Cooperative Equilibria and Conditional Obligations

Allard Tamminga* Barteld Kooi*

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Abstract

We investigate group actions and their functioning in weak action obligations of the form "Group $\mathcal{G}$ of agents cannot do better than to perform action K" and in conditional weak action obligations of the form "If group $\mathcal{H}$ of agents performs action L, then group $\mathcal{G}$ of agents cannot do better than to perform action K". Conditional weak action obligations are exactly what we need to express Nash-equilibria for n-player non-cooperative games. Moreover, our definition of Nash-equilibria in terms of conditional obligations can be generalized to obtain a solution concept for cooperative games.

1 Game Models

Definition 1 (Game Models) A *game model* $\mathfrak{M}$ is an ordered pair $\langle \mathfrak{S}, \mathfrak{I} \rangle$, where $\mathfrak{S}$ is a choice structure and $\mathfrak{I}$ an interpretation.

Definition 2 (Choice Structures) A *choice structure* $\mathfrak{S}$ is a triple $\langle W, A, Choice \rangle$, where W is a non-empty set of possible worlds, A a finite set of agents, and $Choice$ a choice function.

Choice sets of *individual agents* are given by a function $Choice$ from individual agents to sets of sets of possible worlds, meeting the conditions that (1) for each individual agent a in A it holds that $Choice(a)$ is a partition of W, and (2) for each selection function s assigning to each individual agent a in A a set of possible worlds $s(a)$ such that $s(a) \in Choice(a)$ it holds that $\bigcap_{a \in A} s(a)$ is non-empty.

*Department of Theoretical Philosophy, Faculty of Philosophy, University of Groningen, Oude Boteringestraat 52, 9712 GL Groningen, Netherlands. E-mail: A.M.Tamminga@rug.nl and B.P.Kooi@rug.nl.

Next, we extend this choice function for individual agents to a function $Choice$ for *groups of agents*, given the set $Select$ of selection functions s assigning to each individual agent a in A an option $s(a)$ in $Choice(a)$:

$$Choice(\mathcal{G}) = \{\bigcap_{a \in \mathcal{G}} s(a) : s \in Select\},$$

if $\mathcal{G}$ is non-empty. Otherwise, $Choice(\mathcal{G}) = \{W\}$.

Definition 3 (Interpretations) An interpretation $\mathfrak{I}$ is a utility function $u : \wp(A) \times W \mapsto [-5, 5]$.

The utility an *individual agent* a assigns to a possible world w is given by a real number between, say, -5 and 5. We write $u(a, w) = 4$, if an individual agent a assigns to a possible world w a utility of 4.

The utility a *group of agents* $\mathcal{G}$ assigns to a possible world w is given by the arithmetical mean of the individual utilities the individual agents in $\mathcal{G}$ assign to w:

$$u(\mathcal{G}, w) = \frac{1}{|\mathcal{G}|} \sum_{a \in \mathcal{G}} u(a, w),$$

if $\mathcal{G}$ is non-empty. Otherwise, $u(\mathcal{G}, w) = 0$. Thus, if $u(a, w) = 4$ and $u(b, w) = 0$, then $u(\{a, b\}, w) = 2$.

Definition 4 ($\mathcal{G}$-Dominance) Let $\mathfrak{M}$ be a game model. Let $\mathcal{G} \subseteq A$ and $K, K' \in Choice(\mathcal{G})$ and $\emptyset \neq x \subseteq K$ and $\emptyset \neq x' \subseteq K'$. Then

$x \succeq_{\mathcal{G}} x'$ iff for all $S \in Choice(A - \mathcal{G})$ and for all $w, w' \in W$ it holds that if $w \in x \cap S$ and $w' \in x' \cap S$, then $u(\mathcal{G}, w) \geq u(\mathcal{G}, w')$.

Moreover, $x \succ_{\mathcal{G}} x'$ if and only if $x \succeq_{\mathcal{G}} x'$ and $x' \not\succeq_{\mathcal{G}} x$.

2 Language

To study game models formally, we define for each game model $\mathfrak{M}$ a propositional modal language $\mathfrak{L}_{\mathfrak{M}}$ built from a countable set $\mathfrak{A}_{\mathfrak{M}} = \{\alpha_{\mathcal{G}}^{K} : \mathcal{G} \subseteq A \text{ and } K \in Choice(\mathcal{G})\}$ of atomic group actions.[1] $\mathfrak{L}_{\mathfrak{M}}$ is the smallest set (in terms of set-theoretical inclusion) satisfying the conditions (i) through (v):

(i) $\mathfrak{A}_{\mathfrak{M}} \subseteq \mathfrak{L}_{\mathfrak{M}}$
(ii) If $\varphi \in \mathfrak{L}_{\mathfrak{M}}$, then $\neg\varphi \in \mathfrak{L}_{\mathfrak{M}}$
(iii) If $\varphi, \psi \in \mathfrak{L}_{\mathfrak{M}}$, then $\varphi \wedge \psi \in \mathfrak{L}_{\mathfrak{M}}$
(iv) If $\alpha_{\mathcal{G}}^{K} \in \mathfrak{A}_{\mathfrak{M}}$, then $\circledcirc\alpha_{\mathcal{G}}^{K} \in \mathfrak{L}_{\mathfrak{M}}$
(v) If $\alpha_{\mathcal{G}}^{K}, \alpha_{\mathcal{H}}^{L} \in \mathfrak{A}_{\mathfrak{M}}$ and $\mathcal{H} \subseteq A - \mathcal{G}$, then $\circledcirc(\alpha_{\mathcal{G}}^{K}/\alpha_{\mathcal{H}}^{L}) \in \mathfrak{L}_{\mathfrak{M}}$.

[1] We owe the idea of action constants to (Horty 2001, p. 83).

