Aggregation of Incomplete Preferences under Risk

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Wohrkshop Risk and Social Decisions
Introduction

Aim

Aggregating (possibly) incomplete individual preferences into a (possibly) incomplete social preference

Why?

- Individuals’ preferences intrinsically incomplete
- Paretian preferences among subgroups
- Individuals’ (complete) preferences partially observable
- It might be too demanding to ask for complete social preferences

Framework

Preferences under risk, represented by utility sets (à la Harsanyi)
Harsanyi (1955): risk, individual and social preferences are vNM (complete)

Weymark (1984): arrovian approach. Individual preferences are complete, but social preferences are not.

Baucells & Shapley: risk, individual preferences are complete and vNM, social preferences might be incomplete VNM. Partial agreements among pairs imply completeness of social preferences


welfarist aggregation of incomplete individual preferences?
Road map

1. Incomplete preferences under risk
2. The single profile approach Harsanyi’s theorem without completeness
3. The multiprofile approach: interval utilitarianism and utilitarianism
**Individual preferences**

**Domain**

- $X$ : finite set of alternative, $\Delta X$ : set of lotteries on $X$
- $\mathcal{U}$ : set of all non-empty, compact, convex subsets of $\mathbb{R}^X$
- $N = \{1, \ldots, n\}$: individuals, with preferences $\succeq_i$; (0: society)
## Individual preferences

### Domain
- $X$: finite set of alternative, $\Delta X$: set of lotteries on $X$
- $U$: set of all non-empty, compact, convex subsets of $\mathbb{R}^X$
- $N = \{1, \ldots, n\}$: individuals, with preferences $\succeq_i$; ($0$: society)

### Axioms
- $\succeq$ is reflexive and transitive
- Continuity: $\{\alpha \in [0, 1] \mid p \succeq \alpha q + (1 - \alpha) r\}$ closed
- Independence: $p \succeq q \iff \alpha p + (1 - \alpha) r \succeq \alpha q + (1 - \alpha) r$

### Representation (DMO, 2004; BS, 2008)
- $\exists U \in U, \forall p, q \in \Delta X, p \succeq q \iff [Eu(p) \geq Eu(q), \forall u \in U]$. 
  $U$ can be taken to be a singleton iff $\succeq$ is complete.
Harsanyi’s theorem without completeness

Axiom (Pareto)

\[ \forall i \in N, \ p \succsim_i \Rightarrow p \succsim_0 q \]

Theorem (Harsanyi’s theorem without completeness)

\( \succsim_0 \) satisfies Pareto iff there exists a nonempty closed and convex set \( T \subseteq \mathbb{R}^n_+ \times U_1 \times \cdots \times U_n \) such that:

\[
U_0 = \left\{ \sum_{i=1}^{n} \lambda_i u_i \mid (\lambda_1, \ldots, \lambda_n, u_1, \ldots, u_n) \in T \right\}.
\]
Harsanyi’s theorem without completeness

\[ U_0 = \left\{ \sum_{i=1}^{n} \lambda_i u_i \mid (\lambda_1, \ldots, \lambda_n, u_1, \ldots, u_n) \in T \right\} \]

Special cases

1. Pareto ordering: \( T = \mathbb{R}_+^n \times U_1 \times \cdots \times U_n \)
2. “Utilitarian”: \( T = \{ \lambda \} \times \{ u_1 \} \times \cdots \times \{ u_n \} \)

Corollary

\( U_1, \ldots, U_n \) are singletons.

\[ U_0 = \bigcup_{\lambda \in \Lambda} \sum_{i=1}^{n} \lambda_i U_i. \]

with \( \Lambda \subseteq \mathbb{R}_+^n \).
Setup

Social welfare function

\[ f : \mathcal{U}^n \rightarrow \mathcal{U} \]

Axiom (Unanimity)

For all \( U \in \mathcal{U} \), \( f(U, \ldots, U) = U \)

Axiom (Monotonicity)

\[ [U_i \subseteq U'_i, \ \forall i] \Rightarrow f(U) \subseteq f(U') \]

Axiom (Weak Pareto)

\[ [Eu_i(p) \geq Eu_i(q), \ \forall i \text{ and } u_i \in U_i] \Rightarrow Eu(p) \geq Eu(q), \ \forall u \in f(U) \]

Lemma

\[ [Unanimity + Monotonicity] \Rightarrow \text{Weak Pareto} \]
Interval neutrality

