Assessing Risky Social Situations

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1 Introduction

No satisfactory approach?

1) Harsanyi's utilitarianism: $E\sum_i u_i = \sum_i Eu_i$

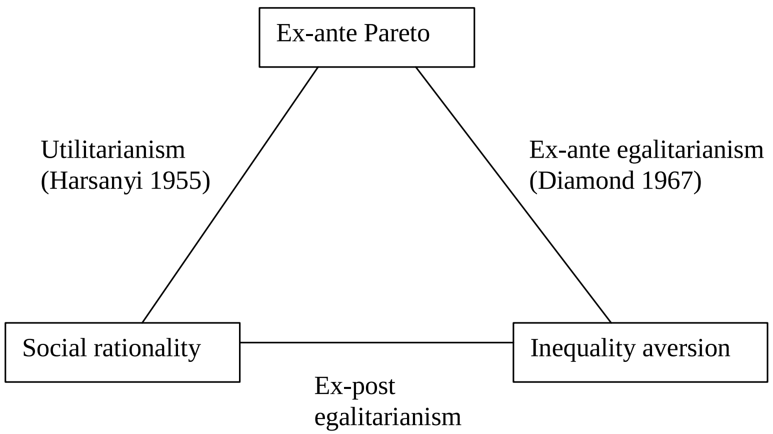
Pb: no aversion to inequality in utilities

2) Ex ante egalitarianism: $\sum_i \varphi(Eu_i)$

Pb: a lottery may be good even if it reduces $\sum_i \varphi(u_i)$ ex post for sure

3) Ex post egalitarianism: $E\sum_i \varphi(u_i)$

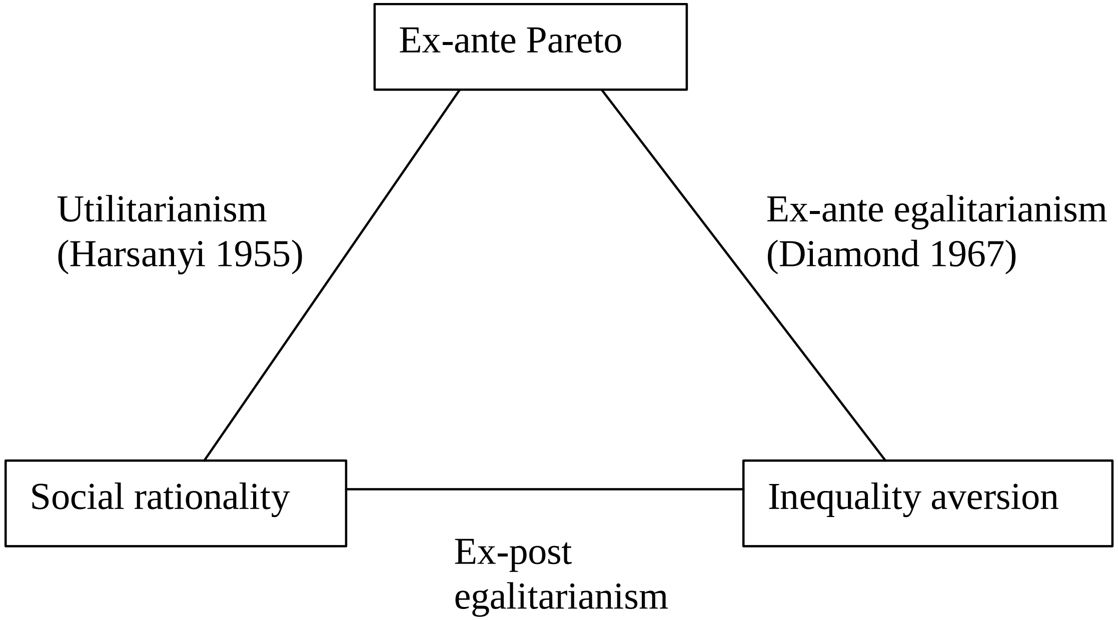
Pb: never respects Eu_i (even for a single individual)



4) Broome's and Hammond's utilitarianism: u_i is the evaluator's measure
Pb: reduced-form, incomplete theory: how to construct u_i?
5) Convex combination of 2 and 3: $\sum_i \varphi(Eu_i) + E \sum_i \varphi(u_i)$
Pb: adds the drawbacks of 2 and 3.

