Aggregation of Incomplete Preferences under Risk

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Wohrkshop Risk and Social Decisions

Work in progress...

Introduction

Aim

Aggregating (possibly) incomplete individual preferences into a (possibly) incomplete social preference

Why?

- Individuals' preferences intrinsically incomplete
- Paretian preferences among subgroups
- individuals' (complete) preferences partially observable
- It might be too demanding to ask for complete social preferences

Framework

Preferences under risk, represented by utility sets (*la* Harsanyi)

Related literature

- Harsanyi (1955): risk, individual and social preferences are vNM (complete)
- Weymark (1984): arrovian approach. Individual preferences are complete, but social preferences are not.
- Baucells & Shapley: risk, individual preferences are complete and vNM, social preferences might be incomplete VNM. Partial agreements among pairs imply completness of social preferences
- Konczak & Lang (2005): voting. Individual preferences might be incomplete. Possible and necessary winners.
- welfarist aggregation of incomplete individual preferences?

Road map

1. Incomplete preferences under risk
2. The single profile approach Harsanyi's theorem without completeness
3. The multiprofile approach: interval utilitarianism and utilitarianism

Individual preferences

Domain

- X : finite set of alternative, ΔX : set of lotteries on X
- $\mathscr{U}$: set of all non-empty, compact, convex subsets of $\mathbb{R}^X$
- $N = \{1, \ldots, n\}$: individuals, with preferences $\succcurlyeq_i$ (0: society)

Individual preferences

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Axioms

- $\succcurlyeq$ is reflexive and transitive
- Continuity: $\{\alpha \in [0,1] \,|\, p \succcurlyeq \alpha q + (1-\alpha) r\}$ closed
- Independence: $p \succcurlyeq q \Leftrightarrow \alpha p + (1-\alpha) r \succcurlyeq \alpha q + (1-\alpha) r$

Representation (DMO, 2004; BS, 2008)

$\exists U \in \mathscr{U}, \forall p, q \in \Delta X, \; p \succcurlyeq q \Leftrightarrow [Eu(p) \geq Eu(q), \; \forall u \in U]$.
U can be taken to be a singleton iff $\succcurlyeq$ is complete.

Harsanyi's theorem without completeness

Axiom (Pareto)

$[\forall i \in N,\ p \succcurlyeq_i] \Rightarrow p \succcurlyeq_0 q$

Theorem (Harsanyi's theorem without completeness)

$\succcurlyeq_0$ satisfies Pareto iff there exists a nonempty closed and convex set $T \subseteq \mathbb{R}^n_+ \times U_1 \times \cdots \times U_n$ such that:

$$U_0 = \left\{ \sum_{i=1}^{n} \lambda_i u_i \,|(\lambda_1, \ldots, \lambda_n, u_1, \ldots, u_n) \in T \right\}.$$

Harsanyi's theorem without completeness

$$U_0 = \left\{ \sum_{i=1}^{n} \lambda_i u_i \,|(\lambda_1, \ldots, \lambda_n, u_1, \ldots, u_n) \in T \right\}$$

Special cases

1. Pareto ordering: $T = \mathbb{R}^n_+ \times U_1 \cdots \times U_n$
2. "Utilitarian": $T = \{\lambda\} \times \{u_1\} \times \cdots \times \{u_n\}$

Corollary

$U_1, \ldots, U_n$ are singletons.

$$U_0 = \bigcup_{\lambda \in \Lambda} \sum_{i=1}^{n} \lambda_i U_i.$$

with $\Lambda \subseteq \mathbb{R}^n_+$.

