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Topological Theories Of Social Choice

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Abstract

This chapter presents in a simple way the main results of topological social choice theory due to Chichilnisky. They all impose continuity on a social welfare function and some sort of unanimity property. The justification of continuity is critically discussed. Finally, a topological proof of Arrow's theorem is discussed.

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1 Introduction

The purpose of this chapter is to survey topological theories of social choice. The original contributions to this area of social choice came in a series of papers by Chichilnisky (Chichilnisky (1979, 1980, 1982, 1982a, 1982b, 1983 and 1993). Since there are already three excellent surveys, Mehta (1997), Lauwers (2000 and 2009), and a useful introduction and overview in Heal (1997)[1], this survey is highly selective. It has two main objectives. One is to give a simplified presentation of the key results for which the prerequisites are minimal. Most sections should be well within reach for senior undergraduates. The other purpose is to offer a critical discussion of the defining property of topological social choice, namely continuity. Footnotes or references to the literature provide further detail. Most of the exposition is limited to the simple case of 2 agents and two commodities.

Section 2 presents an elementary introduction to Chichilnisky's impossibility theorem and section 3 presents a domain restriction that provides a possible escape. Further results are presented in sections 4 and 5. Section 6 discusses continuity in a critical way since it is the key property in the earlier sections. Section 7 presents a very different sort of result, namely a highly original proof of Arrow's famous impossibility theorem. Finally, section 8 briefly presents conclusions.

2 Chichilnisky's Theorem: an elementary introduction

This section presents a simple version of the seminal result of topological social choice theory. The mathematical concepts from algebraic topology are explained in an intuitive way before using them in two related proofs of the theorem.

2.1 Linear Preferences

Interpret $\mathbb{R}^2_+$ as a commodity space of bundles of two collective goods. A linear preference on $\mathbb{R}^2_+$ can be represented by straight line indifference curves, two of which are shown in figure 1, or by a vector of unit length perpendicular to an indifference curve at an arbitrary bundle.

[1] Mehta's survey is concise and very good for the technical details of the results it presents. Lauwers (2000) is comprehensive and Lauwers (2009) is more selective, but simpler.

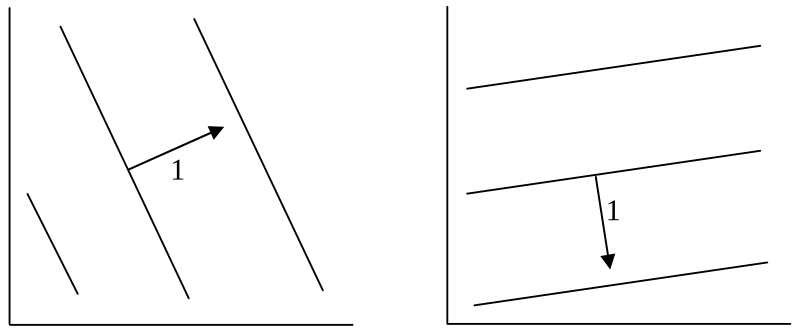

figure 1

The bundle at which the unit vector is based does not matter, given the linearity of the preferences. Its direction shows the direction of preference, so that two "opposite" linear preferences have the same indifference curves, but their unit vectors go in opposite directions. Since vectors are normalized to have unit length, there is no role for preference intensity or interpersonal comparisons.

Now imagine lifting the unit vectors from the diagrams in figure 1 and placing them in a circle of unit radius centered at the origin as shown in figure 2.

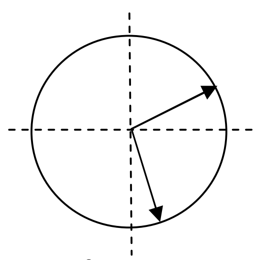

figure 2

Doing this for all possible linear preferences uniquely associates a linear preference with a point on a unit circle and vice versa. Thus, the set of points on S^1, the unit circle, may be taken as the set of all linear preferences.

Identifying linear preferences as points in S^1 is key to all that follows and understanding the simple mathematics of unit circles is therefore crucial. A brief review, easily skipped, covers the only essential prerequisites for understanding simple versions of the main results.[2]

[2] Knowledge of elementary operations on sets and functions between sets is assumed.

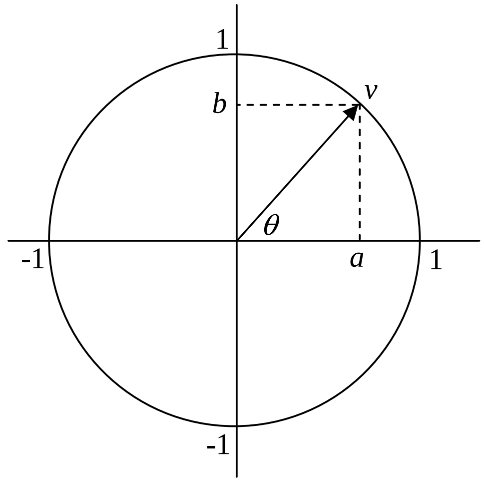

figure 3

Figure 3 shows points on the unit circle centered at the origin given by their Cartesian coordinates as follows: $S^1 = \{(a,b) \in \mathbb{R}^2 : (a^2 + b^2)^{1/2} = 1\}$. These points may also be considered as the set of unit vectors, *v*. Each such vector, *v*, is determined by the angle, θ, between the horizontal axis and *v*. It is often convenient to specify θ by its "circular", or polar, coordinate given by the length of the arc from point (1,0) to *v* in a counter clockwise, or positive, direction. Thus, $\theta = 0$ and $(a,b) = (1,0)$ both denote the same point. Cartesian and polar coordinates are related by: $a = \cos\theta$ and $b = \sin\theta$. Finally, any point in the unit circle may be thought of as a complex number $e^{\theta i} = \cos\theta + i\sin\theta$. This is particularly useful in considering rotations in the positive direction as θ increases from 0 to 2π, and in the negative, clockwise, direction as θ decreases from 0 to -2π. Usually, points in S^1 will be specified by their polar coordinate $\theta \in [0, 2\pi]$, so care should be taken to remember that $\theta = 0$ and $\theta = 2\pi$ specify the same point.

2.2 Chichilnisky's Impossibility Theorem

Consider the special case of 2 agents with linear preferences on $\mathbb{R}^2_+$. A *social welfare function* $f : S^1 \times S^1 \to S^1$ then aggregates agents' preferences $(\theta_1, \theta_2) \in S^1 \times S^1$ into a social preference $f(\theta_1, \theta_2) \in S^1$. Thus, a social welfare function is a function from a

subset of one Euclidean space to a subset of another Euclidean space and continuity is defined in the usual way for such functions. [3]

A simple example is given by a *constant function* that assigns the same $\theta^* \in S^1$ to all pairs $(\theta_1, \theta_2) \in S^1 \times S^1$ of agents' preferences. Another example is given by a social welfare function for which the social preference is the same as agent 1's preference as follows. For all $(\theta_1, \theta_2) \in S^1 \times S^1$, $f(\theta_1, \theta_2) = \theta_1$. This social welfare function is a *dictatorship* of agent 1. All constant and dictatorial social welfare functions are continuous so that the social preference does not jump from one point to another as agents' preferences change.

For a constant function, social preferences are completely insensitive to the preferences of all agents'. For a dictatorial social welfare function, social preferences are completely insensitive to the preferences of all agents' except the dictator. Thus, for constant and dictatorial social welfare functions, social preferences are insensitive to agents' preferences.

Two properties of social welfare functions rule out such insensitivities, namely Unanimity and Anonymity. A social welfare function is *Unanimous* (UN) if and only if the social preference is the same as any unanimously held agents' preference. That is, for all $\theta \in S^1$, $f(\theta, \theta) = \theta$. A social welfare function is *Anonymous* (AN) if and only if it is invariant to reassignments of preferences among agents. That is, for all $\theta, \theta' \in S^1$, $f(\theta, \theta') = f(\theta', \theta)$. Dictatorial social welfare functions are UN but not AN. Constant social welfare functions are AN but not UN. However, both are continuous.

The next example is both UN and AN, but not continuous. This example has the flavor of "averaging" agents' preferences. First, the social preference is defined for agents' preferences that are not given by opposite points on the circle. Thus, for preferences that are not opposites, let the social preference be given by the midpoint of the shortest arc between them, as shown in figure 4. For preferences that are not opposites, this social welfare function is continuous.

[3] Imagine agents pointing to a touch screen monitor with an additional monitor showing the social preference. If agents move their fingers around S^1 on their screens without taking the fingers off S^1, pointing to the changing social preferences would require keeping a finger, not only on the screen, but also on S^1.

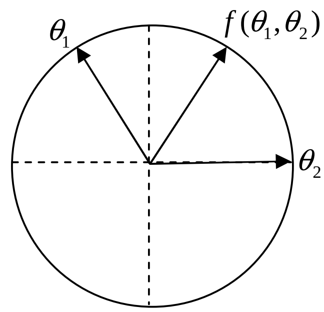

figure 4

It is for opposite preferences that continuity fails. However the social preference is defined for opposite preferences, there must be a discontinuity. To see this, hold θ_2 constant in figure 4 and let θ_1 rotate positively. As θ_1 approaches the point opposite θ_2, the social preference tends to the "North Pole", (0,1), or equivalently, $\pi / 2$. Immediately the rotation passes beyond the point opposite θ_2, the social preference jumps to a point close to (0, -1), or equivalently, $3\pi / 2$. Thus, this social welfare function is discontinuous at all opposite preferences. It is, however, UN and AN.

These examples suggest that it may be difficult to find a continuous social welfare function that is UN and AN. The seminal result in topological social choice theory establishes that such social welfare functions do not exist.

Theorem 1 (Chichilnisky (1979, 1980, 1982)): *There is no continuous social welfare function* $f : S^1 \times S^1 \to S^1$ that has the UN and AN properties.

The proof is postponed till after some mathematical tools have been developed. This result generalizes straightforwardly from circles S^1 to spheres S^m, $1 \le m < \infty$, and to any finite number of two or more agents. See Chichilnisky (1982), Mehta (1997), and Lauwers (2000, 2009).

2.3 Loops

This section offers an informal presentation of the main topological ideas required to prove Theorem 1.