This paper:
1) Propose a new approach that is *less* subject to the four drawbacks (inequality neutral, irrational, disrespectful of Eu_i, incomplete)
2) It is flexible about inequality aversion
3) It has a cost: subgroup separability
4) Illustration with applications

2 Risk as imperfect information



Ex-ante Pareto is not always compelling

L	H	T
Ann	4	0
Bob	0	4

R	H	T
Ann	3	1
Bob	1	3

A rational, inequality-averse observer must prefer R to L.

Here there is no relevant uncertainty for the observer: R dominates L.

The individuals are indifferent only because of their ignorance: their unanimity is not compelling.

If they knew the state of nature, the observer should prefer R. It is better to serve the individuals' informed preferences than their uninformed preferences.

L'	H	T
Ann	4	0
Bob	4	0

R'	H	T
Ann	2	2
Bob	2	2

The observer and the individuals face the same imperfection of information.

The ex-ante Pareto criterion is acceptable in this case (equal risk).

3 A general solution

Population $N = \{1, ..., n\}$
States of nature $S = \{1, ..., m\}$
The evaluator has a fixed probability vector $\pi = (\pi^s)_{s \in S}$, with $\sum_{s \in S} \pi^s = 1$ and $\pi^s > 0$ for all $s \in S$.

Problem: rank lotteries $U = (U_i^s)_{i \in N, s \in S} \in \mathbb{R}^{nm} = \mathcal{L}$

$$U = \begin{array}{cc} \begin{pmatrix} U_1^1 & \cdots & U_1^s & \cdots & U_1^m \\ \vdots & \ddots & \vdots & & \vdots \\ U_i^1 & \cdots & U_i^s & \cdots & U_i^m \\ \vdots & & \vdots & \ddots & \vdots \\ U_n^1 & \cdots & U_n^s & \cdots & U_n^m \end{pmatrix} & \leftarrow (i) \\ \begin{array}{c} \uparrow \\ (s) \end{array} & \end{array}$$

The evaluator's ordering over $\mathcal{L}$ is denoted R (with strict preference P and indifference I).

$$U_i = \left(U_i^1, \cdots, U_i^m\right), \quad U^s = \begin{pmatrix} U_1^s \\ \vdots \\ U_n^s \end{pmatrix}$$

U egalitarian if all U_i equal; U riskless if all U^s equal.
$[U^s]$: riskless lottery in which vector U^s occurs in all states of nature:

$$[U^s] = \begin{pmatrix} U_1^s & & U_1^s \\ \vdots & \cdots & \vdots \\ U_n^s & & U_n^s \end{pmatrix}.$$

Key assumptions:

1) U^s contains all the relevant information in order to evaluate U if s is realized. This rules out Diamond's criticism:

L''	H	T
Ann	4	4
Bob	0	0

R''	H	T
Ann	4	0
Bob	0	4

An impartial and rational evaluator must be indifferent between L'' and R''.

This means that there is some incompleteness left in the theory, if U_i^s must take account of what would have happened to i in other states.

2) The U_i's are comparable across individuals without restriction.

3) EU_i is the correct evaluation of i's ex ante individual interests.

Axiom 1 (Weak Dominance) *For all* $U, U' \in \mathcal{L}$, *one has* $U \, R \, U'$ *if for all* $s \in S$, $[U^s] \ R \ [U'^s]$.

Axiom 2 (Strict Dominance) *For all* $U, U' \in \mathcal{L}$, *one has* $U \, P \, U'$ *if for all* $s \in S$, $[U^s] \ P \ [U'^s]$.

Axiom 3 (Weak Pareto for Equal Risk) *For all egalitarian* $U, U' \in \mathcal{L}$, *one has* $U \, P \, U'$ *if for all* $i \in N$, $EU_i > EU'_i$.

Axiom 4 (Weak Pareto for No Risk) *For all riskless* $U, U' \in \mathcal{L}$, *one has* $U \, P \, U'$ *if for all* $i \in N$, $U_i > U'_i$.

