Propositionwise judgment aggregation: the general case

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Abstract

In the theory of judgment aggregation, it is known for which agendas of propositions it is possible to aggregate individual judgments into collective ones in accordance with the Arrow-inspired requirements of universal domain, collective rationality, unanimity preservation, non-dictatorship and propositionwise independence. But it is only partially known (e.g., only in the monotonic case) for which agendas it is possible to respect additional requirements, notably non-oligarchy, anonymity, no individual veto power, or implication preservation. We fully characterize the agendas for which there are such possibilities, thereby answering the most salient open questions about propositionwise judgment aggregation. Our results build on earlier results by Nehring and Puppe (2002), Nehring (2006), Dietrich and List (2007a) and Dokow and Holzman (2010a).

1 Introduction

Many democratically organized groups, such as electorates, legislatures, committees, juries and expert panels, are faced with the problem of *judgment aggregation*: They have to make collective judgments on certain propositions on the basis of the group members' individual judgments on them, for example on whether to pursue a particular policy proposal, to hold a defendant guilty, or to find that global warming poses a threat of a certain magnitude. In such cases, it is natural to expect that the group's judgment on any proposition should be determined by the individual members' judgments on it. Call this the idea of *propositionwise aggregation*, or technically, *independence*. This idea is naturally reflected in the way in which we normally make decisions in committee meetings, conduct referenda or take votes on issues we want to adjudicate collectively. Propositionwise aggregation can further be shown to be necessary for the non-manipulability of the decision process, both by strategic voting (Dietrich and List 2007b, see also Nehring and Puppe 2002) and by strategic agenda setting (Dietrich 2006a, List 2004). Yet the recent literature on judgment aggregation demonstrates that

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propositionwise aggregation is surprisingly hard to reconcile with the rationality of the resulting group judgments. A sequence of by-now much-discussed results (beginning with List and Pettit 2002, 2004) shows that, for many decision problems, only dictatorial or otherwise unattractive aggregation rules fulfil the requirement of propositionwise aggregation while also ensuring rational group judgments (for a review, see below and List and Puppe 2009; see also a 2010 symposium in *JET*). The classic illustration of what can go wrong is given by the *discursive dilemma* (Pettit 2001, building on Kornhauser and Sager 1986). If individual judgments are as shown in Table 1, for example, majority voting, the paradigmatic case of a propositionwise aggregation rule, generates logically inconsistent group judgments. The results in the literature on judgment aggregation have generalized this finding well beyond majority voting.

	a	$a \to b$	b
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
Majority	True	True	False

Table 1: A discursive dilemma

While this clearly highlights the need to find plausible aggregation rules that lift the restriction of propositionwise aggregation (and the literature already contains some work on this, as discussed at the end of this paper), there are still – surprisingly – a number of open questions on the classic, propositionwise case. The aim of this paper is to answer the most salient such questions. We prove new results on the existence of propositionwise aggregation rules which are non-oligarchic, anonymous, give no individual veto power, or are implication-preserving, as defined below.

To give a more careful overview of our results, it is helpful to review the most closely related existing results. We begin by introducing the classic background conditions imposed on propositionwise aggregation; formal definitions are given later. Call an aggregation rule *regular* if it accepts as admissible input all combinations of fully rational individual judgments (*universal domain*) and produces as its output fully rational collective judgments (*collective rationality*). Call it *unanimity-preserving* if, in the event that all individuals hold the same judgments on all propositions, these judgments become the collective ones. The case of regular, unanimity-preserving and propositionwise judgment aggregation is interesting since it naturally generalizes Arrow's famous conditions on preference aggregation to the context of judgment aggregation (List and Pettit 2004, Dietrich and List 2007a, Dokow and Holzman 2010a).

A much-cited result shows that, if (and only if) the decision problem under consideration, called the *agenda*, has two combinatorial properties, as defined below, the only judgment aggregation rules satisfying these conditions are the dictatorships (Dokow and Holzman 2010a; the 'if' part was independently obtained by Dietrich and List 2007a), which can be shown to generalize Arrow's classic theorem. This result, in turn, builds on an earlier, seminal result on abstract aggregation by Nehring and Puppe (2002).[1] Nehring and Puppe's result requires the aggregation rule to satisfy the further condition of *monotonicity*, according to which a proposition's collective acceptance is

[1] Nehring and Puppe's results were originally formulated in the context of strategy-proof social choice but are translatable into the frameworks of abstract aggregation as well as judgment aggregation in the

never reversed by increased individual support, but applies to a larger class of agendas with only one of the two combinatorial properties just mentioned. Another pair of results addresses the case in which the aggregation rule satisfies an additional *neutrality* condition, requiring equal treatment of all propositions. The conjunction of propositionwise independence and neutrality is called *systematicity*.[2] Here Dietrich and List (2007a) characterize the class of agendas for which only dictatorial aggregation rules are possible, while Nehring and Puppe's earlier paper (2002) provides the analogous characterization in the case in which monotonicity is required as well. Nehring and Puppe (2005) and Nehring (2006) also characterize the classes of agendas for which all regular, unanimity-preserving, propositionwise and monotonic aggregation rules are (i) oligarchic, (ii) oligarchic but non-dictatorial, (iii) give some individual veto power, (iv) violate anonymity, or (v) violate a requirement of neutrality between propositions and their negations.

With the exception of case (v), however, the analogous results without requiring monotonicity are not yet known (on case (v), see Dietrich and List forthcoming-a). The case without the monotonicity requirement, where aggregation rules can but need not be monotonic, is significant from both substantive and technical perspectives. Substantively, monotonicity is neither part of the standard 'package' of Arrovian conditions, nor is it included in the early impossibility theorems on judgment aggregation. Technically, a key tool for the generation of characterization results, namely Nehring and Puppe's so-called 'intersection property' (2002), is not available without requiring monotonicity, and thus the proof of characterization results without this requirement presents an important challenge. Turning to another issue, distinct from monotonicity, a further condition called *implication preservation*, which is inspired by recent work on probabilistic opinion pooling and strengthens unanimity preservation, has not yet been investigated at all in the context of judgment aggregation. Roughly speaking, implication preservation requires that if all individuals agree that a given proposition materially implies another, this agreement should be preserved collectively. This condition is strong, but of technical interest.

