Aggregation of preferences under risk and uncertainty

Thibault Gajdos

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Risk and Uncertainty

Luce & Raiffa, 1957

- Certainty: each action is known to lead invariably to a specific outcome
- Risk: each action leads to one of a set of possible specific outcomes, each outcome occurring with a known probability. The probabilities are assumed to be known to the decision maker
- Uncertainty: each action has as its consequence a set of possible specific outcomes, but the probability of these outcomes are completely unknown or are not even meaningful (p.13)

The Aggregation Problem

$$(x_E, E; x_{\bar{E}}, \bar{E})$$

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$$
\begin{array}{c}
(x_E, E; x_{\bar{E}}, \bar{E}) \\
\Big\downarrow \\
\begin{bmatrix} u_1(x_E), p_1(E) & u_2(x_E), p_2(E) \\ u_1(x_{\bar{E}}), p_1(\bar{E}) & u_2(x_{\bar{E}}), p_2(\bar{E}) \end{bmatrix}
\end{array}
$$

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$$V_1(x_E, E; x_{\bar{E}}, \bar{E})$$
$$V_2(x_E, E; x_{\bar{E}}, \bar{E})$$

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$$\begin{gathered} V_1(x_E, E; x_{\bar{E}}, \bar{E}) \\ V_2(x_E, E; x_{\bar{E}}, \bar{E}) \end{gathered}$$

$$\downarrow$$

$$W\left(V_1(\cdot), V_2(\cdot)\right)$$

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\begin{array}{ccc}
(x_E, E; x_{\bar{E}}, \bar{E}) & & \\
\downarrow & & \\
\left[\begin{array}{cc} u_1(x_E), p_1(E) & u_2(x_E), p_2(E) \\ u_1(x_{\bar{E}}), p_1(\bar{E}) & u_2(x_{\bar{E}}), p_2(\bar{E}) \end{array}\right] & \longrightarrow & \left[\begin{array}{c} u_0(x_E), p_0(E) \\ u_0(x_{\bar{E}}), p_0(\bar{E}) \end{array}\right] \\
\downarrow & & \\
\begin{array}{c} V_1(x_E, E; x_{\bar{E}}, \bar{E}) \\ V_2(x_E, E; x_{\bar{E}}, \bar{E}) \end{array} & & \\
\downarrow & & \\
W\left(V_1(\cdot), V_2(\cdot)\right) & &
\end{array}
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$$
\begin{tikzcd}
(x_E, E; x_{\bar{E}}, \bar{E}) \arrow[d] & \\
\left[\begin{array}{cc} u_1(x_E), p_1(E) & u_2(x_E), p_2(E) \\ u_1(x_{\bar{E}}), p_1(\bar{E}) & u_2(x_{\bar{E}}), p_2(\bar{E}) \end{array}\right] \arrow[r] \arrow[d] & \left[\begin{array}{c} u_0(x_E), p_0(E) \\ u_0(x_{\bar{E}}), p_0(\bar{E}) \end{array}\right] \arrow[dd] \\
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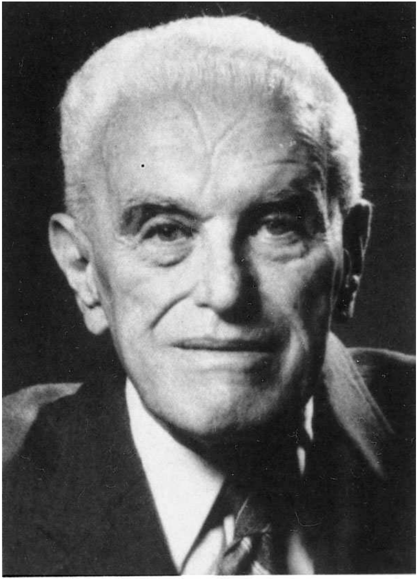
\begin{matrix} V_1(x_E, E; x_{\bar{E}}, \bar{E}) \\ V_2(x_E, E; x_{\bar{E}}, \bar{E}) \end{matrix} \arrow[d] & \\
W\left(V_1(\cdot), V_2(\cdot)\right) \arrow[r, leftrightarrow, "?"] & V_0(x_E, E; x_{\bar{E}}, \bar{E})
\end{tikzcd}
$$



John C. Harsanyi

The aggregation problem

Aggregating n preferences into one that:

1. satisfies the same "rationality" requirements as individuals' preferences
2. is non dictatorial
3. does not provoke unanimous opposition

Road map

1. The vNM case: Harsanyi's aggregation theorem
2. The Subjective Expected Utility case
3. Uncertainty: the (almost) general case