Lemma 1 *Let R satisfy Weak Pareto for No Risk. For all $U \in \mathcal{L}$, all $s \in S$, there is a unique $e(U^s) \in \mathbb{R}$ such that for all $\varepsilon \in \mathbb{R}_{++}$,*

$$[U^s] \ P \ [(e(U^s) - \varepsilon, ..., e(U^s) - \varepsilon)],$$
$$[(e(U^s) + \varepsilon, ..., e(U^s) + \varepsilon)] \ P \ [U^s].$$

For all $x \in \mathbb{R}$, $e(x, ..., x) = x$. Call e the equally-distributed quasi-equivalent (EDQE).

Proof. Take $U^s \in \mathbb{R}^n$. By Weak Pareto for No Risk, the sets $\{x \in \mathbb{R} \mid [U^s] \ P \ [(x, ..., x)]\}$ and $\{x \in \mathbb{R} \mid [(x, ..., x)] \ P \ [U^s]\}$ are not empty.

One shows that necessarily

$$\begin{aligned} \sup\{x \in \mathbb{R} \mid [U^s] \ P \ [(x, ..., x)]\} &= \inf\{x \in \mathbb{R} \mid [(x, ..., x)] \ P \ [U^s]\} \\ &: = e(x, ..., x). \end{aligned}$$

■

Theorem 1 *Let R satisfy Weak or Strict Dominance, Weak Pareto for Equal Risk and Weak Pareto for No Risk. For all $U, U' \in \mathcal{L}$, one has $U \, P \, U'$ if*

$$\sum_{s \in S} \pi^s e\left(U^s\right) > \sum_{s \in S} \pi^s e\left(U'^s\right),$$

where e is the EDQE function.

Proof. (under continuity)
Weak Dominance $\Rightarrow$ for all lotteries U, one can replace each vector U^s by $\left(e\left(U^s\right), ..., e\left(U^s\right)\right)$
Weak Pareto for Equal Risk $\Rightarrow$ expected utility for such lotteries. ■

Call it the "expected equally distributed equivalent" (EEDE).
Example: $E\varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\varphi\left(u_i\right)\right)$.
It behaves like Eu_i for equal risks and like ex post egalitarianism when the distribution of utility is sure.

4 A generalized Gini family

Non reranking lottery: for all $s, t \in S$ and all $i, j \in N$,

$$\left(U_i^s - U_j^s\right)\left(U_i^t - U_j^t\right) \geq 0.$$

(includes equal risk and no risk)

Axiom 5 (Strong Pareto for Non Reranking Risk) *For all non-reranking $U, U' \in \mathcal{L}$, one has $U\, R\, U'$ if for all $i \in N$, $EU_i \geq EU_i'$; in addition one has $U\, P\, U'$ if for some $i \in N$, $EU_i > EU_i'$.*

Axiom 6 (Hammond Equity) *For all $U, U' \in \mathcal{L}$, one has $U\, R\, U'$ if for some $i, j \in N$,*

$$U_i' \gg U_i \geq U_j \gg U_j',$$

while $U_k = U_k'$ for all $k \neq i, j$.

Axiom 7 (Anonymity) *For all $U, U' \in \mathcal{L}$, one has $U\, I\, U'$ if U' differs from U only by permuting the vectors U_i.*

The ex-post leximin social ordering:

$U^s_{(i)}$ utility of ith rank (by increasing order) in U^s,
$U_{(i)}$ denotes $\left(U^s_{(i)}\right)_{s\in S}$
$\geq_{lex}$ denotes the ordinary leximin criterion
The ex-post leximin criterion weakly prefers U to U' iff

$$\left(EU_{(i)}\right)_{i\in N} \geq_{lex} \left(EU'_{(i)}\right)_{i\in N}.$$

I.e, leximin applied to the expected value of the utilities of ith rank in U.

