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Introducing Intransitivities in Social Choice Theory

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Abstract

This expository note offers two diagrams that may be helpful when introducing students to social choice theory. The first diagram may help students get a feel for preferences and the Kemeny distance between preferences. This diagram is then used to discuss intransitivities arising from aggregating the preferences of two individuals. It is argued that this provides a better introduction to Arrow's theorem than the common use of Condorcet cycles. An alternative diagram that can be used for the same purpose or for further exercises is also offered.

Keywords

Social choice, Kemeny distance, Saari triangle

Introduction

Contrary to widespread practice, this expository note argues against drawing too much on Condorcet cycles as a starting point in introductions to social choice theory, and suggests an alternative. A particular diagram introduced in Kemeny (1959) is used, and another diagram using Saari triangles, introduced in Saari (1994, 1995), is offered as an alternative.

It is assumed that the major objective of an introduction to social choice is to provide a good foundation for understanding Arrow's theorem (Arrow, 1952). Thus, a good introduction will quickly acquaint students, perhaps with little or no formal training, with the main tools as well as develop sharp intuitions about Arrow's theorem. All this should be done with the simplest possible structure. It will be argued that using Condorcet cycles contains some redundancy in relation to these desiderata and that the intuitions it provides are misleading at best and

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incorrect at worst. For the purposes of this note, it is presumed that readers are acquainted with the transitivity property of preferences and with Simple Majority Rule.

Kemeny's graph is given in the second section. Intransitivities are discussed in the third section. The fourth section introduces Saari triangles and briefly suggests how it may be used to consolidate what is learned from Kemeny's graph. The fifth section concludes this expository note.

Kemeny's Graph

The first diagram, Kemeny's graph, is given in Figure 1 and shows all preferences over three alternatives x, y and z. The preference with universal indifference is shown at the centre of Kemeny's graph. The strict preferences, with no indifference between distinct alternatives are shown against dark grey or light grey backgrounds, and will be referred to as either D or L preferences. Some pairs of preferences are linked by straight lines to each of which are assigned either 1 or 2. These numbers denote the distances between the linked pairs of preferences. They show the extent by which the linked preferences differ from each other and are calculated as follows.

For any pair of preferences, consider the ways in which they may differ on a pair of alternatives. They may differ either by having opposite strict preferences or by one being a strict preference and other being indifferent. If they differ by one

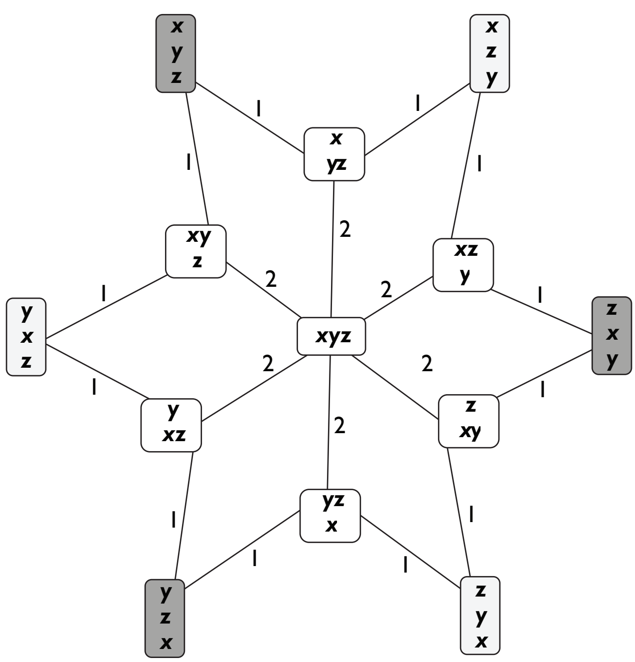

Figure 1. Kemeny's Graph

having a strict preference and the other being indifferent, the preferences are taken to differ by 1 on that pair of preferences. This is the minimum possible difference between a pair of preferences on a pair of alternatives. If the pair of preferences has opposite strict preferences on a pair of alternatives then the preferences are taken to differ by 2 on that pair of preferences. The Kemeny distance between one preference and another is the sum of their differences over all pairs of alternatives.

Not all pairs of preferences are linked. However, all pairs of preferences are linked by paths passing through other preferences. The distances along such paths may be summed to give the distance 'travelled' along that path. For any pair of preferences, the Kemeny distance between them is given by the minimum distance along the paths between them. As one might expect, the maximum distance from any preference is the difference to its opposite preference.

Intransitivities

While the smallest number of individuals required for a Condorcet cycle is three, the smallest number required by Arrow's theorem is two. The use of Condorcet cycles in introducing Arrow's theorem is usually to illustrate intransitivities in the Simple Majority relation. However, as will be shown in this section, only two individuals are required for this purpose.

First, consider any strict preference in Kemeny's graph and consider its opposite strict preference. If two individuals have opposite strict preferences, no distinction exists between alternatives unless either individuals or alternatives are treated differently. In that case, universal indifference seems reasonable as a social preference. Many familiar aggregation procedures do exactly this.

For example, Simple Majority Rule does this as does the Pareto Extension Rule. The Pareto Extension Rule 'resolves' opposite strict preferences on any pair of alternatives by indifference between that pair of alternatives. Thus, there is certainly something to be said for universal indifference for an aggregation of two preferences that have the maximum possible Kemeny distance between them. Of course, universal indifference is a transitive preference. That is, maximal dispersion or maximal departures from a perfect consensus are not generally a source of intransitives. So, what can lead to intransitivities?

