Decision Making Under Uncertainty

Third Masterclass

Outline

- Recap
- Decision making without probabilities
- The Anscombe-Aumann framework
- Maximin EU
- The ‘Smooth’ model
- Incomplete preferences

Recap

Main assumptions of SEU theory (‘Bayesian decision theory’)

- Complete preferences
- Separability / the ‘Sure-thing’ Principle
  - Independence
  - Uncertainty neutrality
- Precise probabilistic beliefs

Beyond Bayesianism

- Alternatives to precise Probabilism
  - Dempster-Shafer belief functions (convex capacities, non-additive probabilities)
  - Sets of probabilities (intervals, lower-upper probabilities)
  - Second-order measures (confidences, reliabilities)

- What does this mean for how we make decisions?
  - Choquet expected utility
  - Maximin EU
  - Unanimity preferences and defaults
  - Smooth ambiguity

Decision Making under Ignorance

In cases of ignorance your information consists of a utility on consequences and nothing more.

States

Actions	**S_1**	**S_2**	**…**	**S_n**
A	$U(a_1)$	$U(a_2)$	…	$U(a_n)$
B	$U(b_1)$	$U(b_2)$	…	$U(b_n)$

Ordinal Rules

- Dominance: Don't choose dominated alternatives.

$$\forall i, U(a_i) \geq U(b_i) \Rightarrow A \gtrsim B$$

- Maximin: Choose the action with best worst outcome.

$$A \gtrsim B \Leftrightarrow min[U(a_i)] \geq min[U(b_i)]$$

- Maximax: Choose the action with best best outcome.

$$A \gtrsim B \Leftrightarrow max[U(a_i)] \geq max[U(b_i)]$$

- These rules require only ordinal utility information

- Maximin (maximax) is unreasonably pessimistic (optimistic)
- Dominance can conflict with both Maximin and Maximax e.g. Car dominates Take Bus
  - Leximin and Leximax avoid this problem
- Dominance assumes independence of states from acts, but can this be assumed under ignorance?

Traffic States

Actions	Heavy	Medium	Light
Take Bus	-2	0	10
Walk	-1	-1	-1
Car	-2	2	10

Hurwicz Criterion

- Let h be a measure of your pessimism taking values in [0,1]
- **Hurwicz Index**: $\mathrm{H(A)} = h.Min[U(a_i)] + (1-h)Max[U(a_i)]$
- Choose the action which maximises the Hurwicz index

e.g. Take Bus is strictly preferred to Walk iff h < 2/3

Traffic States

Actions	Heavy	Medium	Light
Take Bus	-2	-2	1
Walk	-1	-1	-1

Hurwicz Criterion (cont.)

Issues:

- Is pessimism a stable psychological attitude?
- Gives invariant advice (only) if utilities are cardinally measurable
- Also conflicts with dominance

Traffic States

Actions	**Heavy**	**Medium**	**Light**
Take Bus	-2	-2	1
Walk	-1	-1	-1
Car	-2	1	1

Maximean

- Choose the action with greatest average benefit:

$$V_{mean}(\mathrm{A}) = \frac{\sum_{i=1}^{n} U(a_i)}{n}$$

- Main problem: not partition-independent e.g. {heavy, medium, light} versus {heavy, not-heavy}

Traffic States

Actions	**Heavy**	**Medium**	**Light**
Take Bus	-4	1	1
Walk	-1	-1	-1

Minimising Regret

- Regret Index: $\mathrm{r}(a_i) = \max(u(x_i)) - u(a_i)$

Minimax Regret: Choose the action which minimises the maximum regret.

Traffic States

Actions	Heavy	Medium	Light
Take Bus	-2 [1]	-2 [1]	1 [0]
Walk	-1 [0]	-1 [0]	-1 [2]

Minimising Regret

Conflicts with independence of irrelevant alternatives

	Traffic States		
Actions	**Heavy**	**Medium**	**Light**
Take Bus	-2 [1]	-2 [3]	1 [0]
Walk	-1 [0]	-1 [2]	-1 [2]
Car	-2 [1]	1 [0]	1 [0]

Uncertainty regarding Risks

- In many problems we can distinguish hypotheses about the risks (objective probabilities or chances) from our subjective beliefs about them.
- Turn to Anscombe-Aumann framework to model the relationship between the two
- On the Anscombe-Aumann theory rationality requires maximisation of the subjective expectation of (objective) expected utility.

The Anscombe-Aumann Theory

- The A&A model allows for both objective and subjective probabilities and hence uncertainty about risks.
  - Set of states S with an event algebra Ω
  - Set Z of basic outcomes ('prizes')
  - Set Δ of lotteries (probabilities) over Z
  - Set F of acts: mappings from states to lotteries
- As usual the agent has probabilistic beliefs about states and utilities for outcomes, with respect to which she maximises expected benefit.

