999999996						1.000000000						4.18616453						4.18616453										

Note. All power indices have been calculated to an accuracy of eight decimal places. Relative errors greater than 0.0001 are shown. The exact indices have been computed by the method of generating functions.

Table 2 Voting Power in the IMF

Country	$m =$	0	15	0	1	2	3	4	5	10
	Weight %	Shapley-Shubik index		Relative error %						
$i = 1$ USA	17.55	0.2067	0.2061	0.30	0.30	0.22	0.13	0.09	0.05	0.01
$i = 2$ Japan	6.30	0.0640	0.0628	1.89	0.06	0.06	0.04	0.03	0.02	0.00
$i = 3$ Germany	6.15	0.0624	0.0613	1.88	0.06	0.04	0.04	0.03	0.02	0.00
$i = 10$ The Netherlands	2.45	0.0239	0.0235	1.65	0.08	0.06	0.04	0.03	0.01	0.00
$i = 50$ Greece	0.40	0.0038	0.0038	1.55	0.09	0.06	0.04	0.03	0.02	0.00
$i = 100$ Lithuania	0.08	0.0008	0.0008	1.53	0.09	0.06	0.04	0.03	0.02	0.00
$i = 178$ Marshall Islands	0.01	0.0001	0.0001	1.52	0.09	0.07	0.04	0.03	0.02	0.00
	Weight %	Nonnormalized Bz index		Relative error %						
$i = 1$ USA	17.55	0.7706	0.7636	0.92	0.92	0.63	0.34	0.21	0.09	0.01
$i = 2$ Japan	6.30	0.2258	0.1672	35.11	-3.47	-3.47	-1.62	-0.95	-0.36	-0.04
$i = 3$ Germany	6.15	0.2204	0.1638	34.55	-3.57	-1.55	-1.55	-0.91	-0.35	-0.04
$i = 10$ The Netherlands	2.45	0.0859	0.0670	28.18	-3.47	-2.02	-0.94	-0.59	-0.19	-0.02
$i = 50$ Greece	0.40	0.0140	0.0110	27.40	-3.29	-1.93	-0.91	-0.57	-0.20	-0.02
$i = 100$ Lithuania	0.08	0.0028	0.0022	27.38	-3.29	-1.93	-0.90	-0.57	-0.20	-0.02
$i = 178$ Marshall Islands	0.01	0.0005	0.0004	27.38	-3.29	-1.93	-0.90	-0.57	-0.20	-0.02
	Weight %	Normalized Bz index		Relative error %						
$i = 1$ USA	17.55	0.2098	0.2538	-17.34	3.34	2.04	1.00	0.62	0.24	0.03
$i = 2$ Japan	6.30	0.0615	0.0556	10.66	-1.16	-2.12	-0.97	-0.54	-0.22	-0.02
$i = 3$ Germany	6.15	0.0600	0.0545	10.20	-1.26	-0.18	-0.91	-0.50	-0.20	-0.02
$i = 10$ The Netherlands	2.45	0.0234	0.0223	4.98	-1.15	-0.65	-0.29	-0.18	-0.04	-0.01
$i = 50$ Greece	0.40	0.0038	0.0037	4.34	-0.97	-0.56	-0.26	-0.16	-0.05	-0.01
$i = 100$ Lithuania	0.08	0.0008	0.0007	4.33	-0.97	-0.55	-0.25	-0.16	-0.05	-0.01
$i = 178$ Marshall Islands	0.01	0.0001	0.0001	4.32	-0.96	-0.56	-0.25	-0.16	-0.05	-0.01

Note. These results are for the 1999 IMF with $n = 178$.

The IMF Board of Governors. In 1999, the IMF's Board of Governors contained 178 member countries. Voting weights (based on each nation's financial contribution) are shown in Table 2 for selected countries. Here, we see that the United States, with a weight of 17.55%, has a voting weight almost three times that of the second largest member, Japan, who has a weight of 6.30%. Approximate power indices for these selected countries were calculated using a variety of values of m , assuming a simple majority-voting rule ($q = 50\%$). The results are summarized in Table 2. Because this game is too large to be analyzed exactly, the relative errors in the table are calculated taking the values given by taking $m = 15$ as a proxy for the exact values.

Here, we see that the results for Owen's method ($m = 0$) are inaccurate, particularly for the Banzhaf indices. This is not surprising as the large voting weight for the United States leads the normal

approximation in (8) or (9) to be not terribly accurate. The errors rapidly diminish as m increases and the largest voting members are treated exactly. For more discussion of this example, see Leech (2002b).

7. Conclusion

This paper develops a new method for computing power indices for large voting games. The method is a hybrid between the direct application of the definitions of the indices, which are feasible for only small games, and approximation methods due to Owen where the accuracy of the approximation may be limited when voting weights are concentrated. By treating a small number of members with large weights differently from the minor members, the new method achieves a significant reduction in approximation error with little computational effort.

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