Table 2 summarizes what is and is not known on propositionwise aggregation. The table leaves out some early notable non-characterization results (including List and Pettit 2002, Pauly and van Hees 2006, Dietrich 2006a and Mongin 2008) and some results on truth-functional agendas (e.g., Nehring and Puppe 2008, Dokow and Holzman 2009a). The headings of the rows and columns indicate the conditions imposed on the aggregation rule, and the corresponding entries indicate the classes of agendas for which the given conditions are impossible to satisfy. By implication, for all agendas without the indicated properties, the conditions on the aggregation rule *can* be satisfied. The family of *blockedness* conditions – properly defined below – was first introduced in a related framework by Nehring and Puppe (2002).

The present paper fills the five blanks in Table 2, where there are still question marks. In each case, we fully characterize the class of agendas for which the indicated conditions lead to an impossibility, which, as noted, simultaneously characterizes the class of agendas for which they can be met. We also obtain several subsidiary results.

present, logic-based sense. For a statement of the results in an abstract aggregation framework, see Nehring and Puppe (2010), which we also recommend to readers whenever we refer to their 2002 paper. The relationship between the various frameworks is discussed in List and Puppe (2009).

[2] This condition was introduced in List and Pettit's (2002) original impossibility result.

Conditions on an aggregation rule (in addition to regularity, unanimity preservation & prop'wise aggregation)	**Monotonicity not required**	**Monotonicity required**
Non-dictatorship	Totally blocked & even-number negatable (Dokow&Holzman 2010a, sufficiency also in Dietrich&List 2007a)	Totally blocked (Nehring&Puppe 2002)
Non-dictatorship & neutrality between prop's and negations	Non-simple & even-number negatable & non-separable (Dietrich&List 2007a)	Non-simple & non-separable (Nehring&Puppe 2005)
Non-dictatorship & neutrality	Non-simple & even-number negatable (Dietrich&List forthcoming-a)	Non-simple (Nehring&Puppe 2002)
Non-oligarchy	?	Semi-blocked & non-trivial (Nehring 2006)
Anonymity	?	Blocked (Nehring&Puppe 2005)
No veto power	?	Minimally blocked (Nehring&Puppe 2002)
Implication preservation	?	?

Table 2: Classes of agendas generating an impossibility

Surprisingly, while in all previously studied cases – i.e., the three non-dictatorship cases in Table 2 – the move from an impossibility result with the condition of monotonicity to one without it has required the addition of a further condition on the agenda (*even-number negatability*), all but one of our present results do not require this addition.

The paper is structured as follows. In Section 2, we introduce the formal model, following List and Pettit (2002) and Dietrich (2007). In Section 3, we present our results in answer to the question marks in Table 2, devoting one subsection to each of our main results. Our last result (on implication preservation) covers two question marks at once. In Section 4, we give an overview of the logical relationships between the various classes of agendas, partially ordering them by inclusion, and draw some general lessons from our findings. All proofs are given in the Appendix.

2 Model

We consider a finite set of three or more individuals $N = \{1, ..., n\}$ faced with a judgment aggregation problem. The propositions on which judgments are made are represented in a suitable language **L**. A simple example is given by propositional logic. Here **L** consists of some 'atomic' propositions $a, b, c, ...$ and all 'compound' propositions constructible

from them using the connectives $\neg$ ('not'), $\wedge$ ('and'), $\vee$ ('or'), $\rightarrow$ (material 'if-then') etc., such as $a \wedge b$, $a \vee b$, $(a \wedge b) \rightarrow c$. Richer languages, which are often needed to express realistic decision problems, may also include quantifiers ('for all' and 'there exists') or non-truth-functional connectives (e.g., 'it is possible that', 'it is necessary that', 'if _ were the case, then _ would be the case').

Generally, a *language* $\mathbf{L}$ for our purposes is a set of sentences, called *propositions*, that is endowed with a *negation operator* $\neg$ and a notion of *consistency.* Both are well-behaved: The language is closed under negation (i.e., if $p \in \mathbf{L}$, then $\neg p \in \mathbf{L}$), and each set of propositions $S \subseteq \mathbf{L}$ is either *consistent* or *inconsistent* (but not both), subject to standard properties.[3] We say that a set $S \subseteq \mathbf{L}$ *entails* a proposition $p \in \mathbf{L}$, written $S \vdash p$, if $S \cup \{\neg p\}$ is inconsistent.[4]

A decision problem is represented by the *agenda* of propositions under consideration, defined as a non-empty set

$$X = \{p, \neg p : p \in X_+\},$$

where $X_+ \subseteq \mathbf{L}$ contains no propositions beginning with the negation operator $\neg$. In the example of Table 1, the agenda is

$$X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}.$$

We assume that X is finite and that every proposition $p \in X$ is *contingent*, i.e., $\{p\}$ and $\{\neg p\}$ are each consistent. We further assume that double negations cancel each other out.[5]

A *judgment set* is a subset $A \subseteq X$ of the agenda ('A' for set of 'accepted' propositions). It is called

- *complete* if it contains a member of each proposition-negation pair $p, \neg p \in X$; and
- *consistent* if it is a consistent set in $\mathbf{L}$.

Let $\mathbf{U}$ denote the set of all complete and consistent ('fully rational') judgment sets. A *profile* is an n-tuple $(A_1,...,A_n)$ of judgment sets across the individuals in N.

An *aggregation rule* is a function F that assigns to each profile of individual judgment sets $(A_1, ..., A_n)$ from some non-empty domain of admissible profiles a resulting collective judgment set $A = F(A_1, ..., A_n) \subseteq X$. We restrict our attention to *regular* aggregation rules, defined as functions $F : \mathbf{U}^n \rightarrow \mathbf{U}$, which accept all profiles of complete and consistent individual judgment sets as admissible input (*universal domain*) and generate complete and consistent collective judgment sets as output (*collective rationality*).

3 Results

As a background to our results, we first recapitulate the analogue of Arrow's theorem in judgment aggregation. While the conditions of universal domain and collective ra-

[3] Firstly, every proposition-negation pair $\{p, \neg p\} \subseteq \mathbf{L}$ is inconsistent. Secondly, subsets of consistent sets $S \subseteq \mathbf{L}$ are still consistent. Thirdly, the empty set $\emptyset$ is consistent, and every consistent set $S \subseteq \mathbf{L}$ has a consistent superset $T \subseteq \mathbf{L}$ that contains a member of each proposition-negation pair $\{p, \neg p\} \subseteq \mathbf{L}$.

[4] Our model allows one to interpret consistency either semantically (as satisfiability) or syntactically (as non-derivability of a contradiction). Thus the derivative notion of entailment has either a semantic or a syntactic interpretation. In the former case the symbol '$\vDash$' is more common than our symbol '$\vdash$'.