**Notation**

\[ \forall U \in \mathcal{U}, \ p \in \Delta X, \ U(p) = \{ Eu(p) \mid u \in U \} \]

**Axiom (Coulhon & Mongin’s extended neutrality)**

\[ \forall u, u' \in (\mathbb{R}^X)^n, \ p, q \in \Delta X, \]

\[ Eu(p) = Eu'(q) \Rightarrow f(\{u\})(p) = f(\{u'\})(q) \]
Interval neutrality

**Notation**

\[ \forall U \in \mathcal{U}, p \in \Delta X, U(p) = \{ Eu(p) | u \in U \} \]

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**Axiom (Interval neutrality)**

\[ \forall U, U' \in \mathcal{U}^n, p, q \in \Delta X, \\
U(p) = U'(q) \Rightarrow f(U)(p) = f(U')(q) \]
Incomplete preferences under risk
Harsanyi’s theorem without completeness
Interval utilitarianism
Utilitarianism

Correlation

\[ U(p) \quad \text{and} \quad U(q) \]
Correlation

\[ U(p) \quad \text{and} \quad U(q) \]
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Correlation

U(p) \{ \}

U(q) \{ \}

\begin{align*}
\end{align*}
Correlation

\[ U(p) \]

\[ u_1 \]

\[ U(q) \]
Correlation
Correlation
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Correlation

\[ U(p) \{ u_1, u_2, u_3, u_4 \} \quad U(q) \]
Correlation consistency

**Definition (Uncorrelated lotteries)**

$p \neq q \in \Delta X$ are uncorrelated in $U \in \mathcal{U}$ if for all $c \in U(p)$ and all $d \in U(q)$ there exists $u \in U$ such that $Eu(p) = c$ and $Eu(q) = d$.

**Axiom (Correlation consistency)**

For all $U \in \mathcal{U}^n$ and $p \neq q \in \Delta X$, if $p$ and $q$ are uncorrelated in $U_i$ for all $i$, then they are uncorrelated in $f(U)$.

**Axiom (Completeness consistency)**

For all $U \in \mathcal{U}^n$, if $U_i$ are singletons, then $f(U)$ is a singleton.
Theorem

Assume $|X| \geq 2$, and that $f$ satisfies Unanimity and Monotonicity. Then $f$ satisfies Interval neutrality, Correlation consistency and Completeness consistency iff there exists a unique $(\lambda_1, \ldots, \lambda_n) \in \Delta(\{1, \ldots, n\})$ s.t.

$$\forall U \in \mathcal{U}^n, \ p \in \Delta X, \ f(U)(p) = \sum_i \lambda_i U_i(p).$$
**Theorem**

Assume $|X| \geq 2$, and that $f$ satisfies Unanimity and Monotonicity. Then $f$ satisfies Interval neutrality, Correlation consistency and Completeness consistency iff there exists a unique $(\lambda_1, \ldots, \lambda_n) \in \Delta(\{1, \ldots, n\})$ s.t.

$$\forall U \in \mathcal{U}^n, \ p \in \Delta X, \ f(U)(p) = \sum_i \lambda_i U_i(p).$$

**Remark**

$f$ can be interval-utilitarian without satisfying unanimity or monotonicity.
Interval utilitarianism

Example

- $X = \{x, y\}$, $n = 2$
- $f(U_1, U_2) = \{u | \forall p \in \Delta X, Eu(p) \in \frac{1}{2}U_1(p) + \frac{1}{2}U_2(p)\}$
- $U_1 = U_2 = \{(c, c) | c \in [0, 1]\}$
- $f(U_1, U_2) = \{u | \forall p \in \Delta X, Eu(p) \in [0, 1]\} = [0, 1]^2$
Interval utilitarianism

$u(tx + (1-t)y)$

1

0 1

$t$

$U_i$
Interval utilitarianism

$u(tx + (1-t)y)$

$U_i(t_0)$

$U_i$
Interval utilitarianism

The graph illustrates the function $u(tx + (1-t)y)$ with $t$ ranging from 0 to 1. The interval utility function $U_i(t_0)$ is plotted for a specific value of $t_0$. The function $f(U_1, U_2)$ is also shown, indicating the combined utility of two individuals.
Interval utilitarianism vs. utilitarianism

- **Interval utilitarianism:** \( f(U)(p) = \sum_i \lambda_i U_i(p), \forall p \in \Delta X \)
- **Utilitarianism:** \( f(U) = \sum_i \lambda_i U_i \)