Setup

- $N' = \{1, \cdots, n\}$ agents, $N = \{0\} \cup N'$ where $0 =$ "society"
- X (sure) social alternatives
- $\mathscr{L} = \left\{ p : X \to [0,1] \,\middle|\, \begin{array}{c} \#\{x | p(x) > 0\} < \infty \\ \sum_{x \in X} p(x) = 1 \end{array} \right\}$ social lotteries

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- individuals and society satisfy vNM axioms: for all $i \in N$, there exists $u_i : X \to \mathbb{R}$ such that:

$$p \succcurlyeq_i q \Leftrightarrow \sum_{x \in X} p(x) u_i(x) \geq \sum_{x \in X} q(x) u_i(x)$$

Moreover, u_i is unique up to an increasing affine transformation.

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Axioms

Weak Pareto (WP)

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Independent Prospect (IP)

For all $k \in N'$, there exist p_k and q_k such that:

$$p_k \succ_k q_k \text{ and } p_k \sim_i q_k, \forall i \in N' \setminus \{k\}$$

Harsanyi's aggregation theorem

Theorem

Assume that $\succcurlyeq_i$ *is represented by a vNM function* u_i $(i \in N)$ *and (IP) is satisfied. Then (WP) holds iff there exist unique* $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0_n\}$, $\mu \in \mathbb{R}$ *such that:*

$$u_0 = \sum_i \lambda_i u_i + \mu.$$

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Remark

In Harsanyi's original (1955) theorem:

- Pareto indifference: sign of coefficients undetermined
- Independent Prospect not assumed: coefficients not unique

Proof (sketch)

Lemma 1

Under vNM, (IP) implies that there exist p^* and p_* such that:

$$p^* \succ_i q_*, \forall i \in N' \tag{MA}$$

◂ Proof

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◂ Proof

Lemma 2 (De Meyer & Mongin, 1995)

Let $X \neq \emptyset$ and $F = (f_0, f_1, \ldots, f_n) : X \to \mathbb{R}^{n+1}$. If $K = F(X)$ is convex and (WP) and (MA) hold, then there exist $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0_n\}$, $\mu \in \mathbb{R}$ such that:

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Lemma 3

Under the assumptions of Lemma 2, (IP) implies unicity of the coefficients.

Proof

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Proof

Remark

This theorem gave rise to a substantial body of work and often passionate debates. For a survey, see Sen (1986) and Weymark (1991). These debates concern:

- *the interpretation of the theorem* more
- *its normative appeal* more

Subjective Expected Utility

Anscombe-Aumann setup

- S finite set of states of nature
- X social outcomes
- Y simple probability distributions over X (roulette lotteries)
- $\mathscr{A} = \{f : S \to X\}$ acts (horse lotteries)
- $\mathscr{A}$ is a mixture space: $(\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s)$

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Subjective Expected Utility

individuals and society satisfy SEU axioms: for all i there exist a probability measure p_i on S and a non-constant vNM function $u_i : Y \to \mathbb{R}$ such that:

$$f \succcurlyeq_i g \Leftrightarrow \sum_{s\in S} p_i(s)u(f(s)) \geq \sum_{s\in S} p_i(s)u(g(s))$$

Moreover, p_i is unique, u_i is unique up to an increasing affine transform.

Aggregation of SEU (Mongin, 1998)

Theorem

Assume that $\succcurlyeq_i$ is represented by a SEU function with utility u_i and beliefs p_i ($i \in N$) and (IP) is satisfied. Then (WP) iff there exist unique $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0_n\}$, $\mu \in \mathbb{R}$ such that for $f \in \mathscr{A}$:

$$\sum_s p_0(s) u_0(f(s)) = \sum_{i \in N'} \lambda_i \left(\sum_s p_i(s) u_i(f(s)) \right) + \mu.$$

Moreover, $p_j = p_k = p_0$ for all $j, k \in J = \{i \in N' \,|\, \lambda_i \neq 0\}$.

Proof

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Wlog, pick $y \in Y$ and let $u_i(y) = 0$, $\forall i \in N'$.
$\mathscr{A}_{y,t} = \{f \in \mathscr{A} \,|\, f(s) = y, \forall s \neq t\}$

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Second part of the theorem

By the first part:

- $p_0(s)u_0 = \sum_{i \in N'} \lambda_i p_i(s) u_i$, $\forall s \in S$ (acts in $\mathscr{A}_{y,s}$)

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- $p_0(s) \left(\sum_{i \in N'} \lambda_i u_i\right) = \sum_{i \in N'} \lambda_i p_i(s) u_i \; \forall s$

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$$\sum_{i \in J} \lambda_i u_i \left(p_0(s) - p_i(s)\right) = 0$$

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u_i ($i \in J$) aff. indep. $\Rightarrow p_0(s) = p_i(s)$, $\forall i \in J$, $s \in S$

What can we do?