For an arbitrary set *X*, a *path in X* is a continuous function $\alpha:[0,1]\to X$ from the unit interval to *X*. Figure 5 shows the image of two paths, α and β, in $\mathbb{R}^2_+$.

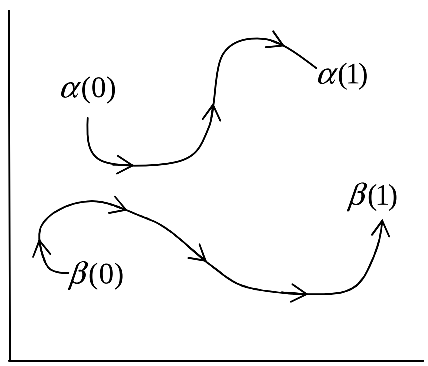

figure 5

Images of paths are orientated in the sense that they “travel” from an initial to a terminal point as shown by the arrows in figure 5. Indeed, reversing the arrows interchanges the initial and terminal points.

Two paths are *homotopic* if one can be continuously deformed into the other. Thus, in figure 5, α and β are homotopic. The homotopy relationship between paths is preserved by composition. That is, if paths $\alpha:[0,1]\to X$ and $\beta:[0,1]\to X$ are homotopic, and $\phi: X\to Y$ is a continuous function, *the compositions* $\phi\circ\alpha$ *and* $\phi\circ\beta$ *are also homotopic*.

A *loop in X* is a closed path in *X*, in the sense that its initial and terminal points are the same. In other words, a path $\alpha:[0,1]\to X$ is a loop in *X* if and only if $\alpha(0)=\alpha(1)$. It is also convenient to regard a loop in *X* equivalently as taking a circle continuously into *X*. That is, a loop in *X* is a continuous function $\alpha: S^1\to X$.

Since loops are paths, they may be homotopic, and figure 6 shows two homotopic loops in $\mathbb{R}^2_+$. Indeed, it is easy to see that all loops in $\mathbb{R}^2_+$ may be continuously deformed into each other and are therefore homotopic. This is not the case for sets with “holes” in them, such as the annulus in figure 7 given by the points bounded by two concentric circles. Indeed, loops β and γ are homotopic, but neither are homotopic to α.

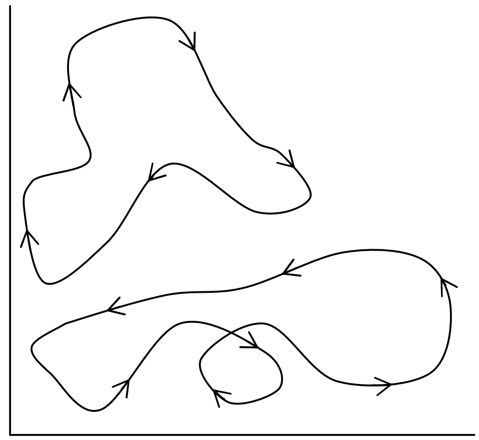

figure 6

figure 7

The reason for these homotopy relations between loops in an annulus is clear. Loop α does not go around the hole while loops β and γ go around the hole once. Note that loops may double back on themselves as shown by γ in figure 7. Even so, it goes around the hole counter clockwise exactly once, as does β. A key intuition is that loops in sets with one hole are homotopic if and only if they go around the hole the same net number of times. While loops in the unit circle will eventually be our main focus of attention, these remarks about loops in an annulus give key intuitions for important facts about loops in the unit circle.

Assume that the circle forming the outer boundary of the annulus in figure 7 is fixed, and imagine the inner circle increasing in radius. As its radius increases the loops are squeezed till eventually they are pressed onto the outer circle. They become loops in the circle, S^1. Squeezed onto S^1, loop α in figure 7 goes part of the way around S^1, but before completing a rotation it returns to its starting point. Loop β when squeezed onto S^1 completes one whole positive rotation as does γ, even though γ temporarily changes to move negatively.

Since loops in S^1 cannot be seen directly in diagrams of S^1, a different diagram is required, namely figure 8.

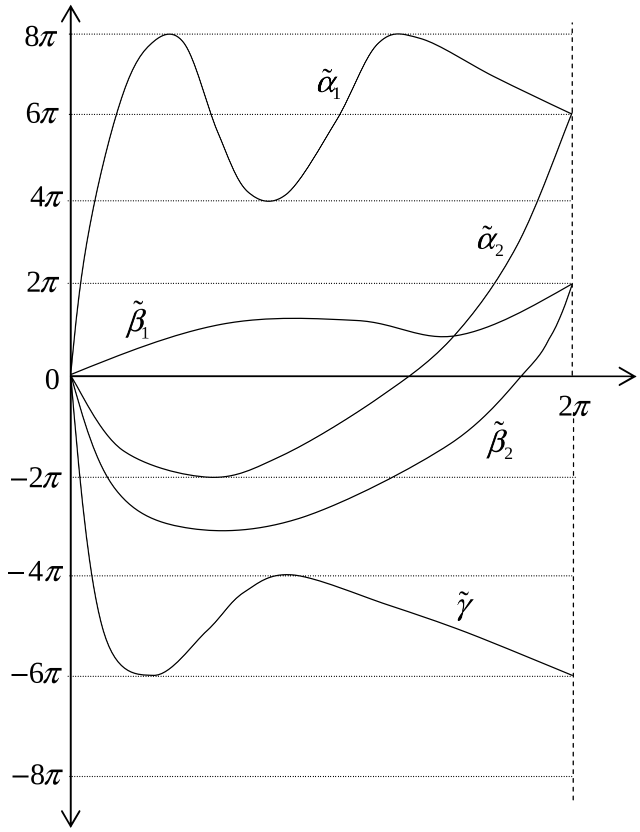

figure 8

Consider a circle S^1 centered at the origin of the plane, $(0,0) \in \mathbb{R}$. Figure 8 shows the behavior of six loops in S^1, taking $(1,0) \in S^1$ as an arbitrary starting point. The vertical axis shows for loop $\alpha : S^1 \to S^1$ how the image $\alpha(\theta) \in S^1$ behaves as θ increases from 0 to 2π along the horizontal axis. For the loop $\alpha_1 : S^1 \to S^1$, $\tilde{\alpha}_1$ shows on the vertical axis that the loop α in S^1 first makes 4 positive rotations, travelling a distance of 8π, then reverses to make 2 negative rotations, reversing again to make 2 positive rotations before finally making one positive rotation. The net number of complete counter clockwise rotations made by α is therefore 4-2+2-1=3. From $\tilde{\alpha}_2$ in figure 8, α_2 in S^1 also makes 3 net positive rotations, though it first makes 1 negative rotation followed by 4 positive rotations.[4]

[4] $\tilde{\alpha}$ is called the *lift* of α. See Armstrong (1983).

The definition of a loop requires that it ends where it begins, having the same initial and terminal points. Therefore, in figure 8, the value of all functions given on the vertical axis at 2π on the horizontal axis is either a positive even multiple of π for a positive net number of rotations or a negative even multiple of π for a negative net number of rotations, or zero. Indeed, calculating $\tilde{\alpha}_1(2\pi)/2\pi = 3$ gives the net number of rotations of α_1.

The net number of rotations of a loop in S^1 is known as its degree. In general, for any loop λ in S^1, represented by a function $\tilde{\lambda}$ as in figure 8, its degree, $\deg(\lambda)$ is given by: $\deg(\lambda) = \tilde{\lambda}(2\pi)/2\pi$. For the loops described in figure 8: $\deg(\alpha_1) = \deg(\alpha_2) = 3$, $\deg(\beta_1) = \deg(\beta_2) = 1$ and $\deg(\gamma) = -3$.

Looking at figure 8, it should be clear that loops having the same degree are homotopic and loops having different degrees are not homotopic. For example, α_1 can clearly be continuously deformed into α_2, but neither of these can be continuously deformed into β_1, β_2 or γ.[5] Preservation of degree by homotopy is a useful result.

Theorem 2: Let α and β be loops in S^1. Then α and β are homotopic if and only if $\deg(\alpha) = \deg(\beta)$.

One more construction concerning loops is required, namely the product of loops. The rough idea is that two loops may be joined together to form another loop. For example, figure 7 shows loops β and γ in $\mathbb{R}^2_+$. The image of the product, $\beta \cdot \gamma$, of β and γ would begin at one of the intersections of their individual images, go around the image of β first and then around the image of γ.

Consider again any loops α and β in S^1. The net number of times that their product, $\alpha \bullet \beta$ rotates positively in S^1 must be equal to the sum of the net number of positive

[5] Actually, figure 8 shows the possibilities for continuous deformations between the lift representations of loops used in the diagram, which are paths and not loops, rather than the possibilities for continuous deformations of the loops themselves. However, loops in S^1 are homotopic if and only if their lifts in figure 8 are homotopic.

rotations that each makes individually. After all, roughly speaking, their produce first "does what α does" and then "does what β does". In other words:

$\deg(\alpha \cdot \beta) = \deg(\alpha) + \deg(\beta)$: *The degree of a product of loops in* S^1 *is equal to the sum of their degrees.*

The key points for later use are:

- homotopy of paths is preserved by composition
- degrees of loops are preserved by homotopy (Theorem 2)
- degree of products of loops is equal to the sum of their degrees

2.4 Fundamental equation of topological social choice theory

This section begins by presenting particular loops and some relationships between them. These lead to the fundamental equation of topological social choice theory, an equation that plays a crucial role in all major results, including theorem 1.

The first loop, λ_1, is agent 1's loop and it shows the social preference as 1's preference makes a positive rotation. However, it is useful to consider this loop as a composition of another loop with the social welfare function. The intuition is given in figure 9.

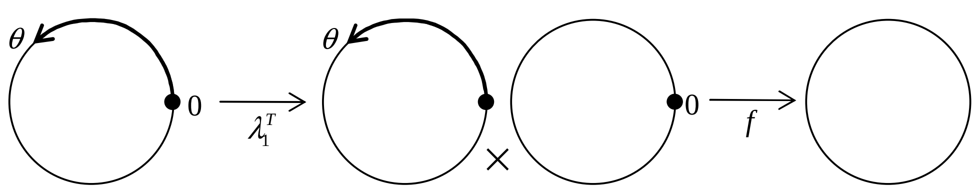

figure 9

Beginning at the left of figure 9, the first circle shows a positive rotation that determines the same positive rotation of 1's preference, shown in the left circle in the center of the diagram. The second of the two central circles shows 2's preference held constant at $\theta = 0$. Note that the far left and two central circles, show a loop in $S^1 \times S^1$. Each pair of points in the central circles is then taken by the social welfare function, $f : S^1 \times S^1 \to S^1$ into a social preference in the final circle on the right of the diagram.