Theorem 2 *The ordering R satisfies Weak Dominance, Strong Pareto for Non Reranking Risk, Hammond Equity and Anonymity if and only if it is the ex-post leximin criterion. It then also satisfies Strict Dominance.*

Proof. By Strong Pareto for Non Reranking Risks, there is an ordering $\tilde{R}$ over $\mathbb{R}^n$ such that for all non-reranking $U, U' \in \mathcal{L}$, $U\, R\, U'$ iff

$$(EU_i)_{i\in N}\ \tilde{R}\ (EU_i')_{i\in N}\,.$$

By Hammond Equity, Anonymity and Strong Pareto for Non Reranking Risk, this ordering satisfies the standard Hammond Equity, Anonymity and Strong Pareto properties over n-vectors.
By Hammond (1979, Th. 4-5), $\tilde{R}$ is then the leximin ordering.
By Weak Dominance and Anonymity, every U^s can be reordered by increasing order. ■

Ex post generalized Gini:

$$\sum_{i \in N} \alpha_i EU_{(i)}$$

for some fixed parameters $(\alpha_i)_{i \in N} \in \mathbb{R}^n_{++}$. Inequality aversion is obtained when $\alpha_1 > ... > \alpha_n$.

Axiom 8 (Continuity) *Let $U, U' \in \mathcal{L}$ and $(U(t))_{t \in \mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ be such that $U(t) \to U$. If $U(t)\, R\, U'$ for all $t \in \mathbb{N}$, then $U\, R\, U'$. If $U'\, R\, U(t)$ for all $t \in \mathbb{N}$, then $U'\, R\, U$.*

Theorem 3 *The ordering R satisfies Weak or Strict Dominance, Strong Pareto for Non Reranking Risk, Anonymity and Continuity if and only if it is an ex-post generalized Gini criterion.*

5 The separability challenge

One would like to extend the Pareto criterion to the case when a subgroup takes an "equal risk" while the rest bears no risk and is not concerned.

Axiom 9 (Weak Pareto for Subgroup Equal Risk) *For all egalitarian* $U, U' \in \mathcal{L}$, *riskless* $V \in \mathcal{L}$, $M \subseteq N$, $(U_M, V_{-M})\ P\ (U'_M, V_{-M})$ *if* $\forall i \in M,\ EU_i > EU'_i$.

Weak Pareto for Subgroup Equal Risk is violated by EEDE criteria displaying strict aversion to inequality.

Under Weak or Strict Dominance, this axiom is incompatible with inequality aversion over riskless prospects. Consider the following prospects, where $\varepsilon > 0$.

V	Heads	Tails
Ann	2	2
Bob	2	2

V'	Heads	Tails
Ann	$4 + \varepsilon$	$0 + \varepsilon$
Bob	2	2

Z	Heads	Tails
Ann	$3 - \varepsilon$	$1 - \varepsilon$
Bob	$3 - \varepsilon$	$1 - \varepsilon$

By Weak Pareto for Subgroup Equal Risk, V' is preferred to V and V is preferred to Z. By transitivity, therefore, V' is preferred to Z.

In the presence of inequality aversion over riskless prospects, for ε small enough, $[Z^s]$ is preferred to $[V'^s]$ for $s =$Heads,Tails. Therefore, by Weak or Strict Dominance, Z is at least as good as V', a contradiction.

One obtains a variant of Harsanyi's theorem

Theorem 4 *Let R satisfy Weak or Strict Dominance, and Weak Pareto for Subgroup Equal Risk. There is $\alpha \in \mathbb{R}^n_+ \setminus \{0\}$ such that for all $U, U' \in \mathcal{L}$, one has $U \, P \, U'$ if $\sum_{i \in N} \alpha_i EU_i > \sum_{i \in N} \alpha_i EU_i'$.*

Comment: Harsanyi's theorem is based on separability across individuals (via Pareto) and separability across states (via EW). Weak Pareto for Subgroup Equal Risk, which is weaker than the usual Pareto axiom, contains enough of both separabilities.

Proof. The EDQE $e(.)$, when only one i takes a risk, generates a function

$$f(z) = e(u_1, ..., u_{i-1}, z, u_{i+1}, ..., u_n)$$

such that for all $x, y \in R$ and all $\varepsilon > 0$,

$$f\left(\pi^1 x + \left(1 - \pi^1\right) y + \varepsilon\right) \geq \pi^1 f(x) + \left(1 - \pi^1\right) f(y) \geq f\left(\pi^1 x + \left(1 - \pi^1\right) y - \varepsilon\right),$$

and then derive that e is affine. ■

Is this a challenge?