- Relaxing Pareto: Gilboa, Samet and Schmeidler (2004)
- Allowing for less restrictive preferences
- After all, SEU is very special: it imposes uncertainty neutrality

Uncertainty aversion

Ellsberg's paradox

90 balls in urn: 30 red, and 60 blue and yellow

Uncertainty aversion

Ellsberg's paradox

90 balls in urn: 30 red, and 60 blue and yellow

	R	Y	B
f	1	0	0
g_1	0	1	0
g_2	1	0	1
h	0	1	1

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Modal preferences

$$f \succ g_1 \;\&\; h \succ g_2$$

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Inconsistent with SEU

- $f \succ g_1 \Rightarrow \Pr(Y) < \frac{1}{3}$
- $h \succ g_2 \Rightarrow \frac{2}{3} > \frac{1}{3} + \Pr(B) = \frac{1}{3} + (\frac{2}{3} - \Pr(Y)) \Rightarrow Pr(Y) > \frac{1}{3}$

Preliminary definitions

Capacity

$\rho : 2^S \longrightarrow [0,1]$ such that:

- $\rho(\emptyset) = 0$ and $\rho(S) = 1$
- $A \subseteq B \Rightarrow \rho(A) \leq \rho(B)$

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Binary acts $\mathscr{B}$

$f \in \mathscr{A}$ st there exists $E \in 2^S$, $x, y \in Y$:

- $f(s) = x, \forall s \in A$
- $f(s) = y, \forall s \in A^c$
- denoted: xAy

Biseparable preference (Ghirardato & Marinacci, 2001)

c-linear biseparable preferences

The preference relation $\succcurlyeq$ is c-linear biseparable iff there exist a function $V : \mathscr{A} \to \mathbb{R}$ and a capacity ρ on 2^S such that:

- $\forall x \succcurlyeq y$, letting $u(x) = V(x)$,

$$V(xAy) = \rho(A)u(x) + (1 - \rho(A))u(y)$$

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- $\forall f \in \mathscr{B}$, $x \in Y$, $\alpha \in [0, 1]$,
$$V(\alpha f + (1 - \alpha)x) = \alpha V(f) + (1 - \alpha)V(x)$$

Moreover ρ is unique and V is unique up to an increasing affine transformation.

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  Moreover ρ is unique and V is unique up to an increasing affine transformation.

Uncertainty aversion

A c-linear bisep. preference is *uncertainty neutral wrt event E* iff $\rho(E) = 1 - \rho(E^c)$.
It is *uncertainty neutral* if it is uncertainty neutral wrt all events.

Biseparable preference (Ghirardato & Marinacci, 2001)

Examples

- Subjective Expected Utility
- Choquet Expected Utility (Schmeidler, 1986)
- Maxmin Expected Utility (Gilboa & Schmeidler, 1989)
- $\alpha-$Maxmin Expected Utility (Jaffray, 1989)

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GTV 08

Generalization of c-linear biseparable preferences: rank dependent additive preferences allow for state dependence.

Aggregation (Gajdos, Tallon, Vergnaud, 2008)

Theorem

Assume that $\succcurlyeq_i$ are c-linear biseparable preferences, represented by functions V_i with capacities ρ_i and that (IP) is satisfied. Then (WP) holds iff there exist unique $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n_+ \setminus \{0_n\}$, $\mu \in \mathbb{R}$ such that for $f \in \mathscr{B}$:

$$V(f) = \sum_{i \in N'} \lambda_i V_i(f) + \mu.$$

Moreover, $\lambda_i \lambda_j \neq 0$ iff i and j are uncertainty neutral

Aggregation (Gajdos, Tallon, Vergnaud, 2008)

Interpretation

- Either social preferences are a linear aggregation of uncertainty neutral individual preferences;
- Or there is a dictator.

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- If social preferences are not neutral towards uncertainty, then there is a dictator;
- It is in some sense stronger than Harsanyi's Theorem, since neutrality towards uncertainty is a consequence, not an assumption.