Thus, 1's loop, λ_1, does indeed show the social preference for a positive rotation of 1's preference.

In the same way, agent 2's loop, λ_2, shows the social preference as 2's preference makes a positive rotation and it too may be considered as the composition of a loop in $S^1 \times S^1$ with the social welfare function. Figure 9 is easily adapted for this case by exchanging the roles of the two central circles.[6]

Considering this loop for constant and dictatorial social welfare functions may help to clarify the analysis. For a constant social welfare function, consider what happens in figure 9 as θ rotates in the circle on the far left. There is an identical rotation in the left of the two central circles while the point in the right of the central circles remains constant at 0. Finally, the social preference given in the circle on the far right of figure 9 remains constant at some point in S^1 that is determined by the social welfare function. For a dictatorial social welfare function, the only rotation that differs from the constant social welfare function just described is for the rotation in the far right circle in figure 9. This rotation is exactly the same as the rotation in the circle on the far left.

The next loop combines λ_1 and λ_2 into their product, $\lambda_{12} = \lambda_1 \bullet \lambda_2$. This loop shows the social preferences for sequential positive rotations of the preferences of agents 1 and 2. With some care, figure 9 may be adjusted for this case too. First, beginning on the left of the diagram as before, consider half a positive rotation of θ from 0 to π. During this half rotation, there is a positive rotation in the left of the two central circles. That is, 1's preference makes one complete positive rotation during half a positive rotation in the far left circle. As the rotation in the far left circle continues to complete a positive rotation, 2's preference in the right of the two central circles makes a complete positive rotation. The circle on the far right then shows the response of the social preference. Thus, the loop λ_{12} does indeed show the response of the social preference on the right to sequential positive rotations, first of agent 1's preference and then of agent 2's preference.

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[6] Of course, obvious notational changes are also required, namely changing λ_1^T to λ_2^T. Similar notational changes required by subsequent variations in figure 9 are left to the reader.

The loop λ_U, shows the response of the social preference to a unanimous positive rotation in both agents' preferences. In figure 9, as a complete positive rotation takes place in the first circle on the far left, the same rotation simultaneously takes place in both central circles. The circle on the far right then shows the response of the social preference to this unanimous positive rotation in agents' preferences.

The loops λ_{12} and λ_U have more than one thing in common. First, they both show the response of the social preference to complete positive rotations in agents' preferences. The only difference in the rotations in agents' preferences is that in λ_U the rotations are simultaneous while in λ_{12} they are sequential. The other thing they have in common is that they each compose a loop in $S^1 \times S^1$, shown by the left arrow in figure 9, with the social welfare function, shown by the right arrow in figure 9. Since the social welfare function is the same for both λ_{12} and λ_U, they therefore only differ because they use different loops in $S^1 \times S^1$.

By considering variations in these loops in $S^1 \times S^1$, it will now be shown that λ_{12} and λ_U can be continuously deformed into each other. That is, they are homotopic. In fact, an explicit homotopy will be constructed using a class of loops λ_δ^T. The intuitive idea is to let 1's preference rotate "faster" than 2's, so that it rotates ahead of 2's preference. The parameter, δ, $0 \le \delta \le 1$, determines the extent to which 1's preference rotates ahead of 2's. The loop λ_δ^T is given in the table.

The first row shows that as θ, given in the first column, goes through the first half of a positive rotation, agent 1's preference rotates ahead of it by $\delta\theta$ and the rotation of agent 2's preference lags behind it by $-\delta\theta$. This is also shown in figure 10. At the end of the first half of the rotation in column 1, row 2 shows $\theta = \pi$, $\theta_1 = (1+\delta)\pi$ and $\theta_2 = (1-\delta)\pi$. The second half of the rotation then follows and is given in the second row of the table. In the second row of the table, 2's rotation is faster then 1's and it just catches it as the rotation is completed. This can be seen by setting $\theta_1 = \theta_2 = 2\pi$ in the second row.

Table for λ_δ^T

θ	θ_1	θ_2
$0 \le \theta \le \pi$	$(1+\delta)\theta$	$(1-\delta)\theta$
$\pi \le \theta \le 2\pi$	$(1+\delta)\pi + (1-\delta)(\theta-\pi)$	$(1-\delta)\pi + (1+\delta)(\theta-\pi)$

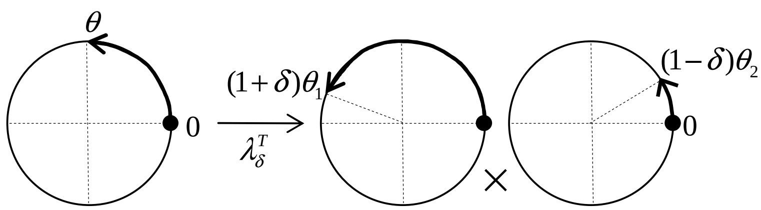

figure 10

To see that a homotopy has been constructed, consider λ_δ^T for $\delta = 0$ and $\delta = 1$. If $\delta = 0$, $\theta_1 = \theta_2 = \theta$ everywhere in the table and this is the loop λ_U^T of unanimous rotations. For $\delta = 1$, the first row of the table shows that $\theta_1 = 2\theta$ and 1's preference completes a positive rotation when $\theta = \pi$. Furthermore, 2's preference remains constant at $\theta_2 = 0$. Substituting $\delta = 1$ in the second row of the table shows 1's preference remaining constant at $2\pi = 0$ and 2's preference making a complete positive rotation. Thus, if $\delta = 1$, λ_δ^T specifies sequential positive rotations in the agents preferences, first for 1 and then for 2, just as required by λ_{12}^T. Indeed, as δ varies from 0 to 1, λ_δ^T continuously changes from λ_U^T to λ_{12}^T showing that λ_U^T and λ_{12}^T are homotopic.

This homotopy is also illustrated in figure 11. Column 1 of the table is given on the horizontal axis and columns 2 and 3 are shown on the vertical axis. The upper and lower thick lines with horizontal segments show the sequential rotations in preferences of 1 and 2 respectively. The diagonal shows the unanimous rotation. Dashed lines show the rotations for $0 < \delta < 1$, with the rotation of agent 1's preference above the diagonal and that of agent 2's preference below it. Clearly, the dashed lines show unanimous rotations continuously deforming into sequential rotations.

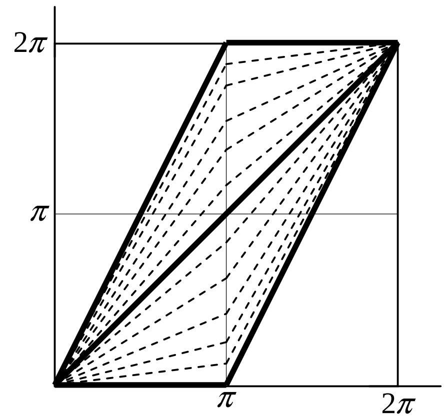

figure 11

Since λ_U^T and λ_{12}^T are homotopic, and taking their compositions with a continuous social welfare function gives λ_U and λ_{12}, it follows that λ_U and λ_{12} are also homotopic since homotopy is preserved by composition. From theorem 2 therefore, it follows that: $\deg(\lambda_U) = \deg(\lambda_{12})$. Substituting $\deg(\lambda_{12}) = \deg(\lambda_1) + \deg(\lambda_2)$ now gives the fundamental equation of topological social choice theory.

(1) $\qquad \deg(\lambda_U) = \deg(\lambda_1) + \deg(\lambda_2)$

It should be emphasized that equation 1 is an implication of continuity alone. Neither UN nor AN are used in its derivation.

2.5 Proof of theorem 1

Consider (1): $\deg(\lambda_U) = \deg(\lambda_1) + \deg(\lambda_2)$. UN and AN yield further restrictions on (1) that quickly lead to a contradiction.

UN requires that an unanimously held preference is also the social preference. Therefore, if agents' preferences make a unanimous (simultaneous) positive rotation, the social preference makes exactly one net positive rotation. That is, $\deg(\lambda_U) = 1$, which substituted in (1) gives: $1 = \deg(\lambda_1) + \deg(\lambda_2)$.

AN requires that the number of rotations made by the social preference is exactly the same in response to a rotation in agent 1's preference as it is to a rotation in agent 2's

preference. That is, $\deg(\lambda_1) = \deg(\lambda_2) = z$ for some integer *z*. Substituting into (1) now gives: $1 = 2z$.

However, there is no integer which, when doubled, is equal to 1. Thus, there is no continuous social welfare function that is unanimous and anonymous, and the proof is complete.

Two observations on this proof are worth making. One concerns the distinct roles of the properties. The other concerns the distinctive role played by information that is summarized by integers.

The fundamental equation of topological social choice theory given in (1) is a consequence of continuity alone. The proof proceeds by using the other properties, UN and AN, to obtain restrictions on the terms in (1). These are that the degree of λ_U is equal to one and the degrees of λ_1 and λ_2 are equal. Other major results will be proved in a similar way, by starting with (1) and then obtaining restrictions on it using other properties.

The terms in (1) all count the net number of preference rotations for specific loops. Therefore, these terms must all be integers. Indeed, the problem of proving theorem 1, as well as other results, can be reduced to an integer problem.[7] It is instructive to give a variation on the proof that makes this reduction explicit.

The first step is to associate the set $\mathbb{Z}$ of integers with loops in S^1 and the set of ordered pairs of integers $\mathbb{Z}\times\mathbb{Z}$ with loops in $S^1\times S^1$. Since degrees of loops are integers, it is no surprise that degrees of loops are used to establish this association. For example, associate with any loop λ in S^1, its degree, $\deg(\lambda)$. Given the homotopy preservation of degree, this integer is also associated with any loop that is homotopic to λ.