Who is unconcerned and safe? $\rightarrow$ Independence of the Utilities of the Dead (IUD)...

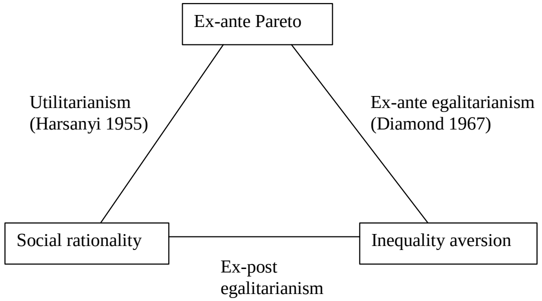
If we apply the EEDE criterion to all (past, present and future) mankind, IUD is violated.

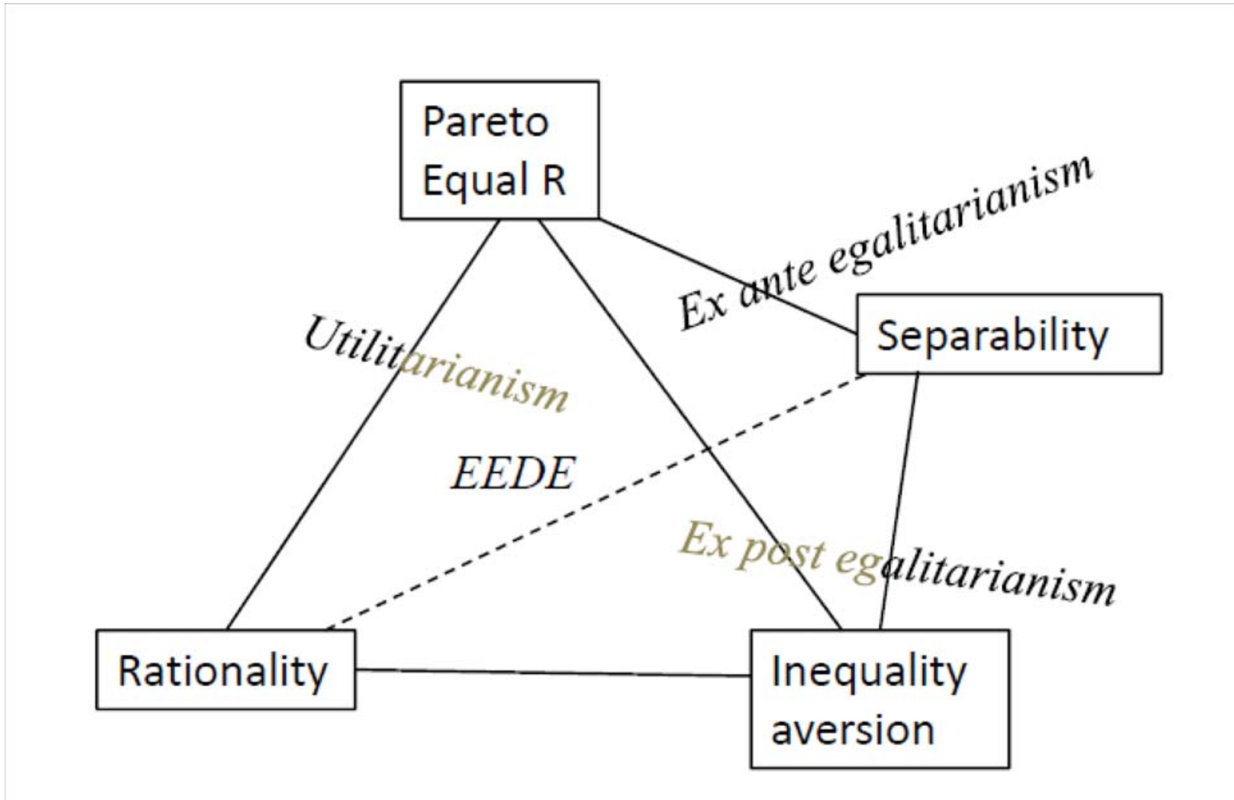
If we apply it to present and future generations only, time consistency is violated.