3 Semantics

Definition 5 (Semantical Rules) Let $\mathfrak{M}$ be a game model. Let $\mathcal{G} \subseteq A$ and let $\mathcal{H} \subseteq A - \mathcal{G}$. Let $w \in W$ and let $\varphi, \psi \in \mathfrak{L}_{\mathfrak{M}}$. Then

(i)	$\mathfrak{M}, w \models \alpha_{\mathcal{G}}^{K}$	iff	$w \in K$,	if $\alpha_{\mathcal{G}}^{K} \in \mathfrak{A}_{\mathfrak{M}}$
(ii)	$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$	
(iii)	$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$	
(iv)	$\mathfrak{M}, w \models \circledcirc\alpha_{\mathcal{G}}^{K}$	iff	for all K' in $Choice(\mathcal{G})$ with $K' \neq K$ it holds that $K \succeq_{\mathcal{G}} K'$	
(v)	$\mathfrak{M}, w \models \circledcirc(\alpha_{\mathcal{G}}^{K}/\alpha_{\mathcal{H}}^{L})$	iff	for all K' in $Choice(\mathcal{G})$ with $K' \neq K$ it holds that $K \cap L \succeq_{\mathcal{G}} K' \cap L$.	

Theorem 1 Let $\mathfrak{M}$ be a game model. Let $\mathcal{G} \subseteq A$ and let $\mathcal{H} \subseteq A - \mathcal{G}$. Let $\alpha_{\mathcal{G}}^{K}, \alpha_{\mathcal{H}}^{L} \in \mathfrak{A}_{\mathfrak{M}}$. Then the following statements are equivalent:

(i) $\mathfrak{M} \models \circledcirc(\alpha_{\mathcal{G}}^{K}/\alpha_{\mathcal{H}}^{L})$ for all $L \in Choice(\mathcal{H})$
(ii) $\mathfrak{M} \models \circledcirc\alpha_{\mathcal{G}}^{K}$.

4 Cooperative Equilibria

The present formalism allows for a definition of Nash-equilibria for n-player non-cooperative games. For example, given a game model $\mathfrak{M}$ consisting of two agents a and b, such that $Choice(a) = \{K_1, K_2\}$ and $Choice(b) = \{L_1, L_2\}$, it holds that $K \cap L$ is a Nash-equilibrium in $\mathfrak{M}$ if and only if $\mathfrak{M} \models \circledcirc(\alpha_a^K/\alpha_b^L) \wedge \circledcirc(\alpha_b^L/\alpha_a^K)$.

Definition 6 (Cooperative Equilibria) Let $\mathfrak{M}$ be a game model. Let $\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$ be a partition of A. Then $x \subseteq W$ is a *cooperative equilibrium for* $\{\mathcal{G}_1, \ldots, \mathcal{G}_n\}$ *in* $\mathfrak{M}$ if and only if

(i) $\mathfrak{M} \models \bigwedge_i \circledcirc(\alpha_{\mathcal{G}_i}^{K_i}/\alpha_{A-\mathcal{G}_i}^{L_i})$
(ii) $x = K_i \cap L_i$ for all i such that $1 \leq i \leq n$.

Note that cooperative equilibria for singleton partitions are Nash-equilibria in the standard sense. Note also that if for each a in A it holds that $Choice(a)$ is finite, then there always is a cooperative equilibrium for the partition $\{A\}$.

We now illustrate the notion of cooperative equilibria for partitions with an example. Let $\mathfrak{M}$ be the following game model, where $Choice(a) = \{K_1, K_2\}$ and $Choice(b) = \{L_1, L_2\}$ and $Choice(c) = \{M_1, M_2\}$:

	L_1	L_2		L_1	L_2	
K_1	(3,3,1)	(0,4,4)		(6,0,0)	(0,1,0)	K_1
K_2	(4,0,3)	(1,1,0)		(2,0,3)	(0,1,1)	K_2
	M_1	M_1		M_2	M_2	

$K_1 \cap L_1 \cap M_1$ is a cooperative equilibrium for $\{\{a, b\}, \{c\}\}$, since it holds that

$$\mathfrak{M} \models \circledcirc(\alpha_{ab}^{K_1 L_1} / \alpha_c^{M_1}) \wedge \circledcirc(\alpha_c^{M_1} / \alpha_{ab}^{K_1 L_1}).$$

$K_2 \cap L_2 \cap M_2$ is a cooperative equilibrium for $\{\{a\}, \{b\}, \{c\}\}$, since it holds that

$$\mathfrak{M} \models \circledcirc(\alpha_a^{K_2} / \alpha_{bc}^{L_2 M_2}) \wedge \circledcirc(\alpha_b^{L_2} / \alpha_{ac}^{K_2 M_2}) \wedge \circledcirc(\alpha_c^{M_2} / \alpha_{ab}^{K_2 L_2}).$$

$K_1 \cap L_2 \cap M_1$ is a cooperative equilibrium for $\{\{a, b, c\}\}$, since it holds that

$$\mathfrak{M} \models \circledcirc\alpha_{abc}^{K_1 L_2 M_1}.$$

Note that the first equilibrium is the most rewarding one for a and b. It is better for them to form the coalition $\{a, b\}$, which results in a utility of 3 for both of them, than to form the coalition $\{a, b, c\}$, which results in a utility less than 3.

5 Future Research

From a logical point of view, our use of Horty's action constants should be abandoned in favor of a definition of a formal language $\mathfrak{L}$ that does not depend on a previous definition of game models. This can be done if basic group actions are seen as nullary modalities.

Moreover, it would be interesting to compare cooperative equilibria for different partitions in terms of the value the respective equilibria have for the individual agents. The example we just discussed teaches us that at least in some cases it is better for individual agents to join groups who are smaller than the whole set of agents.

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