The loop λ_1^T in $S^1\times S^1$, in which 1's preference makes one positive rotation while 2's preference is constant, is associated with $(1,0)\in\mathbb{Z}\times\mathbb{Z}$. Similarly, λ_2^T is associated with $(0,1)\in\mathbb{Z}\times\mathbb{Z}$. Associating the loops λ_U^T and λ_{12}^T with ordered pairs in $\mathbb{Z}\times\mathbb{Z}$ will

[7] Lauwers (2009) is particularly successful in highlighting the "integer approach" that follows in this section. Indeed, his paper establishes some unity in topological social choice theory.

be particularly useful. The sequential positive rotations specified by λ_{12}^T are associated with the sum of the pairs in $\mathbb{Z}\times\mathbb{Z}$ associated with λ_1^T and λ_2^T, $(1,0)+(0,1)\in\mathbb{Z}\times\mathbb{Z}$. Finally, the simultaneous positive rotations in agents' preferences specified by λ_U^T is associated with $(1,1)\in\mathbb{Z}\times\mathbb{Z}$.

Given these associations of loops with points in $\mathbb{Z}\times\mathbb{Z}$, a function $f_*:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ may be associated with a continuous social welfare function, $f:S^1\times S^1\to S^1$. Setting aside some details, consider the following.

(i) $f_*\left((1,0)+(0,1)\right)$: This gives the number of rotations of social preferences in response to sequential positive rotations in agent 1's preference followed by a sequential positive rotation in agent 2's preference.

(ii) $f_*(2,0)$: This gives the number of rotations of social preferences in response to 2 positive rotations in agent 1's preference.

(iii) $2f_*(1,0)$: This is double number of rotations of social preferences in response to one positive rotation in agent 1's preference.

Now, very roughly, AN says that the response of social preferences to rotations in agents' preferences depends on the total number of rotations, not how that total is distributed among agents. It follows that the responses in social preferences in (i), (ii) and (iii) are all equal:

(2) $$f_*\left((1,0)+(0,1)\right)= f_*(2,0)=2f_*(1,0)$$

Since $(1,0)+(0,1)=(1,1)$, it follows from (2) that:

(3) $$f_*(1,1)=2f_*(1,0).$$

UN requires that $f_*(1,1)=1$, since the social preference must make one positive rotation in response to simultaneous rotations in agents' preferences. Therefore (3) becomes:

(4) $$1=2f_*(1,0)$$

Again, this is impossible since there is no integer $f_*(1,0)$ which, when multiplied by 2, is equal to 1.

3 Domain Restriction

This section presents a possibility result for continuous social welfare functions that are Unanimous and Anonymous. Unlike other areas of social choice, the literature in topological social choice has focused almost exclusively on domain variations as a way of escaping the impossibility in theorem 1. The leading result is in Chichilnisky and Heal (1983), where a general class of domains are considered.[8] A rather special case of this class restricts the domain of linear preferences and this is the topic of this section.

3.1 In figure 12, the center of the unit circle is no longer at the origin of $\mathbb{R}^2$. It has been shifted horizontally to the right. Assume that the linear preferences given by points in $\mathbb{R}^2$ to the left of the vertical axis cannot be the preferences of any agent. Thus, only points on the circle that are to the right of the vertical axis are subject to aggregation. These points will be called *admissible*.

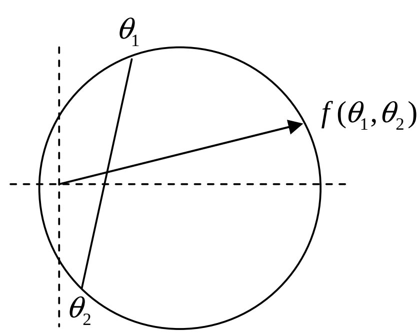

figure 12

To aggregate any pair of admissible points, θ_1 and θ_2, project the mid point of the line, or chord, between them from the origin of $\mathbb{R}^2$ onto the circle. If $\theta_1 = \theta_2 = \theta$, then this construction gives $f(\theta,\theta) = \theta$, so that UN is satisfied. Interchanging θ_1 and θ_2 does not change the chord used for the projection, and therefore AN is satisfied.

[8] Some members of this class are not domain restrictions and some are not easily interpreted as subsets of preferences.

Finally, this aggregation[9] is clearly continuous. On this restricted domain therefore, there is a continuous social welfare function that is Unanimous and Anonymous. Since this construction remains possible even if only a single point of the circle is not admissible, the minimal restriction may be very mild.

3.2 There is a topological property of a set that is responsible for the possibility result obtained by restricting the domain to a proper subset of S^1 that is one of its arcs. Intuitively, this property is that a set can be continuously shrunk to any of its points.

Consider the set of points on or inside a circle, known as a disk. Clearly, it is possible to continuously shrink the disk to any of its points. Indeed, the convexity of the disk may be used to obtain a very simple continuous shrinkage to a point. However, such continuous shrinkages may also be possible in non convex sets and the admissible arc in figure 12 is an example. This can be seen by imagining that the arc is made of string and lifted off the circle and laid on the horizontal axis. As such, it would coincide with an interval of real numbers and this is a convex set. Sets that can be continuously shrunk to any of its points are called *contractible*. Arcs are contractible, including an arc equal to a circle with a single point deleted.

Theorem 3: Let $C(S^1)$ denote an arc of the circle, S^1. Then there exists a continuous function $f: C(S^1)\times C(S^1) \to C(S^1)$ that has the UN and AN properties.

Before leaving this issue, it should be mentioned that the result in Chichilnisky and Heal (1983) shows that, for a general class of domains, contractibility is necessary and sufficient for the existence of continuous aggregations that are Unanimous and Anonymous. However, the use of this particular class is controversial and it is not clear that some of its members can be interpreted as preferences. See Lauwers (2000, 2009) for a discussion and further references.

[9] This aggregation may be given as follows. Let $C(S^1)$ denote the admissible subset of S^1. Then, for all $(\theta_1, \theta_2) \in C(S^1)\times C(S^1)$, $f(\theta_1, \theta_2) = (\theta_1 + \theta_2)/\|\theta_1 + \theta_2\|$.

4 Weak Pareto and No Veto

This section strengthens the Unanimity property of theorem 1 to a Pareto property, and replaces Anonymity with a No Veto property. With these variations on theorem 1, another impossibility result is obtained.

4.1 As before, consider commodity bundles in $\mathbb{R}^2_+$ of collective goods and agents with linear preferences on $\mathbb{R}^2_+$ given by unit vectors. The set $A = \{1, \dots, k\}$, $2 \le k < \infty$, denotes the set of agents[10]. A social welfare $f : (S^1)^k \to S^1$ now assigns a social preference $f(\theta_1, \dots, \theta_k) \in S^1$ to all k-tuples $(\theta_1, \dots, \theta_k) \in (S^1)^k$ of agents' preferences.

Consider figure 13. In the left diagram in figure 13, unbroken lines show indifference curves for two agents' linear preferences intersecting at bundle *y*, and their unit vectors. These are also shown in the center diagram. Dashed lines show indifference curves and unit vectors for different social preferences in left and center diagrams. All the unit vectors in the left and center diagrams are shown in the unit circle in the right diagram.

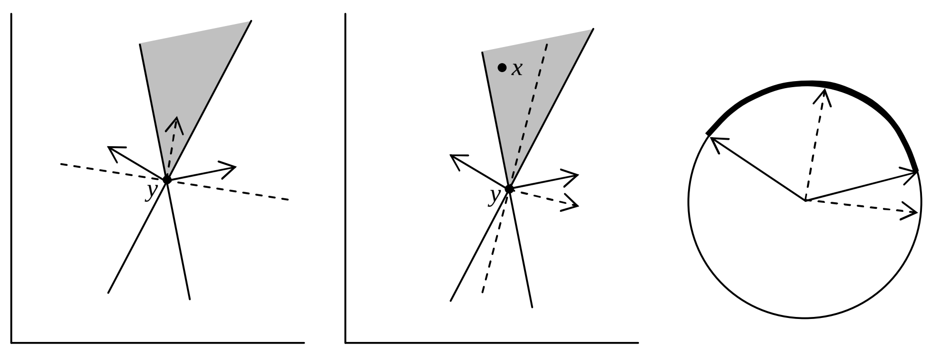

figure 13

In both the center and left diagrams in figure 13, bundles that are strictly preferred to *y* by both agents' preferences are shown by shaded areas. All these bundles are also strictly preferred to *y* by the social preference in the left diagram but not in the center diagram. This has implications for the unit vectors as follows. In the right diagram, an arc is determined by the intersection of the cone given by the unit vectors of agents' preferences and the circle. The unit vector of the social preference in the left diagram is contained in this arc, but the unit vector of the social preference in the center

[10] The proof of the theorem in this section is trivial if there are only 2 agents.

diagram is not. In general, all social preferences with unit vectors contained in the arc determined in this way by a pair of agents' preferences will strictly prefer all bundles to *y* that the agents' preferences do.

A "Cone property" may be now be formulated. A social welfare function $f:(S^1)^k \to S^1$ has the *Cone* property if and only iff, whenever agents preferences are given either by $\theta_1 \in S^1$ or $\theta_2 \in S^1$, the social preference is in the shortest arc between them. In case of unanimity, $\theta_1 = \theta_2 = \theta$, the arc is a single point and the social preference must be θ to satisfy the Cone property. Thus, the Cone property implies UN but not vice versa.

A little more notation is required for the Pareto property. For all $x, y \in \mathbb{R}^2_+$, all $i \in A$ and all $\theta_i \in S^1$, $xP(\theta_i)y$ will be written in case *x* is strictly preferred to *y* according to *i*'s linear preference given by θ_i. For any *k*-tuple of agents' preferences, $(\theta_1,\ldots,\theta_k) \in (S^1)^k$, $xP\left(f(\theta_1,\ldots,\theta_k)\right)y$ will be written in case *x* is strictly preferred to *y* by the social preference for this *k*-tuple of preferences.