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Example

- Non-dictatorial aggregation of Maxmin Expected Utility maximizers (or CEU) is impossible if individuals are uncertainty averse
- True even if they have the same "beliefs"

Sketch of the proof

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Usual arguments show that aggregation must be linear

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- Use (IP) to show that if $\lambda_i \neq 0$ then $\rho_i(E) = 1 - \rho_i(E^c)$ for all E

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- Again use (IP), assuming $\exists \lambda_j > 0, \lambda_k > 0$: $\exists x, y$ st $x \succ_j y$, $y \succ_k x$ and $x \sim_0 y$
  $V_0(xEy) - V_0(yEx) = 0$ (direct computation)

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  $V_0(xEy) - V_0(yEx) = 0$ (direct computation)
  $V_0(xEy) - V_0(yEx) = \sum_i \lambda_i \left(V_i(xEy) - V_i(yEx)\right)$
  Given $\rho_i(E) = 1 - \rho_i(E^c)$, leads $\rho_0(E) = 1 - \rho_0(E^c)$

The End?

Proof of Lemma 1

By (IP) and vNM there exist (p_k, q_k), $k \in N'$ such that:

$$\begin{cases} u_k(p_k) \geq u_k(q_k) \\ u_i(p_k) = u_i(q_k), \forall i \in N' \setminus \{k\} \end{cases}$$

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Let $p^* = \sum_i \frac{1}{n} p_i$ and $p_* = \sum_i \frac{1}{n} q_i$.

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Let $p^* = \sum_i \frac{1}{n} p_i$ and $p_* = \sum_i \frac{1}{n} q_i$.

$$\begin{aligned} u_k(p^*) &= \sum_i \frac{1}{n} u_k(p_i) \\ &= \frac{1}{n} u_k(p_k) + \sum_{i \neq k} \frac{1}{n} u_k(p_i) \\ &> \frac{1}{n} u_k(q_k) + \sum_{i \neq k} \frac{1}{n} u_k(q_i) = u_k(p_*) \end{aligned}$$

Back

Proof of Lemma 2

Definitions

- $R = \{z \in \mathbb{R}^{n+1} \,|z_0 \leq 0,\; z_i > 0 \,\forall \in N'\}$
- $K = (f_0, f_1, \ldots, f_n)(X) = F(X)$ convex
- $K^- = \{z' - z'' \,\big|(z', z'') \in K^2\}$ convex and symmetric wrt 0

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Separation argument

- (WP) $\Leftrightarrow R \cap K^- = \emptyset \Leftrightarrow R \cap \mathrm{Vect}(K^-) = \emptyset$ (K^- conv and sym)

Proof of Lemma 2

Definitions

- $R = \{z \in \mathbb{R}^{n+1} \,|z_0 \leq 0,\ z_i > 0 \,\forall \in N'\}$
- $K = (f_0, f_1, \ldots, f_n)(X) = F(X)$ convex
- $K^- = \{z' - z'' \,\big|(z', z'') \in K^2\}$ convex and symmetric wrt 0

Separation argument

- (WP) $\Leftrightarrow R \cap K^- = \emptyset \Leftrightarrow R \cap \mathrm{Vect}(K^-) = \emptyset$ (K^- conv and sym)
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Proof of Lemma 2

Sign of φ_i, $i \in N'$

- $\gamma e_j + \sum_{i \in N'} e_i \in R, \forall \gamma > 0$
- $\left\langle \varphi, \gamma e_j + \sum_{i \in N'} e_i \right\rangle > 0$
- $\varphi_j(1+\gamma) + \sum_{i \in N' \setminus \{j\}} \varphi_i > 0, \forall \gamma > 0$
- Thus $\varphi \neq 0$. Let $\gamma \to \infty$: $\varphi_j \geq 0$

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Sign of φ_0

- [(MA) and (WP)] $\Rightarrow$ there exist $(\theta_0, \theta_1, \ldots, \theta_n) \in K^-$ s.t. $\theta_i > 0$ for all i
- $\varphi_0 \theta_0 = \sum_{i \in N'} -\varphi_i \theta_i$
- Thus $\varphi_0 < 0$

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Proof of Lemma 3

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- $u_0(p_k) - u_0(q_k) = \lambda_i \left(u_k(p_k) - u_k(q_k)\right)$
- Thus λ_k unique (true for all $k \in N'$)
- Thus μ unique

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Diamond's critics

p	$Pr(\theta_1) = \frac{1}{2}$	$Pr(\theta_2) = \frac{1}{2}$
u_a	1	0
u_b	0	1

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The Independence assumption is unacceptable for the social preferences

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The identification problem

Sen (1986), Weymark (1991)

- Let $\tilde{u}_i = \alpha_i u_i$
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Weights are meaningless from a normative point of view

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