[5] To be precise, for any $p \in X$, where p belongs to the proposition-negation pair $\{q, \neg q\} \subseteq \mathbf{L}$ (with $q \in X_+$), we write '$\neg p$' to refer to the other member of that pair. This ensures that $\neg p$ is still in X.

tionality satisfied by a regular judgment aggregation rule are the analogues of Arrow's equally named conditions, Arrow's conditions of independence of irrelevant alternatives, the weak Pareto principle and non-dictatorship have the following three analogues.

Propositionwise independence. For all $p \in X$ and all admissible profiles $(A_1, ..., A_n)$, $(A'_1, ..., A'_n)$, if $p \in A_i \Leftrightarrow p \in A'_i$ for all individuals i, then $p \in F(A_1, ..., A_n) \Leftrightarrow p \in F(A'_1, ..., A'_n)$.

Unanimity preservation. For all admissible unanimous profiles $(A, ..., A)$, we have $F(A, ..., A) = A$.

Non-dictatorship. There exists no individual $i \in N$ (a *dictator*) such that $F(A_1, ..., A_n) = A_i$ for every admissible profile $(A_1, ..., A_n)$.

Aggregation rules satisfying these conditions are respectively called *propositionwise*, *unanimity-preserving* and *non-dictatorial*. The regular judgment aggregation rules satisfying all three conditions are precisely the analogues of preference aggregation rules satisfying Arrow's classic conditions, i.e., social welfare functions. For which decision problems can we find such rules?

While Arrow's theorem tells us that in the case of preference aggregation there are such rules if and only if there are at most two alternatives (and, by implication, none once there are three or more alternatives), the necessary and sufficient conditions for the existence (or non-existence) of such rules in the case of judgment aggregation are more complicated. To introduce these conditions, we must begin with some preliminary terminology. We say that a proposition $p \in X$ *conditionally entails* another proposition $q \in X$, written $p \vdash^* q$, if

$$\{p\} \cup Y \vdash q \text{ for some set } Y \subseteq X \text{ consistent with } p \text{ and with } \neg q.$$

Further, for $p, q \in X$, we write $p \vdash\vdash^* q$ if

$$\text{there is a sequence of propositions } p_1,...,p_k \in X \text{ such that } p{=}p_1 \vdash^* p_2 \vdash^* ... \vdash^* p_k{=}q.$$

So $\vdash\vdash^*$ is the transitive closure of $\vdash^*$.

Definition 1 *An agenda X is* totally blocked *if, for all propositions $p, q \in X$, $p \vdash\vdash^* q$ (Nehring and Puppe 2002).*

Total blockedness requires that any two propositions in X can be linked by a path of conditional entailments. Accordingly, it is sometimes also called *path-connectedness*. To define the next condition, call a set of propositions $S \subseteq \mathbf{L}$ *minimal inconsistent* if it is inconsistent but all its subsets are consistent.

Definition 2 *An agenda X is* even-number negatable *if there is a minimal inconsistent set $Y \subseteq X$ with a subset $Z \subseteq Y$ of even size such that $(Y \backslash Z) \cup \{\neg p : p \in Z\}$ is consistent (Dietrich 2007, Dietrich and List 2007a).*

This condition could also be stated by replacing 'of even size' with 'of size two', as we note in the following remark.

Remark 1 *An equivalent statement of even-number negatability is the following: There is a minimal inconsistent set $Y \subseteq X$ with distinct elements $p, q \in Y$ such that $(Y \backslash \{p, q\}) \cup \{\neg p, \neg q\}$ is consistent.*

Even-number negatability requires that the agenda include a minimal inconsistent set that becomes consistent by negating some *even* number of its members (respectively, some *pair* of its members). Even-number-negatability is equivalent to Dokow and Holzman's (2010a) condition of *non-affineness*, which requires that the set of admissible $\{0,1\}$-evaluations ('truth-value assignments') over the proposition-negation pairs in X should not be an affine subspace of $\{0,1\}^{\frac{|X|}{2}}$. The agenda of our introductory discursive-dilemma example is even-number-negatable, but not totally blocked. By contrast, the so-called *preference agenda* – consisting of all binary ranking propositions of the form 'x is preferable to y' over three or more distinct alternatives x, y, z, ... and subject to the standard rationality constraints on preferences – is both even-number-negatable (Dietrich and List 2007a, Dokow and Holzman 2010a) and totally blocked (Nehring 2003). We are now in a position to state the analogue of Arrow's theorem.

Theorem 1 *If the agenda is totally blocked and even-number negatable, there exists no propositionwise, unanimity-preserving and non-dictatorial aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$. Otherwise there exist such rules.*

In this form, Theorem 1 was proved by Dokow and Holzman (2010a); the impossibility part was also proved by Dietrich and List (2007a). The result builds on an earlier theorem by Nehring and Puppe (2002), in which the aggregation rule is required to satisfy the additional condition of *monotonicity*, while the agenda condition of even-number negatability is not needed. Unlike Arrow's theorem, which shows that preference aggregation in accordance with Arrow's conditions is impossible for all but the most trivial decision problems (namely for all except binary decisions), its analogue in the case of judgment aggregation implies a significant possibility. After all, the conjunction of total blockedness and even-number negatability is quite demanding and violated by many decision problems discussed in the literature on judgment aggregation, including, as noted, the example of Table 1. However, the condition of non-dictatorship is arguably too weak to guarantee fully 'democratic' judgment aggregation in the ordinary sense of the term. In what follows, we consider three ways of strengthening the requirement of non-dictatorship – namely *non-oligarchy*, *anonymity* and *no individual veto power* – and finally one strengthening of unanimity preservation – namely *implication preservation*, thereby addressing all the question marks in Table 2.

3.1 Non-oligarchic aggregation

To introduce the condition of non-oligarchy, call an aggregation rule F *oligarchic* if there is a non-empty set $M \subseteq N$ (of *oligarchs*) and a judgment set $D \in \mathbf{U}$ (the *default*) such that, for all $p \in X$ and all admissible profiles $(A_1, ..., A_n)$,

$$p \in F(A_1, ..., A_n) \Leftrightarrow \begin{cases} p \in A_i \text{ for } \textit{all} \text{ oligarchs } i \in M & \text{if } p \in X \backslash D \\ p \in A_i \text{ for } \textit{some} \text{ oligarch } i \in M & \text{if } p \in D. \end{cases}$$

Under this notion of an oligarchy, first defined by Nehring and Puppe (2008), a group of oligarchs has the power to determine the overall collective judgment on any given

proposition p whenever they are unanimous on p and to force the group to revert to a default judgment on p whenever they disagree.[6] A dictatorship is the special case in which the set of oligarchs is singleton.