The most commonly used Pareto Unanimity property is called the Weak Pareto property, following Arrow (1951). It requires that if all agents strictly prefer one bundle to another, so must the social preference. This may be expressed a little more formally as follows. A social welfare function $f:(S^1)^k \to S^1$ has the *Weak Pareto* (WP) property if and only iff, for all $(\theta_1,\ldots,\theta_k) \in (S^1)^k$ and all $x, y \in \mathbb{R}^2_+$: if $xP(\theta_i)y$ for all $i \in A$ then $xP\left(f(\theta_1,\ldots,\theta_k)\right)y$. If there are at most two distinct agents' preferences, as in the discussion of figure 13, WP imposes the same restriction on the social preference as the Cone property. However, WP imposes restrictions on the social preference in many other situations as well in which agents may more than two distinct preferences. Therefore, the WP property implies the Cone property, but not vice versa. In fact, theorem 3 below can be strengthened by replacing WP by the Cone property.

The final property required in this section requires situations in which the preference of one agent is the opposite of the preferences of all other agents. Thus, agents' preferences will be called polarized against agent *i* if all the other agents have exactly

the opposite preference that *i* has. A little more formally: $(\theta_1,\ldots,\theta_k)\in(S^1)^k$ is *Polarized against* $i\in A$ if and only if, for some $\theta\in S^1$, $\theta_j=\theta$ for all $j\in A\setminus\{i\}$ and $\theta_i=-\theta$.

Particularly for larger numbers of agents, it seems reasonable that in the case of polarization against an agent, the social preference should not be the same as the preference of the minority agent. That is, this prevents the social preference being the same as the minority agents' preference whenever this is unanimously opposed with its opposite preference. This property may be expressed as follows. A social welfare function $f:(S^1)^k\to S^1$ has the *No Veto* (NV) property if and only if, for all $i\in A$, all $(\theta_1,\ldots,\theta_k)\in(S^1)^k$ and all $\theta\in S^1$: if $(\theta_1,\ldots,\theta_k)$ is polarized against *i* and $\theta_i=-\theta$ then $f(\theta_1,\ldots,\theta_k)\neq-\theta$.

Theorem 4 may now be expressed as follows.

Theorem 4 (Chichilnisky (1982b): There is no continuous social welfare function $f:(S^1)^k\to S^1$ *that has the WP and NV properties.*

The proof may be easily understood using diagrams similar to the one given in figure 8, section 2.3. Consider unanimous agents' preferences and, for simplicity, let their unanimous preference be identified with the point $(1,0)\in S^1$ or, equivalently, $\theta=0\in S^1$. Now consider the loop λ_i defined as in section 2.4 showing the response of the social preference to a positive rotation in *i*'s preference, holding the preferences of all other agents constant. This is illustrated by the solid line in figure 14. As *i*'s preference rotates from $\theta_i=0$ to $\theta_i=2\pi$, shown on the horizontal axis, the social preference determined by the solid line is shown on the vertical axis.

The shaded areas in figure 14 show the restriction required by the Cone property which is implied by WP. For example, for $0<\theta_i<\pi$, the social preference must be between 0 and θ_i, and this gives the left shaded area. At $\theta_i=\pi$, *i* has a preference that is the opposite preference from all other agents. That is, agents are polarized against *i*.

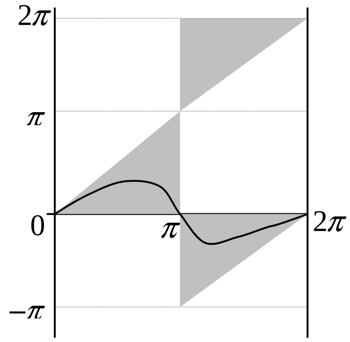

Figure 14

Therefore, at $\theta_i = \pi$, NV requires that the social preference is not the same as the preference of other agents. That is, the social preference is not equal to π. Continuity of the social welfare function then implies that the social preference in figure 14 enter the lower of the 2 shaded areas in the right of the diagram. To enter the higher shaded area, the social preference would have to be the same as that of *i*, namely π, violating NV. Therefore, for $\pi < \theta_i < 2\pi$, continuity of the social welfare function requires that the social preference is in the lower of the shaded areas on the right of figure 14. It follows that at $\theta_i = 2\pi$, the social preference must be 0.

To summarize, as *i*'s preference makes a positive rotation, the net number of times that the social preference rotates is 0. Indeed, in the case illustrated in figure 14, the social preference rotates positively at first, while *i*'s preference makes roughly a quarter of a rotation, after which it reverses direction and rotates negatively for nearly half a rotation before changing direction again and making roughly nearly a quarter of a positive rotation. The important point is that the social preference fails to make a complete rotation. In other words: $\deg(\lambda_i) = 0$. Of course, the argument used for *i* could be made for all other agents, so that $\deg(\lambda_i) = 0$ for all $i \in A$ and certainly $\deg(\lambda_1) + \ldots + \deg(\lambda_k) = 0$.

Since WP implies UN, and UN implies, $\deg(\lambda_U) = 1$, the fundamental equation of topological social choice theory, $\deg(\lambda_U) = \deg(\lambda_1) + \ldots + \deg(\lambda_k)$ therefore requires that $1 = 0$. This contradiction completes the proof.

5 Strategic Manipulators and Homotopic Dictators

This sector presents two closely related results. Both concern continuous social welfare functions that have the Weak Pareto property. Section 4 showed that this property, together with the No Veto property, leads to an impossibility result for continuous social welfare functions. This section shows that the only possibilities created by dropping the No Veto property have strategic manipulators who are also homotopic dictators. However, it is the former that is the more signficant.

5.1 Recall that continuity implies the fundamental equation of topological social choice theory:

(1) $$\deg(\lambda_U) = \deg(\lambda_1) + \ldots + \deg(\lambda_k)$$

Recall also that WP implies the Cone property and consider the implications of this using figure 15.

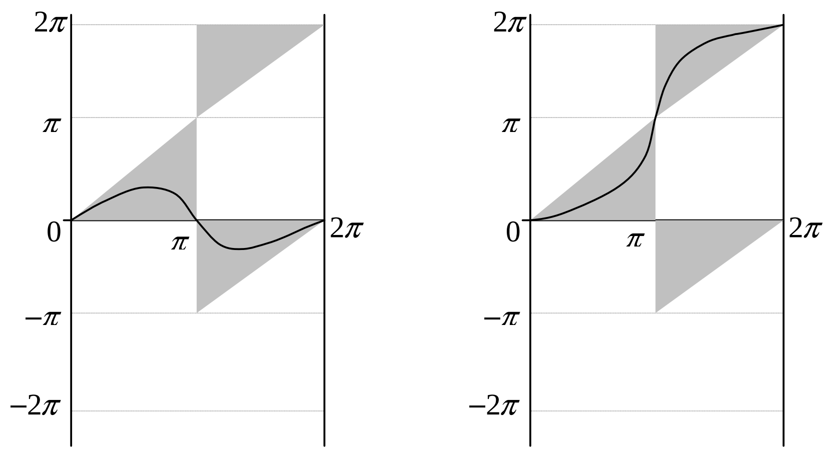

Figure 15

In figure 15 it is assumed that the linear preferences of all agents except $j \in A$ are given by $\theta = 0$ and j's linear preference makes a positive rotation as θ increases from 0 to 2π on the horizontal axis. The social preferences given by a continuous social welfare function for this rotation are shown via the solid lines on the vertical axes. That is, the solid lines are an illustration of the loop λ_j as defined in section 2.3 As in figure 14, section 4, the Cone property restricts the solid lines to the shaded areas.

For $0 \le \theta < \pi$ on the horizontal axis, there is a single shaded area. However, at $\theta = \pi$ a transition must be made to one of the two shaded areas on the right. The left diagram shows a transition to the lower shaded area. In this case, the social preference does not complete a rotation. That is, $\deg(\lambda_j) = 0$. In the right hand diagram, the transition is to the higher of the shaded areas and the social preference makes one positive rotation in response to a positive rotation in 1's preference. That is, $\deg(\lambda_j) = 1$.

Substituting $\deg(\lambda_U) = 1$ in equation (1): $1 = \deg(\lambda_1) + \ldots + \deg(\lambda_k)$. Since, as has just been shown, for all $i \in A$, $\deg(\lambda_i) = 0$ or $\deg(\lambda_j) = 1$, it follows that for exactly one agent *j*, $\deg(\lambda_j) = 1$, and for all other agents *i*, $\deg(\lambda_i) = 0$. In other words, the right diagram in figure 15 holds for one agent and it is the left diagram that holds for all the others.

Figure 16 repeats the right diagram in figure 15 and illustrates the sense in which that agent is favored by the social welfare function.

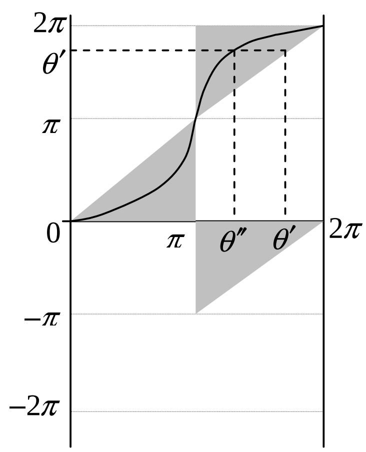

Figure 16

In figure 16, if the agent wants θ' to be the social preference the agent can achieve this by expressing θ''. Indeed, for all possible $\theta' \in S^1$ and $\theta \in S^1$, there exists a $\theta'' \in S^1$ such $f(\theta'', \theta) = \theta$. Such an agent is called a *Strategic Manipulator*. The following result has therefore been established.

Theorem 5 (Chichilnisky (1983, 1993): Let the continuous social welfare function $f:(S^1)^k \to S^1$ *have the WP property. Then there is a Strategic Manipulator.*

In other words, for a continuous Weakly Paretian social welfare function, there is one agent who can get whatever social preference they want, irrespective of the preferences expressed by other agents, and no other agents can do this. It may seem as though such an agent has some sort of “dictatorial” power. Indeed, consider figure 16 again. The solid line can be continuously deformed into the diagonal. The diagonal shows the case of a dictator in the sense that the social preference is always the same as that of the agent. Such an agent is called a *Homotopic Dictator*. See Chichilnisky (1982a).