Violating IUD may be intuitive: it is bad to take a risk that would put future generations below us or our ancestors

"sustainable development"

With the generalized Gini EEDE, ex ante Pareto is fully satisfied when every generation is sure to be better off than its predecessors.


Ex-ante Pareto
Utilitarianism
(Harsanyi 1955)
Ex-ante egalitarianism
(Diamond 1967)
Social rationality
Inequality aversion
Ex-post
egalitarianism


Pareto
Equal R
Ex ante egalitarianism
Separability
Utilitarianism
EEDE
Ex post egalitarianism
Rationality
Inequality
aversion

6 Welfare economics of risk

Limits of the second welfare theorem:

A complete set of markets yields an allocation that is ex ante Pareto efficient.

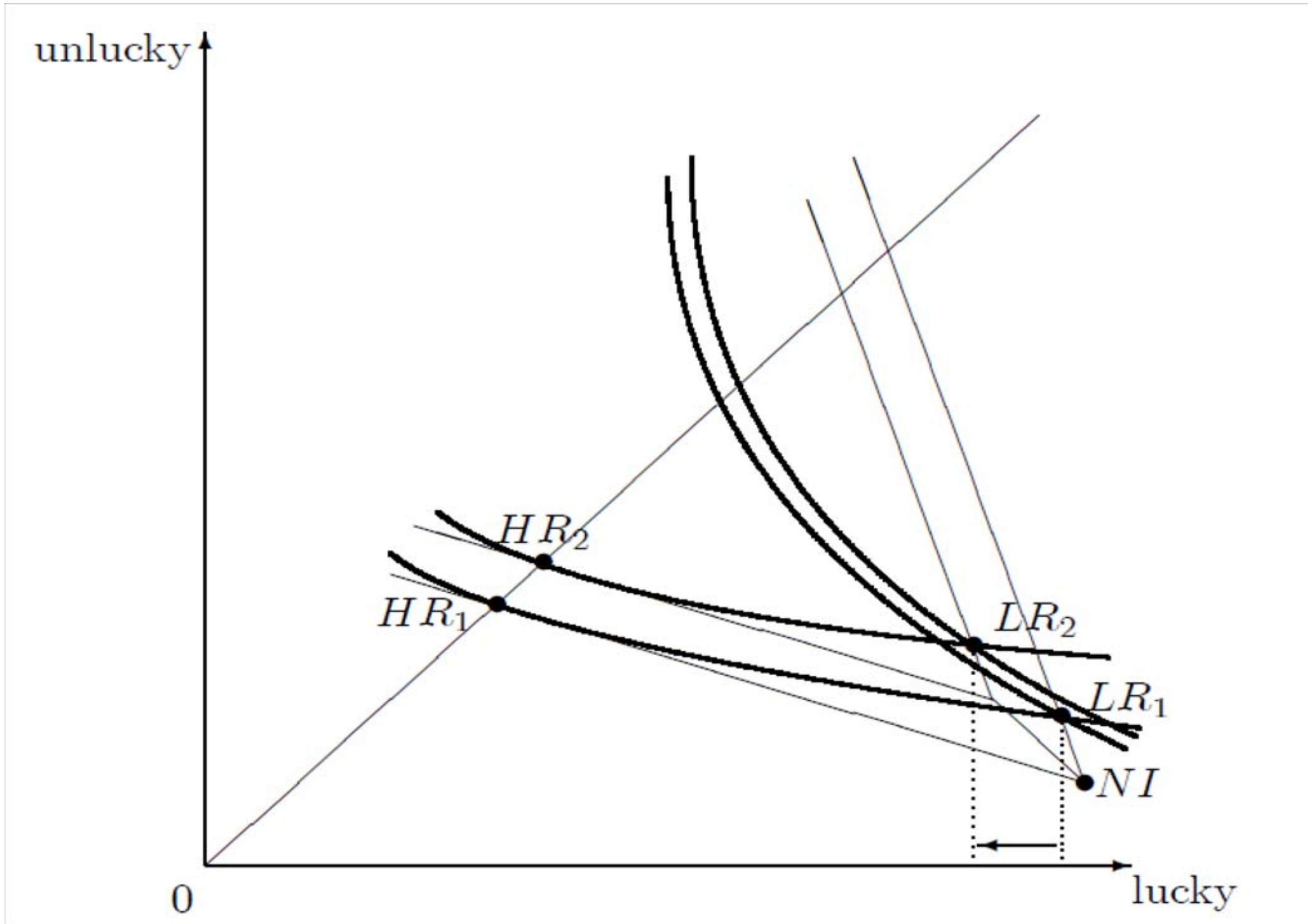
A socially optimal allocation for EEDE may not be achievable with free markets and lump-sum transfers.