It is now reasonable to ask for which agendas we can find aggregation rules satisfying the previous conditions with non-dictatorship strengthened as follows.

Non-oligarchy. The aggregation rule F is not oligarchic.

Definition 3 *An agenda X is* semi-blocked *if, for all propositions $p, q \in X$, [$p \vdash\vdash^* q$ and $q \vdash\vdash^* p$] or [$p \vdash\vdash^* \neg q$ and $\neg q \vdash\vdash^* p$] (Nehring 2006).*

An example of an agenda satisfying this condition is the one in our introductory illustration of the discursive dilemma, i.e., $X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$.[7] To give just one example of the relevant conditional entailments within that agenda, notice that $a \vdash\vdash^* \neg b$ (since $a \vdash^* \neg(a \rightarrow b)$ with $Y = \{\neg b\}$ and $\neg(a \rightarrow b) \vdash^* \neg b$ with $Y = \varnothing$) and $\neg b \vdash\vdash^* a$ (since $\neg b \vdash^* \neg(a \rightarrow b)$ with $Y = \{a\}$ and $\neg(a \rightarrow b) \vdash^* a$ with $Y = \varnothing$). The following theorem applies.

Theorem 2 *If the agenda is semi-blocked and even-number negatable, there exists no propositionwise, unanimity-preserving and non-oligarchic aggregation rule $F : \mathbf{U}^n \rightarrow \mathbf{U}$. Otherwise there exist such rules.*

This theorem continues to hold if we impose the additional condition of monotonicity on the aggregation rule while weakening even-number negatability to the condition that the agenda is *non-trivial* (where an agenda is called *trivial* if it contains only a single proposition-negation pair up to logical equivalence between propositions). This monotonic variant was proved by Nehring (2006).[8] Note that the move from Nehring's result with the condition of monotonicity to the present result without it parallels the move from the existing non-dictatorship results with monotonicity to those without it (recall the three non-dictatorship cases in Table 2). In each of these cases, even-number negatability is essentially substituted for monotonicity. Interestingly, however, the following corollary, *as well as all of our subsequent results*, do not require the agenda condition of even-number negatability despite not imposing monotonicity. This shows that the central move by which previous impossibility results without monotonicity have been obtained, namely the substitution of even-number negatability for monotonicity (familiar from Dokow and Holzman's and Dietrich and List's works), does not generalize to other salient cases.

The announced corollary concerns the case in which non-dictatorship can be achieved but non-oligarchy cannot. The corollary is an immediate consequence of Theorems 1 and 2, together with the following lemma.

[6] Another notion of oligarchy, discussed in Gärdenfors (2006), Dietrich and List (2008) and Dokow and Holzman (forthcoming-b), defines $F(A_1, ..., A_n)$ as $\cap_{i \in M} A_i$, without any default judgments. An oligarchy in this sense typically generates incomplete collective judgments, whereas the one discussed in the present paper guarantees completeness.

[7] The claim that this agenda is semi-blocked requires that the connective $\rightarrow$ be interpreted as the material conditional, as in standard propositional logic. There are other, arguably more realistic interpretations of $\rightarrow$ under which the agenda is not semi-blocked. Thus our example can illustrate both the impossibility and the possibility part of the following theorem, depending on the interpretation of $\rightarrow$.

[8] The non-triviality condition is omitted in Nehring's statement of the result.

Lemma 1 *Every non-trivial agenda that is semi- but not totally blocked is even-number negatable.*

Corollary 1 *All propositionwise and unanimity-preserving aggregation rules $F : \mathbf{U}^n \to \mathbf{U}$ are oligarchic but not all are dictatorial if and only if the agenda is semi- but not totally blocked and non-trivial.*

An instance of an agenda that is semi- but not totally blocked is of course the one in our discursive-dilemma example. The corollary remains true if monotonicity is imposed as an additional condition on the aggregation rule.

3.2 Anonymous aggregation

The next condition to be investigated is anonymity, the requirement of equal treatment of all individuals.

Anonymity. For all admissible profiles $(A_1, ..., A_n)$, $(A'_1, ..., A'_n)$ which are permutations of each other, $F(A_1, ..., A_n) = F(A'_1, ..., A'_n)$.

For which agendas can we find anonymous propositionwise aggregation rules?

Definition 4 *An agenda X is* blocked *if it contains a proposition $p \in X$ such that $p \vdash\vdash^* \neg p$ and $\neg p \vdash\vdash^* p$ (Nehring and Puppe 2002).*

An example of an agenda that is blocked (but neither totally nor semi-blocked) is the one consisting of a, $a \wedge b$, $a \wedge \neg b$, $a \wedge c$ and their negations. It is easy to verify that $a \vdash\vdash^* \neg a$ and $\neg a \vdash\vdash^* a$ (but there is no conditional entailment from any other proposition to $a \wedge c$). The following theorem holds. Notice that, as in Corollary 1 above, the agenda condition of even-number negatability is not required, despite the absence of monotonicity.

Theorem 3 *Let n be even. If the agenda is blocked, there exists no propositionwise, unanimity-perserving and anonymous aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$; otherwise there exist such rules.*

The agenda in our introductory example can be used to illustrate the possibility part of this theorem, since it is not blocked; we can never find a sequence of conditional entailments from a proposition to its negation. Here a propositionwise, unanimity-preserving and anonymous aggregation rule is given by accepting a collectively if and only if it is accepted by all individuals and accepting each of $a \to b$ and b collectively if and only if it is accepted by at least one individual. Consistently with Theorem 2 above, this is an oligarchy with default $D = \{\neg a, a \to b, b\}$.

Theorem 3 remains valid if we add monotonicity as a condition on F. For an odd group size n, the agenda condition for the impossibility is not blockedness but a stronger and very complex condition; we spare the reader with the details. The result with monotonicity added, for both even and odd n, was proved by Nehring and Puppe (2002). Jointly with their result for odd n, we obtain the following corollary, which drops monotonicity from Nehring and Puppe's analogous result.

Corollary 2 *There exist propositionwise, unanimity-perserving and anonymous aggregation rules* $F : \mathbf{U}^n \to \mathbf{U}$ *for all group sizes* n *if and only if the agenda is not blocked.*

3.3 Aggregation without veto power

Note that oligarchic aggregation rules have the special property that all oligarchs have the power to veto (i.e., prevent) any collective judgment set other than the default one. Even anonymous aggregation rules do not automatically avoid the presence of such veto power: In fact, they may give veto power to *every* individual. Just consider the case of anonymous oligarchic rules, in which every individual is an oligarch. These observations suggest that it may be democratically appealing to require the absence of individual veto power.