That is, for continuous Weakly Paretian social welfare functions, there must be a Homotopic Dictator. The significance and interpretation of this result has attracted criticism in Baigent (2002), Baigent (2009), Saari (1997) and Saari and Kronwetter (2009). The issue is whether a homotopic dictator is favored in some way that it is distinct from being a Strategic Manipulator. In particular, are Homotopic Dictators necessarily objectionable in a way that is similar to dictators of the type formulated in Arrow (1950)? Critics do not think so and their argument is illustrated in figure 17.

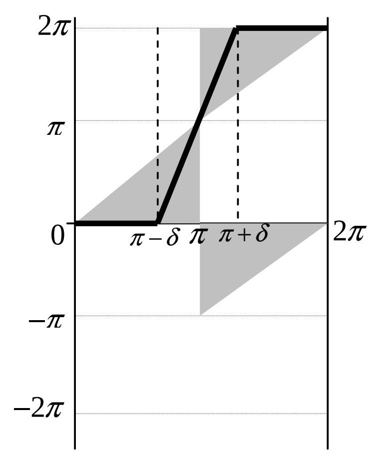

Figure 17

Assume for simplicity that there are only two agents, one of whom must be a Strategic Manipulator. Assume this agent is 1. The heavy solid line shows changes in social preference as 1’s preference makes a positive rotation. For θ close to 0, as the solid

line goes long the horizontal axis, the social preference remains the same as agent 2's preference. It does not respond at all to these changes in 1's preference. For 1's preference θ between $\pi-\delta$ and $\pi+\delta$, the social preference rotates faster than agent 1's preference till, at $\theta=\pi+\delta$, it completes a positive rotation. At this point the social preference is again equal to 2's preference. It remains equal to 2's preference for the remainder of 1's rotation. As δ becomes arbitrarily close to zero, the proportion of the domain on which the social preference is the same as 2's preference, can be made arbitrarily large.

All this may be summed up as follows. The agent that is not the Homotopic Dictator may have almost dictatorial power. Yet, the Homotopic Dictator can always get any particular preference to be the social preference but strategically manipulating.

Section 6 Continuity

A social welfare function is continuous if changes in social preference can be bounded to be arbitrarily small by taking sufficiently small changes in agents' preferences. In sections 2.1, "small" is defined using the Euclidean metric. [11] This property defines topological social choice theory and is therefore subjected to critical scrutiny in this section.[12] The main question is the following: What good reasons are there for imposing continuity on a social welfare function? Closely related issues regarding some of the limitations arising from the formulation of continuity are also briefly addressed.

6.1 Justification

The most striking thing about the justification of continuity is how little attention it has received. In Chichilnisky (1982), after claiming that continuity "can be argued to be a natural property", the following justification is offered:

"One reason for requiring continuity is that it is desirable for the social rule to be relatively insensitive to small changes in individual preferences. This makes mistakes in identifying

[11] More generally, a family T(X) of subsets of a set X containing itself and the empty set is a *topology* on X if and only if it contains arbitrary unions and finite intersects of its members. Members of T(X) are called *open* sets of this topology on X. One way, but not the only way, to define open sets is by using a metric as follows. A set is open if, for all its members, it also contains all elements sufficiently close to it. Given sets X and Y, both with topologies, a function from X to Y is *continuous* if and only if the inverse images of open sets in Y are open in X. Any book on topology will give further explanation.
[12] See also Lauwers (2000 and especially 2009).

preferences less crucial. It also permits one to approximate social preferences on the basis of a sample of individual preferences."

This justification assumes that social choice requires agents' preferences to be known, not just by agents' themselves, but by others. The reason for this is not clear. Certainly, this is not the case in many formulations of social choice problems. For example, in many descriptions of social choice problems, agents' preferences are widely regarded as private information, and the response to this in social choice theory has been design rather than approximation or estimation. Given this, it is not clear what role "mistakes", estimation or "approximation" could be. In the design approach, one role of a social welfare function is evaluative. That is, it establishes what the social choice ought to be for different agents' preferences. However, there is no requirement to establish agents' preferences. Rather, the problem is to design mechanisms such that, whatever agents' preferences happen to be, their strategic choices based on their preferences lead to the same outcomes that the social welfare function has determined ought to be chosen. There is no central planner in this account who needs to "find out" what agents' preferences are or who may make mistakes in establishing agents' preferences.

The point here is that if continuity is meant to solve some sort of practical problem, then an explicit description of a social choice problem in which the alleged problem arises would be a welcome addition to the literature.

This point may also be clarified by considering another much discussed discontinuity in the literature, namely the literature on Sen's Poverty Index. This well known function is not continuous, having a discontinuity at the poverty line. See Sen (1997, section A.6.4) where the issue of measurement errors is discussed. Sen writes: "In the context of *actual use* of poverty indicators, the possibility of measurement errors is indeed a legitimate and serious concern" (emphasis added). However, as argued in the previous paragraph, it is not clear that social welfare functions have an *actual use* in the same sense as poverty indices or any other sense in which mistakes, approximation or estimation arise. Social welfare functions have quite different uses for which it is not clear why continuity is desirable.

If, for some social choice problems, agents' preferences do need to be estimated, the emerging area of Spherical Statistics may seem promising. See, for example,

Jammalamadaka and SenGupta (2001). They define versions of the usual descriptive statistics for observations of “directions”, given by points on a unit circle. Such data may, for example, record the directions of birds migrating. The “averaging” example in section 2.1 is taken as the definition of the mean for such observations. However, for opposite points on the circle, they write: “We say that no mean direction exists” and, “In this case, it is clear that the data do not indicate any preferred or mean direction”. This response suggests that the estimation of social preferences may not be a straightforward exercise.

While some may claim that continuity “can be argued to be a natural property”, it is not clear what makes it natural. Indeed, in Lauwers (2009) it is observed that discontinuity is also a natural phenomenon. To this, it may be added that discontinuity is sometimes created by human beings where it does not exist in nature. For example, in most countries there are two contrived discontinuities every year when clock time changes between summer and winter time. Discontinuities in clock time also occur moving from one time zone to another. Presumably, some see advantages or benefits in these discontinuities since they are not imposed by nature. Such examples warn against any general presumption that discontinuities are undesirable.

Indeed, it may appear far fetched, but perhaps discontinuities in social welfare functions can serve a useful purpose. For example, the “averaging” social welfare function used as an example in section 2 has discontinuities exactly at points at which agents’ linear preferences are completely opposed. This fact may be exploitable in a number of ways. For example, instead of lamenting the loss of continuity whenever there is maximum disagreement, perhaps this should be taken as an opportunity to use non preference information. Another possible use is that further reflection or discussion is required for a preference to be regarded as representing agents’ preferences.

6.2 Limitations of continuity

Recall that continuity of a function requires images of points that are relatively close in its domain to be relatively close in its codomain. However, there are many contexts in which intuitions about closeness are either absent or are not clear and this limits the development of topological social choice theory.

Consider the diversity of the development of social choice theory following Arrow (1950 and 1951). Among other major areas, it led to characterization results that greatly enhanced understanding and appreciate of particular aggregation procedures. It also led to the study of aggregation procedures using interpersonal utility comparisons. A major development considered non binary social choice. In all of these cases, closeness intuitions are a lot less obvious than they are for the linear preferences required in topological social choice theory.

Consider for example, non binary social choice.[13] This area eventually led to assigning choice functions to agents' preferences rather than social preferences. If a choice function always selects single alternatives for agents' preferences and the alternatives are points in Euclidean space, then closeness may be based on clear intuitions. See Chichilnisky (1993) and Lauwers (2009). However, these intuitions quickly loose their clarity if several alternatives are given as acceptable choices for some agents' preferences, and this less restricted formulation is typical elsewhere in social choice theory. See Deb (2010).

The importance of assigning choice functions rather than preferences to agents' preferences was emphasized early in Buchanan (1954) and much later in Sen (1993). The lack of clear intuitions about closeness at least hinders raising these problems in topological social choice theory. Indeed, many of the developments that followed Arrow (1950) will remain beyond the limits of topological social choice theory unless clear intuitions about closeness of the relevant objects are developed.

Finally, it is worth emphasizing that the Kemeny metric for preferences on finite sets of alternatives does embody a clear intuition. This is that the more pairs of alternatives on which preferences disagree and the greater the extent of their disagreement on those pairs, the further they are apart. The use of this metric goes back at least to Dodgson and has been used for results in Arrow's framework that are in the same spirit as theorem 1.[14] Finally, in the non topological framework, there is no limitation on the analysis of non binary choice corresponding to the one in topological social choice. See Baigent (1997) for an example.

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[13] See Deb (2010) for a survey of non binary social choice.
[14] Baigent (1997),

7 Topological Proof of Arrow's theorem

In (1993), Baryshnikov presents a remarkable proof of Arrow's famous theorem that uses topological methods. Some of these have been developed in previous sections, but there is at least one additional concept required. In any case, the topological proof of Arrow's theorem is probably more challenging than the theorems presented earlier. The purpose of this section is to simplify the presentation of Baryshnikov's important contribution as much as possible. It deserves to be more widely appreciated.

7.1 Nerves[15]

The concept of the "nerve of a covering of a set" is the key concept that facilitates the transformation of Arrow's formulation so that topological arguments can be used. For simplicity, consider the set of all strict preferences[16] on the three alternatives, *x*, *y* and *z*. There are 6 such strict preferences as follows:

1	2	3	4	5	6
x	x	y	y	z	z
y	z	x	z	x	y
z	y	z	x	y	x

These preferences are identified subsequently by their numbers in this table.

Preferences 1, 2 and 5 are the only preferences for which *x* is strictly preferred to *y*. Let $(xy+)$ denote this subset of preferences. Likewise, let $(xy-)$ denote the subset of preferences for which *x* is ranked strictly below *y*.[17] The following table gives the subsets of preferences having a specific strict preference on each pair of alternatives:

$(xy+)$	$(yx+)$	$(yz+)$	$(zy+)$	$(xz+)$	$(zx+)$
1	3	1	2	1	4
2	4	3	5	2	5
5	6	4	6	3	6

[15] Slightly different presentations of the material in this section may be found in Baryshnikov (1997) and Lauwers (2000, 2009).
[16] Thus, agents' preferences are complete, reflexive and antisymmetric binary relations on the set of alternatives.
[17] Of course, $(xy+) = (yx-)$.