Setup

Social welfare function

$f : \mathscr{U}^n \to \mathscr{U}$

Axiom (Unanimity)

For all $U \in \mathscr{U}$, $f(U, \dots, U) = U$

Axiom (Monotonicity)

$[U_i \subseteq U_i', \forall i] \Rightarrow f(U) \subseteq f(U')$

Axiom (Weak Pareto)

$[Eu_i(p) \geq Eu_i(q), \forall i \text{ and } u_i \in U_i] \Rightarrow Eu(p) \geq Eu(q), \forall u \in f(U)$

Lemma

$[Unanimity + Monotonicity] \Rightarrow$ *Weak Pareto*

Interval neutrality

Notation

$\forall U \in \mathscr{U}, \ p \in \Delta X, \ U(p) = \{Eu(p) \mid u \in U\}$

Axiom (Coulhon & Mongin's extended neutrality)

$\forall u, u' \in \left(\mathbb{R}^X\right)^n, \ p, q \in \Delta X,$

$$Eu(p) = Eu'(q) \Rightarrow f(\{u\})(p) = f(\{u'\})(q)$$

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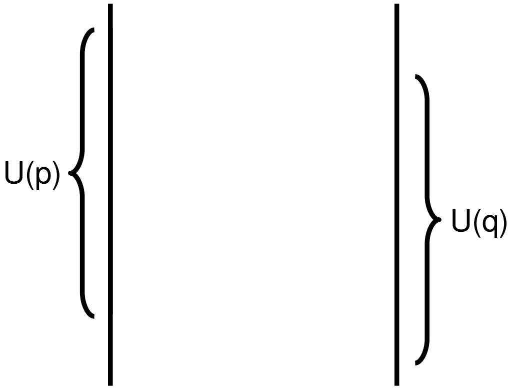
Correlation



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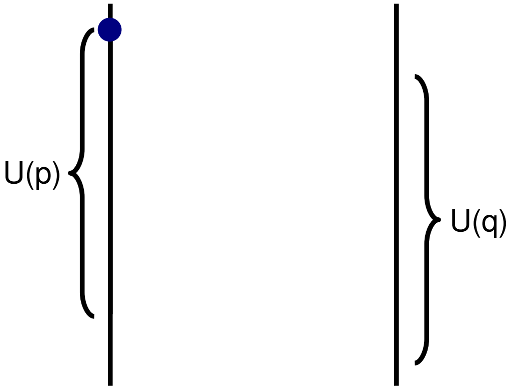

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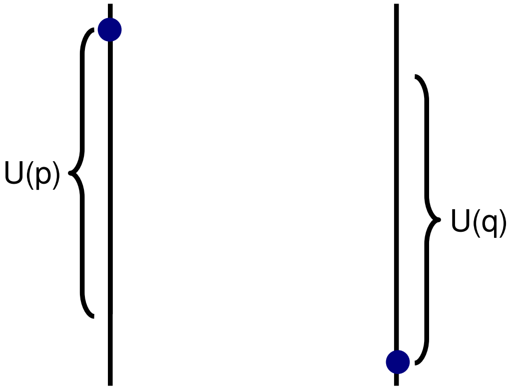

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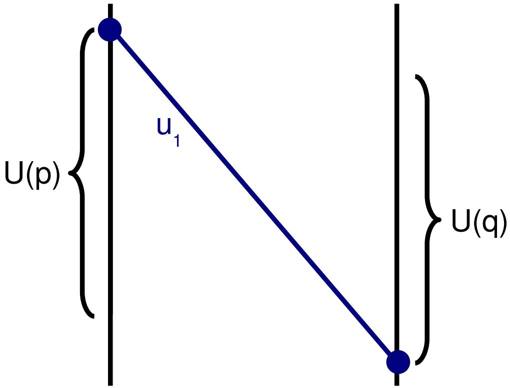

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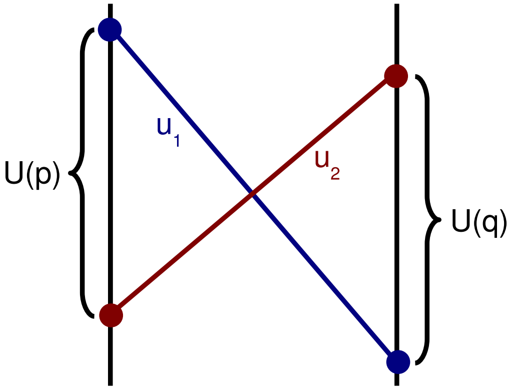