Consider any strict preference and another strict preference that is a second farthest strict preference from it. For example, the second farthest from x top ranked and z bottom ranked is y top ranked and x bottom ranked. On the Kemeny graph, such pairs of preferences are easily obtained by taking not the opposite strict preference but the strict preference nearest to the opposite strict preference. The Simple Majority relation for such pairs of preferences is not transitive. In the example of the previous paragraph, the Simple Majority gives x indifferent to y and to z but y strictly preferred to z, an instransitivity. This is also the aggregate relation given by the Pareto Extension Rules as well. Note that maximally dispersed preferences are not used to obtain the intransitivity in these examples.

To illustrate, for Condorcet cycles a minimum of three individuals is required. Indeed, the two triples of individual preferences for which Simple Majority Rule

gives a strict preference, or Condorcet cycle, are exactly given by the D and L preferences in the Kemeny graph. It is visually obvious that the D preferences are maximally dispersed as are the L preferences.

Simple Majority and Pareto Extension rules have several advantages over Condorcet cycles in introducing Arrow's theorem. One is that using Simple Majority and Pareto Extension examples require fewer individuals than is required for Condorcet cycles. It follows that Condorcet cycles fail to illustrate Arrow's theorem in its simplest setting. Also, the unwary may incorrectly think that maximal dispersion plays a key role in obtaining intransitivity. Of course, some dispersion is essential but it need not be maximal. Indeed, maximal dispersion in the case of two individuals using either Simple Majority or Pareto Extension rules gives universal indifference which is transitive.

Of course, it can be argued that the implications of Condorcet cycles are more drastic than the intransitivities obtained with two individuals using Simple Majority and Pareto Extension rules. Strict preference cycles over x, y and z mean there are no highest ranked alternatives if all three are available. In contrast, although intransitive, the examples using Simple Majority and Pareto Extension rules with two individuals do give highest ranked alternatives even though the social preference is intransitive. But in an introduction to Arrow's theorem, it is probably best to stick to examples using the smallest possible number of individuals. This is not to say that Condorcet cycles may play a useful role at a later stage.

An example of the confusion arising from the use of Condorcet cycles in the context of Arrow's theorem, may be found in Chichilnisky (1982), where a 'trivial outcome' refers to a social preference with universal indifference. 'Evidently, this cannot be considered a satisfactory solution of the social choice problem; in particular, if such trivial outcomes were acceptable, neither Condorcet's nor Arrow's paradoxes would hold.'

As has already been argued, such trivial outcomes (universally indifferent social preference) seem reasonable and, if they are acceptable, both Condorcet's and Arrow's paradoxes continue to hold, at least in Arrow's framework.

Saari Triangle

The Kemeny graph is simple and user friendly. However, it is not the only possible diagram that could be used for the argument in this note. For students, what they learn from one diagram will be consolidated by going over the same argument using another. There are several available. A particularly elegant diagram was introduced in Saari (1994, 1995) and is known as a Saari triangle. It is shown in Figure 2.

A Saari triangle is a triangle decomposed into six faces, each of which is also a triangle. Each face is associated with a strict preference and these are shaded to correspond with the dark and light grey used for these strict preferences in Kemeny's graph. A good exercise for students is to locate the preferences that are

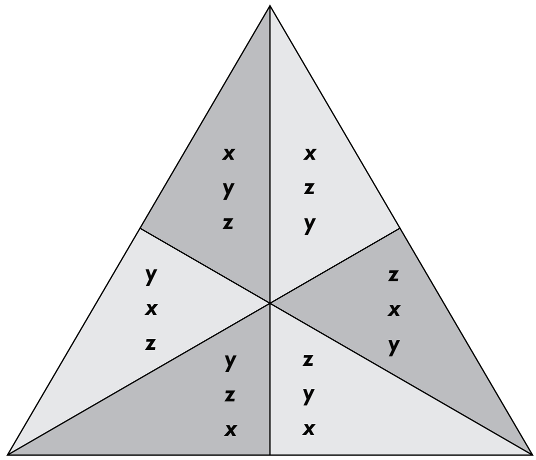

Figure2. A Saari Triangle

not strict preferences on a Saari triangle. It is also a good exercise to show Kemeny distances on a Saari triangle. Only some general guidance is offered here.

Consider pairs of adjacent faces. They share one of their edges in the interior of the larger Saari triangle. The two faces with a shared edge differ only by an opposite strict preference on a single pair of alternatives. Thus, the shared edge is associated with indifference between that pair of alternatives on which the pair of preferences has opposite strict preferences. This corresponds to the single preference in between one strict preference and another of its closest strict preferences in the Kemeny graph. Of course, the universally indifferent preference is located where all edges separating faces meet in the centre of the Saari triangle. Only a little more thought is required to assign preferences to all the points on the boundary of the Saari triangle. Finally, it is possible to establish Kemeny distances between pairs of preferences counting the number of transitions between faces and edges, though some care is required for the distance from universal indifference to any other preference. With all this correctly accomplished, all of the arguments in the previous section may be illustrated using a Saari triangle.

Conclusions

Two diagrams have been presented, Kemeny's graph in Figure 1 and a Saari triangle in Figure 2. Either may be used to display all preferences over three alternatives and Kemeny distances between them. The Kemeny graph was used to illustrate the possibility of intransitivities with only two individuals, the minimum required for Arrow's theorem, using Simple Majority Rule and the Pareto Extension Rule. It was argued that this has advantages over the use of Condorcet cycles for the same purpose, and avoiding possible confusion. Some brief suggestions were offered about doing this with a Saari triangle, leaving details as a worthwhile exercise.

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