Note that the union of all the subsets of preferences contains the set of all strict preferences on *x*, *y* and *z*. That is, this collection of subsets of preferences *covers* the set of all strict preferences on these alternatives.

The Nerve of this cover is a geometrical object called a *simplical complex* constructed from the subsets of preferences given in the previous table. It consists of points called *vertices*, lines called *edges* and triangular sets of points called *faces*.[18] There are 6 vertices, each one associated with a subset of preferences given by a column in the previous table. A pair of vertices is connected by an edge if and only if their subsets of preferences have at least one preference in common. For example, $(xy+)$ and $(xz+)$ are connected by an edge since they have at least one preference in common. In this case, the intersection of $(xy+)$ and $(xz+)$ contains exactly preferences 1 and 2. See columns 1 and 5. However, vertices $(xy+)$ and $(yx+)$ have an empty intersection, having no preference in common. Thus, in the simplical complex under construction there is no edge connecting $(xy+)$ and $(yx+)$. Finally, the triple of vertices $(xy+)$, $(yz+)$ and $(xz+)$ has a non empty intersection shown by preference 1 in columns 1, 3 and 5 of the previous table. Thus, these three vertices give a face of the object under construction. However, there are two triples of vertices whose subsets of preferences have nothing in common. Therefore, there are no faces for these triples. Consideration of the columns in the previous table show these triples to be firstly, $(xy+)$, $(zx+)$ and $(yz+)$, and secondly, $(xz+)$, $(zy+)$ and $(yx+)$. In figure 19, these are the vertices of the two triangular holes. However, it is instructive to consider figure 18 first and then construct figure 19 from it.

Notation is simplified in the figures by writing *xy* instead of $(xy+)$ and *yx* instead of $(yx+)$, and similarly for subsets of preferences determined by a specific strict preference on other pairs of alternatives. In figure 18, edges connect vertices for pairs of subsets of preferences having at least one preference in common, and the triangles are shaded for vertices associated with 3 subsets of preferences having at least one preference in common. As already described therefore, there is an edge connecting *xy* and *xz*, since they have at least one preference in common, but there is no edge connecting *xy* and *yx* since no preference can be in both of these subsets. Preference 2

[18] It may help to take an early look at the figure that will be constructed. See figure 19.

is in the subsets associated with vertices *xy*, *zy* and *xz*, ensuring that, not only are each of the pair-wise vertices connected, but also that their triangle is shaded. However, there is no preference in all of the subsets for *xy*, *yx* and *xz*, so the triangle for these vertices is not shaded. The numbers in figure 18 identify the preferences that are in the intersections of the vertices that give the edges and faces. While edges determine subsets of either one or, in some cases two, preferences, faces determine a unique preference.

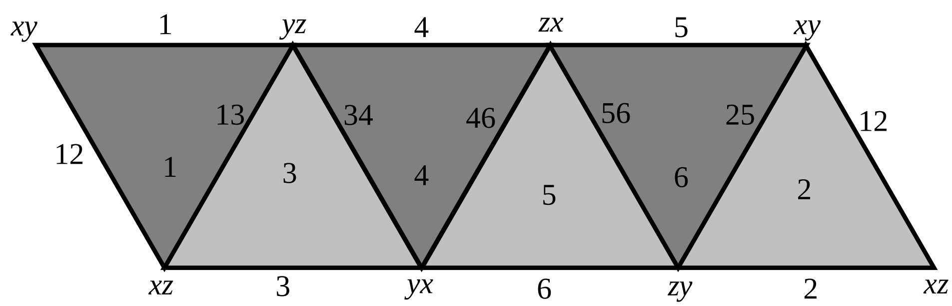

figure 18

To obtain the diagram in figure 19 from figure 18, note that the left and right edges in figure 18 are the same, both connecting *xy* with *xz*. It would be possible to cut out figure 18 and glue these two edges together while making folds along the edges that join all of the shaded triangles. The result is a band, or bracelet with a surface of triangular tiles. In fact, figure 19 illustrates an octahedron with two "opposite" faces deleted.

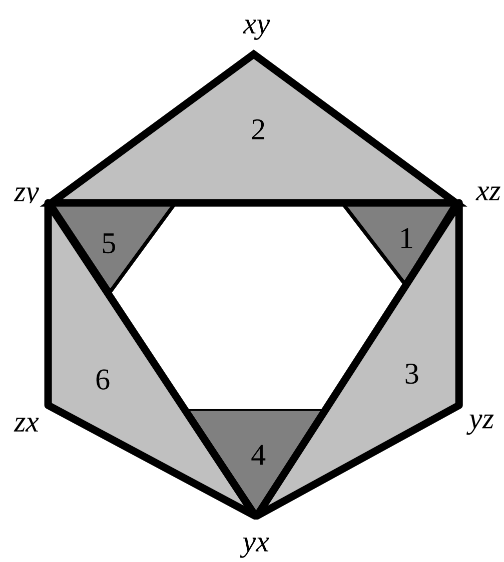

figure 19

The figure illustrated in figure 19 has been built up by joining some vertices and edges into triangles and then joining some triangles together along one of their edges as required by the definition of a *simplical complex*. The particular simplical complex shown in figure 19 is the *nerve* $N(\mathcal{P})$ *of* $\mathcal{P}$, where $\mathcal{P}$ denotes the set of all strict preferences on the alternatives. Baryshnikov's proof of Arrow's theorem also requires the nerve of a cover of another set, namely $\mathcal{P}\times\mathcal{P}$, the set of all pairs of strict preferences.

For any pair of alternatives, *x* and *y*, let $(xy++)$ denote the subset of pairs of strict preferences in $\mathcal{P}\times\mathcal{P}$ for which both agents strictly prefer *x* to *y*; let $(xy+-)$ denote the subset of pairs of strict preferences in $\mathcal{P}\times\mathcal{P}$ for which agent 1 strictly prefers *x* to *y* and agent 2 strictly prefers *y* to *x*; let $(xy-+)$ denote the pair of strict preferences in $\mathcal{P}\times\mathcal{P}$ for which agent 1 strictly prefers *y* to *x* and agent 2 strictly prefers *x* to *y*; and $(xy--)$ denotes the subset of pairs of strict preferences for which both agents strictly prefer *y* to *x*. A similar notation is used for the other pairs of alternatives, *y* and *z*, and *x* and *z*.

The union of all these subsets of ordered pairs of preferences contains the set of all ordered pairs of preferences. Thus, these subsets of ordered pairs of preferences cover the set $\mathcal{P}\times\mathcal{P}$. The previous method of constructing $N(\mathcal{P})$ is easily extended to give the nerve $N(\mathcal{P}\times\mathcal{P})$ of this cover of $\mathcal{P}\times\mathcal{P}$. Each of the subsets, $(xy++)$, $(xy+-)$, ... , $(xz-+)$ and $(xz--)$, is associated with a vertex. If any pair of these subsets have an ordered pair of preferences in common then they have an edge connecting them and so on for faces determined by more than two vertices. Since there are 4 vertices for each ordered pair of alternatives and 3 alternatives, there are 12 vertices.

In the following table, each row and column corresponds to the strict preferences shown. It follows that each cell corresponds to an ordered pair of preferences and the entries in the cells show which of the subsets of ordered pairs of preferences contains that pair of preferences. From this information, it is easy to determine all of the edges and faces of the nerve $N(\mathcal{P}\times\mathcal{P})$ of this cover of ordered pairs of preferences. For example, $(xy--)$ and $(xz++)$ are connected by an edge since both are present in at least one cell in the table. Since there are 3 vertices given in each cell of the table, the triangular faces are easily determined.

	x y z	x z y	y x z	y z x	z x y	z y x
x y z	$(x,y,+,+)$ $(y,z,+,+)$ $(x,z,+,+)$	$(x,y,+,+)$ $(y,z,+,-)$ $(x,z,+,+)$	$(x,y,+,-)$ $(y,z,+,+)$ $(x,z,+,+)$	$(x,y,+,-)$ $(y,z,+,+)$ $(x,z,+,-)$	$(x,y,+,+)$ $(y,z,+,-)$ $(x,z,+,-)$	$(x,y,+,-)$ $(y,z,+,-)$ $(x,z,+,-)$
x z y	$(x,y,+,+)$ $(y,z,-,+)$ $(x,z,+,+)$	$(x,y,+,+)$ $(y,z,-,-)$ $(x,z,+,+)$	$(x,y,+,-)$ $(y,z,-,+)$ $(x,z,+,+)$	$(x,y,+,-)$ $(y,z,-,+)$ $(x,z,+,-)$	$(x,y,+,+)$ $(y,z,-,-)$ $(x,z,+,-)$	$(x,y,+,-)$ $(y,z,-,-)$ $(x,z,+,-)$
y x z	$(x,y,-,+)$ $(y,z,+,+)$ $(x,z,+,+)$	$(x,y,-,+)$ $(y,z,+,-)$ $(x,z,+,+)$	$(x,y,-,-)$ $(y,z,+,+)$ $(x,z,+,+)$	$(x,y,-,-)$ $(y,z,+,+)$ $(x,z,+,-)$	$(x,y,-,+)$ $(y,z,+,-)$ $(x,z,+,-)$	$(x,y,-,-)$ $(y,z,+,-)$ $(x,z,+,-)$
y z x	$(x,y,-,+)$ $(y,z,+,+)$ $(x,z,-,+)$	$(x,y,-,+)$ $(y,z,+,-)$ $(x,z,-,+)$	$(x,y,-,-)$ $(y,z,+,+)$ $(x,z,-,+)$	$(x,y,-,-)$ $(y,z,+,+)$ $(x,z,-,-)$	$(x,y,-,+)$ $(y,z,+,-)$ $(x,z,-,-)$	$(x,y,-,-)$ $(y,z,+,-)$ $(x,z,-,-)$
z x y	$(x,y,+,+)$ $(y,z,-,+)$ $(x,z,-,+)$	$(x,y,+,+)$ $(y,z,-,-)$ $(x,z,-,+)$	$(x,y,+,-)$ $(y,z,-,+)$ $(x,z,-,+)$	$(x,y,+,-)$ $(y,z,-,+)$ $(x,z,-,-)$	$(x,y,+,+)$ $(y,z,-,-)$ $(x,z,-,-)$	$(x,y,+,-)$ $(y,z,-,-)$ $(x,z,-,-)$
z y x	$(x,y,-,+)$ $(y,z,-,+)$ $(x,z,-,+)$	$(x,y,-,+)$ $(y,z,-,-)$ $(x,z,-,+)$	$(x,y,-,-)$ $(x,z,-,+)$ $(x,z,-,+)$	$(x,y,-,-)$ $(x,z,-,+)$ $(x,z,-,-)$	$(x,y,-,+)$ $(y,z,-,-)$ $(x,z,-,-)$	$(x,y,-,-)$ $(y,z,-,-)$ $(x,z,-,-)$

$N(\mathcal{P})$ and $N(\mathcal{P}\times\mathcal{P})$, are the nerves required for Baryshnikov's proof of Arrow's theorem.