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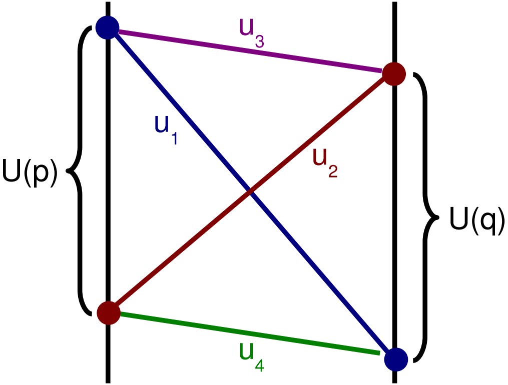

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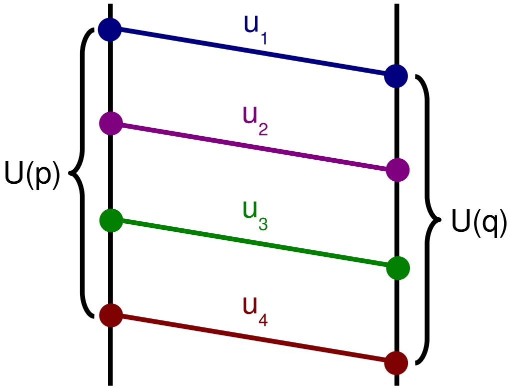

Correlation consistency

Definition (Uncorrelated lotteries)

$p \neq q \in \Delta X$ are uncorrelated in $U \in \mathscr{U}$ if for all $c \in U(p)$ and all $d \in U(q)$ there exists $u \in U$ such that $Eu(p) = c$ and $Eu(q) = d$.

Axiom (Correlation consistency)

For all $U \in \mathscr{U}^n$ and $p \neq q \in \Delta X$, if p and q are uncorrelated in U_i for all i, then they are uncorrelated in $f(U)$.

Axiom (Completness consistency)

For all $U \in \mathscr{U}^n$, if U_i are singletons, then $f(U)$ is a singleton.

Interval utilitarianism

Theorem

Assume $|X| \geq 2$, and that f satisfies Unanimity and Monotonicity. Then f satisfies Interval neutrality, Correlation consistency and Completness consistency iff there exists a unique $(\lambda_1, \ldots, \lambda_n) \in \Delta(\{1, \ldots, n\})$ s.t.

$$\forall U \in \mathscr{U}^n,\ p \in \Delta X,\ f(U)(p) = \sum_i \lambda_i U_i(p).$$

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Remark

f can be interval-utilitarian without satisfying unanimity or monotonicity.

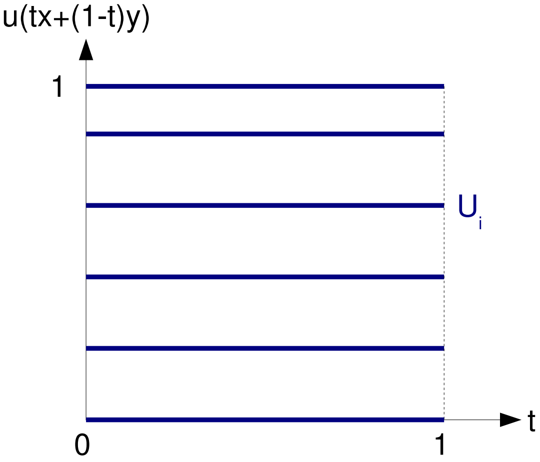
◂ Skip example

Interval utilitarianism

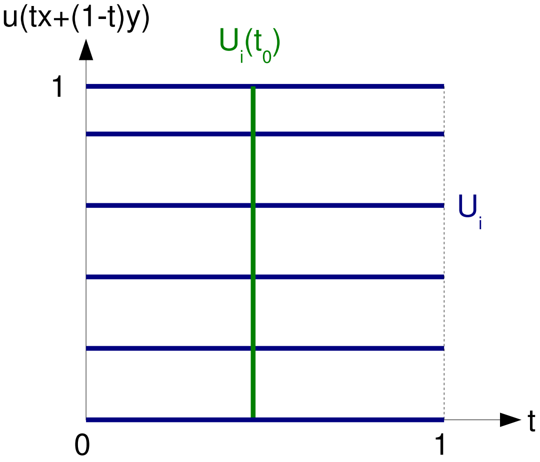
Example

- $X = \{x, y\}$, $n = 2$
- $f(U_1, U_2) = \{u \,|\, \forall p \in \Delta X,\ Eu(p) \in \frac{1}{2}U_1(p) + \frac{1}{2}U_2(p)\}$
- $U_1 = U_2 = \{(c, c) \,|\, c \in [0, 1]\}$
- $f(U_1, U_2) = \{u \,|\, \forall p \in \Delta X,\ Eu(p) \in [0, 1]\} = [0, 1]^2$