No individual veto power. For all admissible profiles $(A_1, ..., A_n)$ in which $n-1$ individual judgment sets coincide, i.e., they are all equal to A, we have $F(A_1, ..., A_n){=}A$.

Informally, the condition of no individual veto power requires that no singleton or empty coalition can ever veto any judgment set. This simultaneously strengthens non-oligarchy (and thereby also non-dictatorship) and unanimity preservation. For which agendas can this condition be met? Unfortunately, the answer is that, for small group sizes, it can *never* be met, while, for sufficiently large group sizes, it can be met only for rather special agendas.

Definition 5 *An agenda* X *is* minimally blocked *if it contains at least two non-equivalent propositions* $p, q \in X$ *such that* $p \vdash\vdash^* q$ *and* $q \vdash\vdash^* p$ *(Nehring and Puppe 2002).*

As in the case of semi-blockedness, the agenda in our introductory discursive-dilemma example is also minimally blocked.[9] Again, we can obtain a general result without requiring the agenda to meet any additional conditions such as even-number negatability. An interesting feature of that result, unlike previous results, is the occurrence of bounds on the group size.

Theorem 4 *If the agenda is minimally blocked, there exists no propositionwise aggregation rule* $F : \mathbf{U}^n \to \mathbf{U}$ *without individual veto power. Otherwise there exist such rules if* $n \geq 2^{\frac{|X|}{2}-1}$ *('large groups') and no such rules if* $n \leq k_X$*, where* k_X *is the size of the largest minimal inconsistent subset of* X *('small groups').*

The theorem continues to hold if we impose the additional conditions of anonymity, monotonicity or unanimity preservation on F (the last condition already follows from no individual veto power). Are the bounds on the group size n in Theorem 4 tight or do the stated (im)possibilities hold even under weaker bounds? The following observation, proved in the Appendix, reinforces the limited possibility of propositionwise judgment aggregation without individual veto power.

[9] Again this requires a material interpretation of $\to$. If, on the other hand, we interpret $\to$ as a strict conditional (roughly speaking, $a \to b$ if and only if $a \vdash b$), the agenda in the example is not minimally blocked and hence falls under the possibility part of the next theorem.

Remark 2 *(a) The upper bound k_X is tight: For every $k > 1$, some agendas X with k_X=k lead to possibility for each group size n>k_X. (b) Any possible replacement of the lower bound $2^{\frac{|X|}{2}-1}$ (as a function of $|X|$) grows exponentially in the agenda size $|X|$.*

We can also simplify Theorem 4 in a way that requires no reference to any bounds.

Corollary 3 *There exist propositionwise aggregation rules $F : \mathbf{U}^n \to \mathbf{U}$ without individual veto power for all sufficiently large group sizes n if and only if the agenda is not minimally blocked.*

Like the theorem, this corollary remains true if anonymity, monotonicity or unanimity preservation are added as conditions on F. With the last two additions, the corollary yields Nehring and Puppe's result on aggregation without individual veto power (2002). Nehring and Puppe state their result as an equivalence between an aggregation possibility and the agenda condition of non-minimal-blockedness. The aggregation possibility must be read as holding *for sufficiently large n*, since the proof requires sufficiently large n. Small n implies impossibility by Theorem 4.

3.4 Implication-preserving aggregation

The conditions investigated so far – non-oligarchy, anonymity and no individual veto power – all strengthen the original condition of non-dictatorship. We have noted that the condition of no individual veto power strengthens unanimity preservation as well. We now turn to a condition that strengthens unanimity preservation alone, against the background of regular propositionwise aggregation. The condition is inspired by recent work on probabilistic opinion pooling (Dietrich and List 2007c).

Implication preservation. For all $p, q \in X$ and all admissible profiles $(A_1, ..., A_n)$, if $p \in A_i \Rightarrow q \in A_i$ for all individuals i, then $p \in F(A_1, ..., A_n) \Rightarrow q \in F(A_1, ..., A_n)$.

Informally, implication preservation requires that if in all individuals' judgments p materially implies q, then p also materially implies q in the collective judgment. If the language $\mathbf{L}$ contains the material conditional $\to$, this can also be expressed as the requirement that, if $A_i \vdash p \to q$ for all individuals i, then we also have $F(A_1, ..., A_n) \vdash p \to q$. Note that an aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$ satisfying implication preservation also satisfies unanimity preservation. (By taking $p = \neg q$ in the condition of implication preservation, we can see that unanimous individual judgments on each proposition must be preserved collectively.)

It turns out that implication-preserving propositionwise aggregation is possible only for an extremely restrictive class of agendas: the 'simple' ones.[10]

Definition 6 *An agenda X is* non-simple *if it has at least one minimal inconsistent subset of size greater than two (in short, if $k_X > 2$).*

Once more, the agenda of our initial example meets this condition; a minimal inconsistent subset of size three is $X = \{a, a \to b, \neg b\}$. In fact, every agenda in which logical interconnections extend beyond pairs of propositions is non-simple.

[10] In Nehring and Puppe's (2002) framework, simple agendas correspond to the 'median spaces'.

Theorem 5 *If the agenda is non-simple, there exists no propositionwise, implication-preserving and non-dictatorial aggregation rule* $F : \mathbf{U}^n \to \mathbf{U}$*. Otherwise there exist such rules.*

As this result shows, by strengthening unanimity preservation to implication preservation, we obtain an impossibility result that holds for most agendas – indeed, for *all* the agendas used in discursive-dilemma examples in the literature. Once again, the result requires no even-number negation condition on the agenda, despite not requiring monotonicity, but remains true if we add monotonicity as a condition on the aggregation rule. Thus Theorem 5 addresses the last two question marks in Table 2 above.