- Note that Δ is convex, i.e. for all f,g in F, there exists an act $\alpha f + (1-\alpha)g$ such that:

$$[\alpha f + (1-\alpha)g](s) = \alpha f(s) + (1-\alpha)g(s)$$

States

Actions	**S1**	**S2**	**S3**
f	[0,1]	[0.5,0.5]	[1,0]
g	[0.2,0.8]	[0.5,0.5]	[0.6,0.4]
	[0.1,0.9]	[0.5,0.5]	[0.8,0.2]

States

Actions	**S1**	**S2**	**S3**
f	[0,1]	[0.5,0.5]	[1,0]
g	[0.2,0.8]	[0.5,0.5]	[0.6,0.4]
f $\frac{1}{2}$ g	[0.1,0.9]	[0.5,0.5]	[0.8,0.2]

Philosophical Issues

- What are lotteries? States, consequences or neither?
- Lotteries model chances: stable features of the world induced by particular ‘set-ups’, e.g. roulette wheels, atomic structure, social stratification, climate systems
- Formally lotteries are possible chance-worlds. Then
  - Subsets of S are factual propositions
  - Subsets of Δ are chance propositions.
- Both kinds of proposition can serve as objects of belief and desire.

Axioms of Preference

- Preference relation $\succsim$ defined on acts and then extended to outcomes
- Preferences over constant acts ($\succsim_\Delta$) give the risk preferences i.e. preferences over lotteries ('the chances').

1. State-Independence: If $\bar{f}(\mathrm{s}) = \mathrm{p}$ and $\bar{g}$(s)=q, then:
$$p \succsim_\Delta q \leftrightarrow \bar{f} \succsim \bar{g}$$

2. Monotonicity: $\forall s, f(s) \succsim_\Delta g(s)\ then\ f \succsim g$

3. Independence:

$$f \gtrsim g \Leftrightarrow \alpha f + (1 - \alpha)h \gtrsim \alpha g + (1 - \alpha)h$$

- Implies both VM Independence and the Sure-Thing Principle.

4. Archimedean: $\exists \alpha, \beta \epsilon [0,1]: \forall f, g, h$ such that $f > g > h$

$$\alpha f + (1 - \alpha)g \gtrsim g \gtrsim \beta f + (1 - \beta)g$$

A&A Representation Theorem

- An agent's preferences satisfy axioms 1 – 4 iff for all acts f and g, $V_{AA}(f) \geq V_{AA}(g) \Leftrightarrow f \succsim g$, where:

$$V_{AA}(f) = \sum_{s \in S} \left[\sum\nolimits_{x \in Z} u(x).f_s(x) \right].\Pr(s)$$

EU of lottery at s

Degree of belief

Rivals to Bayesianism

- Non-additive probabilities and Choquet EU
- Imprecise probabilism and Maximin EU
- Incompleteness, Unanimity and Inertia
- Second order probabilities and the KMM model

Choquet EU

- Model agent's beliefs as capacities
  - Capacity = non-additive probability
  - Intuitively, they are lower-probabilities (betting)
  - Dempster-Shafer belief functions are convex capacities

- Choquet EU maximisation (Schmeidler 1982/89)

$$Choquet\ EU(f) = \sum_{s} [EU(f(s))].v(s)$$

Capacity

Maximin SEU

- Model agent's beliefs as a set of probabilities, C ('imprecise probabilism')
- Pick action with greatest minimum SEU relative to these probabilities

$$Min\ EU(f) = Min_{Pr\in C}\left[\sum\nolimits_s \left[EU\left(f(s)\right)\right].\Pr(s)\right]$$

V_{AA} of f given Pr

- Hurwicz (1951); Levi (1974, 1980); Gardenfors and Sahlin (1982); Gilboa & Schmeidler (1989)

Hedging

- Independence implies ‘uncertainty neutrality’

$$f \backsim g \Leftrightarrow f \backsim \alpha f + (1-\alpha)g \backsim g$$

- But it is sometimes reasonable to hedge one’s risk (as agents seem to do in the Ellsberg set-up)

States

Actions	S1	S2	S3
f	[0,1]	[0.5,0.5]	[1,0]
g	[1,0]	[0.5,0.5]	[0,1]
	[0.5,0.5]	[0.5,0.5]	[0.5,0.5]

States

Actions	S1	S2	S3
f	[0,1]	[0.5,0.5]	[1,0]
g	[1,0]	[0.5,0.5]	[0,1]
f $\frac{1}{2}$ g	[0.5,0.5]	[0.5,0.5]	[0.5,0.5]

- Main implication of Maximin EU is that hedging is rational.

Uncertainty Aversion:

$$f \backsim g \Rightarrow \alpha f + (1-\alpha)g \gtrsim f$$

- Implies a weakening of Independence

Certainty-Independence: For any constant act h,

$$f \gtrsim g \Leftrightarrow \alpha f + (1-\alpha)h \gtrsim \alpha g + (1-\alpha)h$$

- Representation Theorem (Gilboa and Schmeidler)

Preferences satisfy the A&A axioms with Uncertainty Aversion and Certainty Independence instead of Independence iff they maximise the minimum EU.

Incompleteness

- Maximin EU prescribes extreme caution – seems too strong!
- An agent with imprecise beliefs may not have complete preferences over acts.
- Define an incomplete preference relation by:

$$f \gtrsim g \Leftrightarrow \forall (Pr \in C), V_{AA}(f, Pr) \geq V_{AA}(g, Pr)$$

- **Dominance**

It is permissible to choose f iff there is no alternative g such that g > f.

  - In this case, preference does not always determine choice
  - Need to supplement with secondary choice criteria

Bewley (1986) on **inertia**

- Individuals can reasonably follow a course of action until something demonstrably better than it becomes available.
- Default may be path-dependent, so what we should do depends on what we have chosen to do in the past.