Interval utilitarianism



Interval utilitarianism



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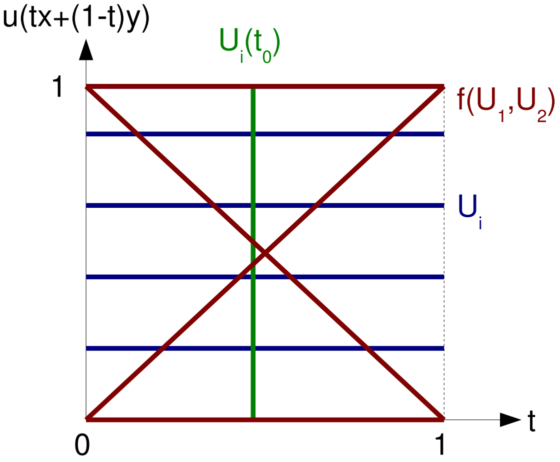


Interval utilitarianism vs. utilitarianism

- Interval utilitarianism: $f(U)(p) = \sum_i \lambda_i U_i(p), \forall p \in \Delta X$
- Utilitarianism: $f(U) = \sum_i \lambda_i U_i$

Remark

- *Utilitarianism* $\Rightarrow$ *Interval utilitarianism*
- *Interval utilitarianism* $\nRightarrow$ *Utilitarianism*

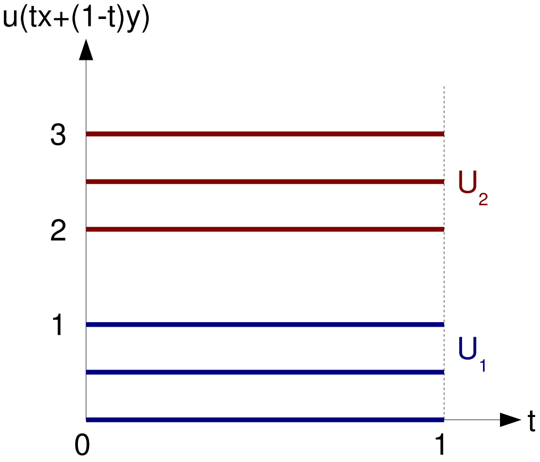
Skip example

Interval utilitarianism vs. utilitarianism

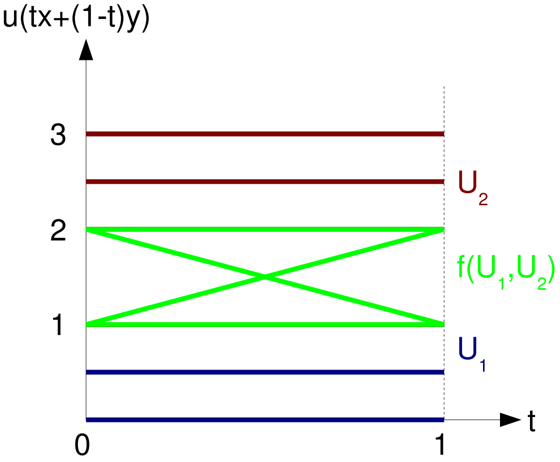
Example

- $f(U_1, U_2) = \{u \,|\, \forall p \in \Delta X,\ Eu(p) \in \frac{1}{2}U_1(p) + \frac{1}{2}U_2(p)\}$
- $f'(U_1, U_2) = f(U_1, U_2) \cap \mathrm{conv}(U_1 \cup U_2)$
- $f'(U_1, U_2)(p) = f(U_1, U_2)(p),\ \forall U_1, U_2 \in \mathscr{U},\ p \in \Delta X$
- $U_1 = \{(c, c) \,|\, c \in [0, 1]\}$ and $U_2 = \{(c, c) \,|\, c \in [2, 3]\}$
- $f'(U_1, U_2) = [1, 2]^2$
- $\frac{1}{2}U_1 + \frac{1}{2}U_2 = \{(c, c) \,|\, c \in [1, 2]\}$