Interestingly, in the case of probabilistic opinion pooling, the directly analogous conditions on an aggregation rule (propositionwise independence, implication preservation and regularity) yield a characterization of linear averaging on the class of non-simple agendas (Dietrich and List 2007c), whereas in the present case of binary judgment aggregation, only degenerate such rules remain, namely dictatorial ones, which give zero weight to all except one individual. One might argue, therefore, that the present impossibility stems not necessarily from an undue strength of implication preservation (which is, after all, satisfied by a common class of aggregation rules in the probabilistic case), but from the informational limitations of binary judgments.[11]

4 Conclusion

We hope to have settled the most salient open questions concerning propositionwise aggregation. Our starting point has been the baseline case of propositionwise judgment aggregation in accordance with Arrow-inspired conditions. We have characterized the classes of agendas for which propositionwise judgment aggregation is possible under various strengthenings of these conditions, requiring, respectively, non-oligarchy, anonymity, no individual veto power and implication preservation. Table 3 summarizes our results. By superimposing Table 3 upon Table 2 above, we are able to fill all the gaps

Conditions on an aggregation rule (in addition to regularity, unanimity preservation & prop'wise aggregation)	**Monotonicity not required**	**Monotonicity required**
Non-oligarchy	Semi-blocked & even-number negatable	(see above)
Anonymity	Blocked	
No veto power	Minimally blocked	
Implication preservation	Non-simple	

Table 3: Classes of agendas generating an impossibility (summary of our results)

in the earlier table. Note that in the last three rows our results subsume the cases with and without requiring monotonicity. Here, unlike in previous results in the literature,

[11] In the probabilistic case, implication preservation is equivalent to *conditional zero-preservation*, the requirement that, for any $p, q \in X$, if all individuals unanimously assign a conditional probability of 0 to p given q, this assignment should be preserved collectively.

monotonicity makes no difference. Contrary to what one might have expected based on previous work, then, the substitution of even-number negatability for monotonicity is *not* the universal recipe for obtaining agenda characterization results without requiring monotonicity. In some cases, the move from a result with the requirement of monotonicity to one without it necessitates the introduction of the additional agenda condition of even-number negatability; in other cases it does not, as our present results show.

Given the large number of agenda conditions occurring in the literature on judgment aggregation and the present paper, as summarized in Tables 2 and 3, it is useful to clarify the logical relationships between the various conditions diagrammatically. Figure 1 partially orders these conditions and the resulting classes of agendas by inclu-

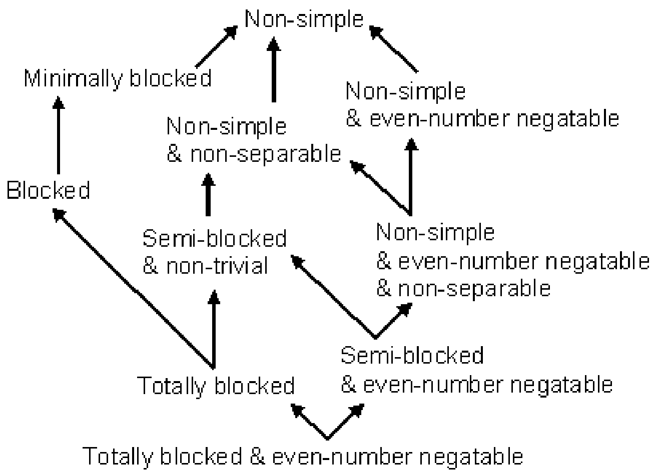

Figure 1: Logical relationships between different agenda conditions

sion. The strongest (most restrictive) condition is at the bottom, the weakest (most permissive) at the top.

What general lessons can we learn from the present results? It is clear that, with increasing strength of the conditions imposed on propositionwise aggregation, we are faced with increasingly general impossibility results, and the classes of agendas for which there remain possibilities become more and more restrictive. Given that genuinely 'democratic' judgment aggregation requires more than non-dictatorship alone, it is fair to conclude that, for many real-world decision problems, classic, propositionwise aggregation is not democratically feasible. This leaves us with three main solutions. We can either (i) relax some of the other Arrow-inspired conditions, notably universal domain and collective rationality, or (ii) search for alternatives to propositionwise aggregation, or (iii) move from binary judgments to more general propositional attitudes, such as non-binary or probabilistic ones, as already mentioned briefly above.

Relaxations of universal domain have been investigated by List (2003), Dietrich and List (2010a) and Pivato (forthcoming), relaxations of collective rationality by several contributions, including List and Pettit (2002), Dietrich and List (2007d, 2008, forthcoming-b), Gärdenfors (2006) and Dokow and Holzman (2010b). The literature also contains some work on aggregation rules that drop the restriction of propositionwise aggregation. Among the proposals investigated are the 'premise-based' aggregation

rules (Pettit 2001, List and Pettit 2002, Bovens and Rabinowicz 2006, Dietrich 2006a, Mongin 2008, Dietrich and Mongin 2010, building also on Kornhauser and Sager 1986), the 'sequential priority' rules (List 2004, Dietrich 2006b) and the 'distance-based' rules (Pigozzi 2006, Miller and Osherson 2009, building also on Konieczny and Pino-Perez 2002). Finally, extensions of the model of judgment aggregation to more general propositional attitudes, such as non-binary or probabilistic ones, have been offered by Dietrich and List (2007c, 2010b) and Dokow and Holzman (2009b), building also on earlier work on abstract aggregation (Rubinstein and Fishburn 1986) and probability aggregation (e.g., Genest and Zidek 1986).

Arguably, the further exploration of non-propositionwise aggregation and the systematic study of more general propositional attitudes are the biggest future challenges in the theory of judgment aggregation. We hope that, by settling the most salient open questions on classic propositionwise aggregation, the present paper inspires the literature to move on to these new challenges.

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A Appendix: proofs

General notation. For all $Z \subseteq Y$ $(\subseteq X)$ we write $Y_{\neg Z} := (Y \backslash Z) \cup \{\neg z : z \in Z\}$. Let $\equiv$ be the (equivalence) relation on X defined by $p \equiv q \Leftrightarrow [p \vdash\vdash^* q$ and $q \vdash\vdash^* p]$. Whenever we consider an aggregation rule F, we denote by $\mathcal{C}_p^F$ or simply $\mathcal{C}_p$ the set of coalitions $C \subseteq N$ that are *winning for* p $(\in X)$, i.e., for which $p \in F(A_1, ..., A_n)$ for all admissible profiles $(A_1, ..., A_n)$ with $\{i : p \in A_i\} = C$. (If F is propositionwise, it is uniquely determined by its family of winning coalitions $(\mathcal{C}_p)_{p \in X}$; if F is also unanimity-preserving resp. monotonic, each set $\mathcal{C}_p$ contains N resp. is closed under enlarging coalitions.)

Proof of Remark 1. We write EN for the agenda condition of even-number negatability, and EN_k for its variant in which 'of even size' is replaced by 'of size k'. We have to show that EN is equivalent to EN_2.