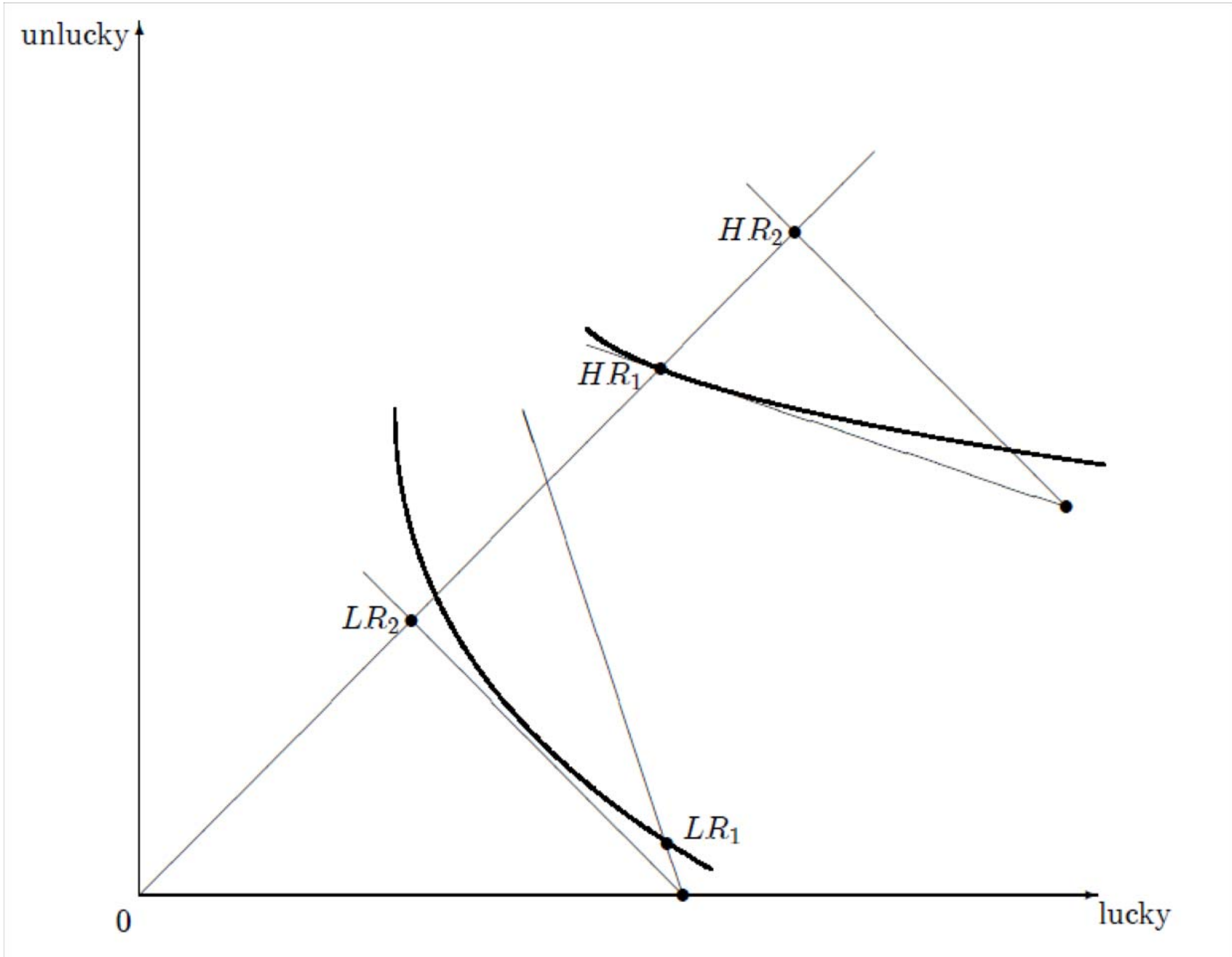
Example: If $U(c, h) = hv(c)$, (c consumption, h health),

individuals may not want to insure ex ante (low marginal utility of c if bad health).