7.2 Loops in nerves

Loops in these nerves may be associated with integers and pairs of integers in the same way as for circles and products of circles in section 2.3. In figure 19 let a loop that makes a net number of counter clockwise (positive) rotations around $N(\mathcal{P})$ be associated with the integer 1. Similarly, a loop that does not complete a rotation is associated with the integer 0. The following examples will be used below.

Example 1: This loop goes from $(xz+)$ to $(zy+)$, then to $(yx+)$ before returning to $(xz+)$. This is the loop along to top edges in figure 18. Since it makes one positive rotation it is associated with the integer 1.

Example 2: This loop goes from $(xz+)$ to $(zy+)$, then to $(xy+)$ before returning to $(xz+)$. This loop does not make a complete rotation and it is therefore associated with the integer 0.

The next example is of a loop in $N(\mathcal{P}\times\mathcal{P})$.

Example 3: This loop goes from $(xz++)$ to $(zy++)$ and then to $(yx+-)$ before returning to $(xz++)$.

Since this loop is in $N(\mathcal{P}\times\mathcal{P})$, the sequence of vertices in the image of the loop are subsets of ordered pairs of preferences. However, this loop combines the loops in examples 1 and 2. Indeed, the only difference between the loops in examples 1 and 2 involves the alternatives *x* and *y* in which the first preferences strictly prefer *y* to *x* and the second preferences are the opposite. Therefore the pair of integers associated with the loop in example 3 is $(1,0)\in\mathbb{Z}\times\mathbb{Z}$. Other pairs of integers are associated with loops in $N(\mathcal{P}\times\mathcal{P})$ in a similar way. For example, $(1,1)\in\mathbb{Z}\times\mathbb{Z}$ is associated with a loop in $N(\mathcal{P}\times\mathcal{P})$ that combines both preferences making a loop like the one in example 1. These associations of loops with integers play a crucial role below.

7.3 Arrow's theorem

For simplicity, continue to consider a set of three alternatives *x*, *y* and *z*, and 2 agents with strict preferences on these alternatives. The social welfare function $f:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ assigns a social strict preference $f(R_1,R_2)$ to the strict preferences $R_1\in\mathcal{P}$ for agent 1 and $R_2\in\mathcal{P}$ for agent 2. The Weak Pareto property was introduced in section 4. It requires that the social preference on any pair of alternatives is the same as any unanimous agents' strict preference on that pair.

The property of Independence of Irrelevant Alternatives (IIA) is also required. IIA requires that the social preference on any pair of alternatives only depend on agents' preferences for that pair. That is, if $R_1\in\mathcal{P}$ ranks a pair of alternatives in the same way as $R_1'\in\mathcal{P}$, and $R_2\in\mathcal{P}$ ranks the same pair in the same way as $R_2'\in\mathcal{P}$, then IIA requires that the social preference $f(R_1,R_2)$ ranks this pair in the same way as the social preference $f(R_1',R_2')$. Social welfare functions that have the IIA property may be decomposed into pair-wise social welfare functions, each one aggregating agents'

rankings of a pair of alternatives.[19] It is IIA that enables the construction of a function $F:(N(\mathcal{P}\times\mathcal{P}))\to N(\mathcal{P})$ corresponding to a social welfare function, $f:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$.

To construct $F:(N(\mathcal{P}\times\mathcal{P}))\to N(\mathcal{P})$, start with the vertices in $N(\mathcal{P}\times\mathcal{P})$ and map them into the vertices of $N(\mathcal{P})$ using the social welfare function $f:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ as follows. Consider for example the vertex, $(xy+-)$. This is the set of all ordered pairs of preferences such that agent 1 strictly prefers *x* to *y* and agent 2 has the opposite strict preference. From IIA, the social preference on *x* and *y* must be same for all ordered pairs of preferences in $(xy+-)$ since agents 1 and 2 rank *x* and *y* in the same way in all of them. If the social preference strictly prefers *x* to *y*, then *F* maps $(xy+-)$ into $(xy+)$, the set of preferences for which *x* is strictly preferred to *y*. The assignment of vertices in $N(\mathcal{P})$ to vertices in $N(\mathcal{P}\times\mathcal{P})$ by $F:(N(\mathcal{P}\times\mathcal{P}))\to N(\mathcal{P})$ is done in the same way. It is worth emphasizing that the Weak Pareto property restricts the images of $(xy++)$ and $(xy--)$ to be $(xy+)$ and $(xy-)$ respectively, and similarly for the other two pairs of alternatives.

The next step is to extend this mapping of vertices to the edges and faces as well in a way that ensures $F:(N(\mathcal{P}\times\mathcal{P}))\to N(\mathcal{P})$ is continuous. This can be done in the following way; vertices are mapped into vertices; vertices that give edges or faces are mapped into vertices that also do this; and finally, the mapping on edges and faces is linear.[20] Such mappings are called *simplical* and they are well known to be continuous.

For Arrow's finite formulation of the social choice problem, all sets are finite and therefore there is no direct use for topological methods. However, the map $F:(N(\mathcal{P}\times\mathcal{P}))\to N(\mathcal{P})$ just obtained from $f:\mathcal{P}\times\mathcal{P}\to\mathcal{P}$ is a continuous mapping between subsets of Euclidean spaces. The final construction is to use the associations of loops in $N(\mathcal{P})$ and $N(\mathcal{P}\times\mathcal{P})$ with integers in $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$ respectively, to

[19] See Blau (1971)
[20] For example, a point in the interior of an edge is a convex combination of the vertices that give the edge and the image of that point must be the same convex combination of the images of these vertices. Similarly for interior points of faces. See Maunder (1996) for details.

associate a function $F_*:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ with $F:\left(N(\mathcal{P}\times\mathcal{P})\right)\to N(\mathcal{P})$. It is no accident that this strikingly resembles the association described in section 2.5.[21]

The function $F_*:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ will now be used to complete this sketch of the proof of Arrow's theorem. The three examples of loops described earlier, and the integers associated with them, will also be required. The sequences of vertices for these loops are:

Loop 1: $(xz+)$, $(zy+)$, $(yx+)$ and back to $(xz+)$

Loop 2: $(xz+)$, $(zy+)$, $(yx-)$ and back to $(xz+)$

Loop 3: $(xz++)$, $(zy++)$, $(yx+-)$ and back to $(xz++)$

Recall that loop 3 is associated with $(1,0)\in\mathbb{Z}\times\mathbb{Z}$. The values that the integer $F_*(1,0)$ may have can be determined by considering first $F(xz++)$, and $F(zy++)$. The Weak Pareto property implies $F(xz++)=(xz+)$, and $F(zy++)=(zy+)$. The two loops, 1 and 2, that are combined in loop 3, only differ for the pair *x* and *y* of alternatives. If $F(yx+-)=(yx+)$ then $F_*(1,0)=1$ and agent 1 is a dictator. If $F(yx+-)=(yx-)$ then $F_*(1,0)=0$ and $F_*(0,1)=1$, since $F_*(1,1)=F_*(1,0)+F_*(0,1)$, $F_*(0,1)=1$ and both terms on the right hand side are either 0 or 1. In this case, 2 is a dictator. Since either 1 or 2 must be a dictator, this completes the sketch of Baryshnikov's proof of Arrow's theorem.

Two observations may be made about Baryshnikov's contribution. First, it is unaffected by any of problems raised in section 6. Indeed, the way continuity of the function $F:\left(N(\mathcal{P}\times\mathcal{P})\right)\to N(\mathcal{P})$ is defined does not need to be intuitively appealing for the purpose of proving Arrow's theorem. Second, Baryshnikov's approach, via the nerves of covers, may also be useful for the analysis of domain restrictions as an escape from Arrow's impossibility. Lauwers (2009) contains some suggestive remarks about deleting faces from $N(\mathcal{P}\times\mathcal{P})$ to obtain the single peaked domain restriction. It may be useful to consider other domain restrictions in a similar way.

[21] The circle and $N(\mathcal{P})$ both have the same first fundamental groups, namely $\mathbb{Z}$ and the product of two circles and $N(\mathcal{P}\times\mathcal{P})$ also have the same fundamental group, namely $\mathbb{Z}\times\mathbb{Z}$. See Maunder (1980).

8 Conclusion

All of the theorems in sections 2, 3, 4 and 5 present the key theorems of topological social choice. It has been shown (theorem 1) that continuity is incompatible with Unanimity and Anonymity, though a very mild domain restriction, contractibility, leads to a possibility result (theorem 3). Theorem 4 weakens Unanimity to the Weak Pareto property or the even weaker Cone property, and shows that this is incompatible with a No Veto property for continuous social welfare functions (theorem 4). Theorem 5 shows that there exists a Strategic Manipulator for continuous social welfare functions that have the Weak Pareto property. These are the main results of topological social choice theory, apart from a very different kind of result, namely Baryshnikov's proof of Arrow's theorem.

All of the results except that of Baryshnikov require a justification of continuity. The discussion in section 6 argues that the justifications given in the literature are not completely convincing because they require a different formulation of the social choice problem than any given in the literature. Some remarks on the naturalness of continuity were discussed along with limitations that may account for the absence of developments similar to those that followed Arrow's theorem. Among these, the absence of a topological analysis of non binary social choice was discussed in more detail.

Overall, the view presented is that the justification of continuity requires more attention; its formulation to include non binary social choice would be an important contribution; and finally, Baryshnikov's proof of Arrow's theorem is a major achievement and may yet have other uses.

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