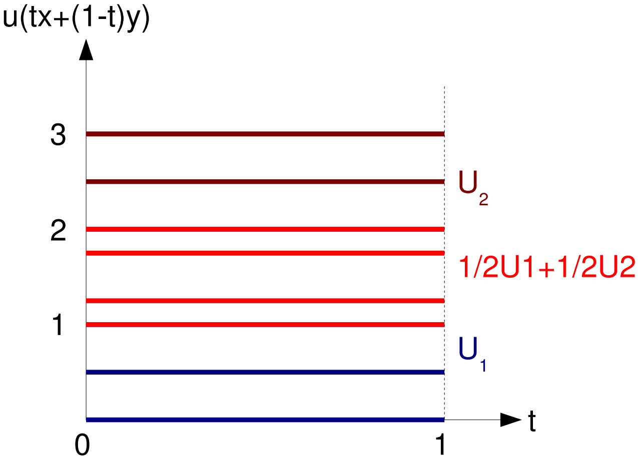
Interval utilitarianism vs. utilitarianism



Interval utilitarianism vs. utilitarianism



Interval utilitarianism vs. utilitarianism



Restricted neutrality

Definition

For all $U \in \mathscr{U}$ and $p \in \Delta X$, we say that U is $p-$univocal whenever $U(p)$ is a singleton.

Axiom (Restricted neutrality)

For all $U, U' \in \mathscr{U}^n$, $p, q \in \Delta X$ such that U_i is $p-$univocal and U'_i is $q-$univocal for all i,

$$U(p) = U(q) \Rightarrow f(U)(p) = f(U')(q).$$

Remark

Interval neutrality $\Rightarrow$ Restricted neutrality

Resolution

Interpretation

You start with U, and eventually learn what is the unambiguous utilty level for p. You update U to some $p-$resolution of U.

Definition

Given $U \in \mathscr{U}$ and $p \in \Delta X$, $U' \in \mathscr{U}$ is a $p-$resolution of U if there exists $c \in U(p)$ such that $U' = \{u \in U \,|\, Eu(p) = c\}$. The set of all $p-$resolutions of U is denoted $\mathscr{R}_p(U)$.

Resolution robustness

Interpretation

Social utility set cannot contain functions that do not correspond to some $p-$resolution.

Axiom (Resolution robustness)

For al $U \in \mathscr{U}^n$, $p \in \Delta X$, $u \in f(U)$, there exists $(U'_1, \ldots, U'_n) \in \mathscr{R}_p(U_1) \times \cdots \times \mathscr{R}_p(U_n)$ such that $u \in f(U')$.

Utilitarianism

Theorem (Utilitarianism)

f satisfies Unanimity, Monotonicity, Completness consistency, Restricted neutrality and Resolution robustness iff there exists a unique $(\lambda_1, \ldots, \lambda_n) \in \Delta(\{1, \ldots, n\})$ s.t.

$$\forall U \in \mathscr{U}^n,\ f(U) = \sum_i \lambda_i U_i$$

Relations among axioms

Remark

Observe that $f(U) = \mathrm{conv}(U_1 \cup \cdots \cup U_n)$ *satisfies:*

- *Unanimity*
- *Monotonicity*
- *Interval neutrality (hence Restricted neutrality)*
- *Resolution robustness*

Relations among axioms

Remark

Observe that $f(U) = \mathrm{conv}(U_1 \cup \cdots \cup U_n)$ *satisfies:*

- *Unanimity*
- *Monotonicity*
- *Interval neutrality (hence Restricted neutrality)*
- *Resolution robustness*

But does not satisfy

- *Completness consistency*
- *Correlation consistency*

Open problems

Question I

Can we find axioms that have an ordinal flavor (in progress)?

Question II

What is the exact nature of the link between individual and social completeness?

Question III

Can we provide and justify axiomatically other sensible social welfare functionals?