**Remark**
- **Utilitarianism \(\Rightarrow\) Interval utilitarianism**
- **Interval utilitarianism \(\not\Rightarrow\) Utilitarianism**
Interval utilitarianism vs. utilitarianism

Example

- \( f(U_1, U_2) = \{ u \mid \forall p \in \Delta X, \, Eu(p) \in \frac{1}{2} U_1(p) + \frac{1}{2} U_2(p) \} \)
- \( f'(U_1, U_2) = f(U_1, U_2) \cap \text{conv}(U_1 \cup U_2) \)
- \( f'(U_1, U_2)(p) = f(U_1, U_2)(p), \, \forall U_1, U_2 \in \mathcal{U} \), \( p \in \Delta X \)
- \( U_1 = \{(c, c) \mid c \in [0, 1]\} \) and \( U_2 = \{(c, c) \mid c \in [2, 3]\} \)
- \( f'(U_1, U_2) = [1, 2]^2 \)
- \( \frac{1}{2} U_1 + \frac{1}{2} U_2 = \{(c, c) \mid c \in [1, 2]\} \)
Interval utilitarianism vs. utilitarianism

\[ u(tx + (1-t)y) \]

\[ U_1 \]

\[ U_2 \]
Interval utilitarianism vs. utilitarianism

\[ u(tx + (1-t)y) \]

\[ U_1 \quad U_2 \]

\[ f(U_1, U_2) \]

\[ 0 \quad 1 \]

\[ 0 \quad 1 \quad 2 \quad 3 \]
Interval utilitarianism vs. utilitarianism

\[ u(tx + (1-t)y) \]

- \( U_1 \)
- \( \frac{1}{2}U_1 + \frac{1}{2}U_2 \)
- \( U_2 \)
### Restricted neutrality

#### Definition

For all $U \in \mathcal{U}$ and $p \in \Delta X$, we say that $U$ is $p$–univocal whenever $U(p)$ is a singleton.

#### Axiom (Restricted neutrality)

For all $U, U' \in \mathcal{U}^n$, $p, q \in \Delta X$ such that $U_i$ is $p$–univocal and $U'_i$ is $q$–univocal for all $i$,

$$U(p) = U(q) \Rightarrow f(U)(p) = f(U')(q).$$

#### Remark

Interval neutrality $\Rightarrow$ Restricted neutrality
Resolution

Interpretation
You start with $U$, and eventually learn what is the unambiguous utility level for $p$. You update $U$ to some $p$–resolution of $U$.

Definition
Given $U \in \mathcal{U}$ and $p \in \Delta X$, $U' \in \mathcal{U}$ is a $p$–resolution of $U$ if there exists $c \in U(p)$ such that $U' = \{u \in U | Eu(p) = c\}$. The set of all $p$–resolutions of $U$ is denoted $\mathcal{R}_p(U)$.
Resolution robustness

Interpretation
Social utility set cannot contain functions that do not correspond to some $p$–resolution.

Axiom (Resolution robustness)
For all $U \in \mathcal{U}^n$, $p \in \Delta X$, $u \in f(U)$, there exists $(U'_1, \ldots, U'_n) \in \mathcal{R}_p(U_1) \times \cdots \times \mathcal{R}_p(U_n)$ such that $u \in f(U')$. 
Utilitarianism

Theorem (Utilitarianism)

\( f \) satisfies Unanimity, Monotonicity, Completeness consistency, Restricted neutrality and Resolution robustness iff there exists a unique \((\lambda_1, \ldots, \lambda_n) \in \Delta(\{1, \ldots, n\})\) s.t.

\[
\forall U \in \mathcal{U}^n, \quad f(U) = \sum_i \lambda_i U_i
\]
Remark

Observe that $f(U) = \text{conv}(U_1 \cup \cdots \cup U_n)$ satisfies:

- **Unanimity**
- **Monotonicity**
- **Interval neutrality** (hence Restricted neutrality)
- **Resolution robustness**
Remark

Observe that \( f(U) = \text{conv}(U_1 \cup \cdots \cup U_n) \) satisfies:

- Unanimity
- Monotonicity
- Interval neutrality (hence Restricted neutrality)
- Resolution robustness

But does not satisfy

- Completeness consistency
- Correlation consistency
Open problems

**Question I**
Can we find axioms that have an ordinal flavor (in progress)?

**Question II**
What is the exact nature of the link between individual and social completeness?

**Question III**
Can we provide and justify axiomatically other sensible social welfare functionals?