Clearly, EN_2 implies EN. Now suppose EN_2 is false. We have to show that EN is false, i.e., that all of EN_2, EN_4, EN_6, and so on, are false. We proceed by induction. (Closely related arguments are made by Dokow and Holzman 2010a.)

First, EN_2 is false by assumption. Now suppose that EN_2, EN_4, ..., EN_{2k} are all false (for a given $k \in \{1, 2, ...\}$). To show that $\text{EN}_{2(k+1)}$ is also false, consider any minimal inconsistent set $Y \subseteq X$ and any subset $Z \subseteq Y$ of size $2(k+1)$. Obviously, Z can be written as $Z = Z' \cup \{p, q\}$ for a set $Z' \subseteq Z$ of size $2k$ and $p, q \in Z$.

Claim 1. $Y_{\neg\{p,q\}}$ is minimal inconsistent.

As EN_2 is false, $Y_{\neg\{p,q\}}$ is inconsistent, so has a minimal inconsistent subset W; we have to show that $W = Y_{\neg\{p,q\}}$. W contains $\neg p$ as otherwise W is included in the (consistent) set $Y_{\neg\{q\}}$. Analogously, W contains $\neg q$. So we may write $W = W' \cup \{\neg p, \neg q\}$ for some $W' \subseteq Y \backslash \{p, q\}$. As EN_2 is false, the set $W' \cup \{p, q\}$ $(= W_{\neg\{\neg p, \neg q\}})$ is inconsistent. So, as $W' \cup \{p, q\}$ is included in the minimal inconsistent set Y, we have $W' \cup \{p, q\} = Y$. Hence, $W = Y_{\neg\{p,q\}}$.

Claim 2. $Y_{\neg Z}$ is inconsistent. (This completes the proof.)

We have $Y_{\neg Z} = (Y_{\neg\{p,q\}})_{\neg Z'}$. This set is inconsistent because EN_{2k} is false and because $Y_{\neg\{p,q\}}$ is minimal inconsistent (by Claim 1) and Z' has size $2k$. ■

A.1 Proof of Theorem 2 on non-oligarchic aggregation

To proof begins with two lemmas (the first of which is known[12]), in addition to Lemma 1 stated in the main text.

Lemma 2 *If the aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$ is propositionwise and unanimity-preserving, then $p \vdash^* q \Rightarrow \mathcal{C}_p \subseteq \mathcal{C}_q$ for all $p, q \in X$.*

Proof. Although known, we recall the simple argument. For F as specified, consider $p, q \in X$ with $p \vdash^* q$. Let $C \in \mathcal{C}_p$. By $p \vdash^* q$ there is $Y \subseteq X$ such that $Y \cup \{p, \neg q\}$ is inconsistent but $Y \cup \{p\}$ and $Y \cup \{\neg q\}$ are consistent. It follows that $Y \cup \{p, q\}$ and $Y \cup \{\neg p, \neg q\}$ are consistent. So, there is an $(A_1, ..., A_n) \in \mathbf{U}^n$ such that each A_i, $i \in C$, includes $Y \cup \{p, q\}$ and each A_i, $i \notin C$, includes $Y \cup \{\neg p, \neg q\}$. Now $F(A_1, ..., A_n)$ contains p (by $C \in \mathcal{C}_p$) and all $y \in Y$ (by $N \in \mathcal{C}_y$), hence it contains q (by $\{p\} \cup Y \vdash q$ and $F(A_1, ..., A_n) \in \mathbf{U}$). So, $C \in \mathcal{C}_q$ as F is propositionwise. ■

Lemma 3 *Every even-number negatable agenda is non-trivial.*

Proof. Let X be even-number negatable. Then there exists a minimal inconsistent $Y \subseteq X$ such that $Y_{\neg Z}$ is consistent for an even-sized $Z \subseteq Y$. So there are distinct $p, q \in Z$. Now X is non-trivial because p is not logically equivalent to q (otherwise Y would remain inconsistent after removing q) and not logically equivalent to $\neg q$ (otherwise $\{\neg p, \neg q\}$ would be inconsistent, violating the consistency of $Y_{\neg Z}$). ■

Proof of Lemma 1. Let X be non-trivial, semi-blocked and not totally blocked. As $\equiv$ is an equivalence relation, X is partitioned into equivalence classes. By assumption on X,

(i) there are exactly two $\equiv$-equivalence classes, each containing exactly one member of each pair $p, \neg p \in X$.

Moreover,

(ii) there is a minimal inconsistent $Y \subseteq X$ such that $|Y| \geq 3$,

since otherwise every conditional entailment in X is in fact an unconditional entailment, so that each $\equiv$-equivalence class consists of logically equivalent propositions, which by (i) implies that X is trivial, a contradiction. Further, one of the two $\equiv$-equivalence classes in (i) satisfies $p \nvdash^* q$ for all p in this class and all q in the other class, since otherwise $p \equiv q$ for p and q from different classes; hence,

(iii) some $\equiv$-equivalence class shares at most one element with each minimal inconsistent set $Y \subseteq X$.

The simple properties (i)-(iii) allow us to prove a key fact:

(iv) for every minimal inconsistent set $Y \subseteq X$, $Y_{\neg Z}$ is consistent for each non-empty subset $Z \subseteq Y$ of pairwise $\equiv$-equivalent propositions.

[12] See Dietrich and List (2007a) and Dokow and Holzman (forthcoming), and earlier Nehring and Puppe (2002), who also assume monotonicity.

To show this, let Y and and Z be as in (iv). If Z is singleton, $Y_{\neg Z}$ is obviously consistent (by Y's minimal inconsistency). Now let $|Z| \geq 2$. Suppose for a contradiction that $Y_{\neg Z}$ is inconsistent. Let Y' be a minimal inconsistent subset of $Y_{\neg Z}$. Let V be the $\equiv$-equivalence class with $Z \subseteq V$, and W the other $\equiv$-equivalence class. By $|Y \cap V| \geq 2$ and (iii), $|Y' \cap W| \leq 1$. So $|Y' \cap \{\neg z : z \in Z\}| \leq 1$ (as $\{\neg z : z \in Z\} \subseteq W$ by (i)). So $Y' \subseteq (Y \backslash Z) \cup \{\neg z\}$ for some $z \in Z$. But $(Y \backslash Z) \cup \{\neg z\}$ is consistent (by Y's minimal inconsistency). So Y' is consistent, a contradiction.

To complete the proof, let Y be as in (ii). By $|Y| \geq 3$ and (i), Y contains two distinct $\equiv$-equivalent p, q. So, by (iv), even-number negatability holds with this Y and with $Z := \{p, q\}$. ■

Proof of Theorem 2. We prove each direction of the implication.