A system of compulsory health insurance protects the unlucky.

Adverse selection and inequalities:




unlucky
HR_2
HR_1
LR_2
LR_1
0
lucky

Value of life:

Is it worth reducing p, the probability of a particular hazard? If it hits, $f(l)$ is the PDF of longevities, otherwise it is $g(l)$. Marginal PDFs $f_i(l_i)$, $g_i(l_i)$.

A standard ex ante or ex post criterion is only sensitive to the change in marginal distributions of risk for individuals whereas the EEDE is sensitive to correlations.

An ex-ante criterion computes the derivative of social welfare with respect to $-p$ as:

$$\sum_{i=1}^{n} \varphi'(EU_i) \int_{a_i}^{+\infty} U(w_i(l_i), l_i) [g_i(l_i) - f_i(l_i)] \, dl_i.$$

A standard ex-post criterion computes it as:

$$\sum_{i=1}^{n} \int_{a_i}^{+\infty} \varphi(U(w_i(l_i), l_i)) [g_i(l_i) - f_i(l_i)] \, dl_i.$$

The EEDE criterion computes it as:

$$\int_{a_1}^{+\infty} ... \int_{a_n}^{+\infty} \varphi^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} \varphi(U(w_i(l_i), l_i)) \right) [g(l) - f(l)] \, dl_n ... dl_1.$$

Example: if it hits, two individuals face 50% chance each of dying. This information is sufficient for ex ante and ex post criteria. But the EEDE considers the hazard increasingly worse if:

- the two individuals will be killed together (with 50% chance)
- they have independent risks
- exactly one (yet unknown) will be killed.

7 Conclusion

Extensions to be done:

1) Fairness in lotteries: how should U_i^s depend on what would happen in other states?

2) Personal responsibility for risk-taking (see Fleurbaey 2008, ch. 6)

3) Economic models in which individuals are not compared in terms of VNM utilities (fair allocation theory)

8 FAQ

Responsibility and ex-ante Pareto.

Hammond (1983), Kolm (1998): individuals have the right to take risks and should be able to assume the consequences.

If anything, responsibility implies disregarding expected utility (for which the agents are responsible).

$\rightarrow$ laissez-faire, but not ex-ante approach.

Risk equity versus catastrophe avoidance.

Keeney (1980):

"risk equity"= preference for a more equal distribution of independent risks of death $(p_1, ..., p_n)$

"catastrophe avoidance"= for a given expected number of fatalities, one prefers a smaller number of actual fatalities

For an ex-post criterion $\sum_{f=1}^{n} \pi_f u(f)$:

risk equity $\Rightarrow$ convex (risk prone): ex-post egalitarianism

catastrophe avoidance $\Rightarrow$ concave (risk averse): ex-post anti-egalitarianism

Decision theory and Savage's sure-thing principle.

In decision theory, weak dominance = "statewise dominance" (Quiggin 1989).

Savage's sure-thing principle: If the decision-maker weakly prefers act $\mathbf{f}$ to act $\mathbf{g}$ conditionally on event E being realized, and also conditionally on event E not being realized, then he must weakly prefer $\mathbf{f}$ to $\mathbf{g}$.

P2: If $\mathbf{f}$ is weakly preferred to $\mathbf{g}$, $f(s) = f'(s)$ for all $s \in E$, $g(s) = g'(s)$ for all $s \in E$, $f(s) = g(s)$ for all $s \notin E$, $f'(s) = g'(s)$ for all $s \notin E$, then $\mathbf{f}'$ is weakly preferred to $\mathbf{g}'$.

P2⇒sure-thing principle⇒statewise dominance.

Maximin satisfies sure-thing principle but not P2.

Minimize probability of $u < u^*$ otherwise maximize Eu : satisfies statewise dominance but not sure-thing principle.