1. First, suppose the agenda X is semi-blocked and even-number negatable. Let $F : \mathbf{U}^n \to \mathbf{U}$ be propositionwise and unanimity-preserving. We show that F is oligarchic.

Case 1: X is totally blocked. Then F is dictatorial (hence oligarchic) by Theorem 1.

Case 2: X is not totally blocked. So, as X is also non-trivial by Lemma 3, the assumptions of Lemma 1 are satisfied. Hence X has the properties (i)-(iv) shown in the proof of Lemma 1; we shall use some of these properties. Let $W \subseteq X$ be the $\equiv$-equivalence class in (iii), and $V := X \backslash W$ the only other $\equiv$-equivalence class (by (i)). Now

(v) there is a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$ such that $|Y \cap W| = 1$.

Suppose the contrary. Then $Y \cap W \neq \emptyset$ only for minimal inconsistent sets $Y \subseteq Y$ of size 2. So every conditional entailment $p \vdash^* q$ with $p \in W$ satisfies $p \vdash q$ and $q \in W$ (the latter since otherwise $\neg q \in W$, implying $|W \cap \{p, \neg q\}| = 2$). Hence the members of W are connected by paths of *un*conditional entailments, so are pairwise logically equivalent. So $V = X \backslash W$ $(= \{\neg w : w \in W\})$ also consists of pairwise logically equivalent propositions. Hence X is trivial, a contradiction by Lemma 2.

Let Y be as in (v). Let w be the element in $Y \cap W$, and v, v' two distinct elements in $Y \cap V$. By Lemma 2, the set of coalitions $\mathcal{C}_p$ is the same for all $p \in V$; call it $\mathcal{C}$. We now prove a first closure-property of $\mathcal{C}$:

(vi) $C, C' \in \mathcal{C} \Rightarrow C \cap C' \in \mathcal{C}$ (*intersection-closedness*).

Let $C, C' \in \mathcal{C}$. Each of the sets $Y_{\neg\{w\}}$, $Y_{\neg\{v'\}}$, $Y_{\neg\{v\}}$ and $Y_{\neg\{v,v'\}}$ is consistent (the first three by Y's minimal inconsistency, the fourth by (iv)). So, there is a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ such that

- $Y_{\neg\{w\}} \subseteq A_i$ for all $i \in C \cap C'$,
- $Y_{\neg\{v'\}} \subseteq A_i$ for all $i \in C \backslash C'$,
- $Y_{\neg\{v\}} \subseteq A_i$ for all $i \in C' \backslash C$,
- $Y_{\neg\{v,v'\}} \subseteq A_i$ for all $i \in N \backslash (C \cup C')$.

Now $F(A_1, ..., A_n)$ contains v since $C \in \mathcal{C}$ and $v \in V$, contains v' since $C' \in \mathcal{C}$ and $v' \in V$, and contains all $y \in Y \backslash \{v, v', w\}$ by unanimity preservation. In summary, $Y \backslash \{w\} \subseteq F(A_1, ..., A_n)$. So, as $Y \backslash \{w\} \vdash \neg w$, $F(A_1, ..., A_n)$ contains $\neg w$. Hence $C \cap C' \in \mathcal{C}_{\neg w}$, i.e., $C \cap C' \in \mathcal{C}$ (as $\neg w \in V$ by $w \in W$), as required.

Next, we prove a second closure property of $\mathcal{C}$:

(vii) $C \in \mathcal{C} \& C \subseteq C' \subseteq N \Rightarrow C' \in \mathcal{C}$ (*superset-closedness*).

Assume $C \in \mathcal{C} \& C \subseteq C' \subseteq N$. We distinguish two cases.

- First, suppose $Y_{\neg\{v,w\}}$ is consistent. Then there exists a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ in which
  - all $i \in C$ accept all propositions in $Y_{\neg\{w\}}$,
  - all $i \in C' \backslash C$ accept all propositions in $Y_{\neg\{v,w\}}$,
  - all $i \in N \backslash C'$ accept all propositions in $Y_{\neg\{v\}}$.

  $F(A_1, ..., A_n)$ contains v by $C \in \mathcal{C}$ and $v \in V$, and contains all $y \in Y \backslash \{v, w\}$ by unanimity preservation. In summary, $Y \backslash \{w\} \subseteq F(A_1, ..., A_n)$. So, by $Y \backslash \{w\} \vdash \neg w$, $\neg w \in F(A_1, ..., A_n)$. Hence $C' \in \mathcal{C}_{\neg w}$, i.e., $C' \in \mathcal{C}$ (by $\neg w \in V$), as required.
- Second, suppose $Y_{\neg\{v,w\}}$ is inconsistent. We consider a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ in which
  - all $i \in C$ accept all propositions in $Y_{\neg\{w\}}$,
  - all $i \in C' \backslash C$ accept all propositions in $Y_{\neg(Y \backslash \{v,w\})}$ (which is consistent by (iv)),
  - all $i \in N \backslash C$ accept all propositions in $Y_{\neg(Y \backslash \{w\})}$ (which is consistent, again by (iv)).

  $F(A_1, ..., A_n)$ contains $\neg w$ by $C \in \mathcal{C}$ and $\neg w \in V$, and contains all $y \in Y \backslash \{v, w\}$, again by $C \in \mathcal{C}$. In summary, $Y_{\neg\{w\}} \backslash \{v\} \subseteq F(A_1, ..., A_n)$. So, as $Y_{\neg\{w\}} \backslash \{v\} \vdash v$ (by the case-B assumption), $v \in F(A_1, ..., A_n)$. Hence, $C' \in \mathcal{C}_v$, i.e., $C' \in \mathcal{C}$ (by $v \in V$), as required.

By (vi) and (vii), $\mathcal{C} = \{C \subseteq N : M \subseteq C\}$ for $M = \cap_{C \in \mathcal{C}} C$, where $M \neq \emptyset$ by unanimity preservation. So F is oligarchic with default W and set of oligarchs M, which completes the impossibility proof.

2. Conversely, suppose the agenda X is *not* semi-blocked or *not* even-number negatable.

Case 1: X is non-trivial. If X is not semi-blocked, then by Nehring (2006) there exists a non-oligarchic aggregation rule satisfying all properties (and even monotonicity). If X is semi-blocked, then by assumption it is not even-number negatable (hence totally blocked by Lemma 1). So, the parity rule $F : \mathbf{U}^n \to \mathcal{P}(X)$ among any odd-sized subgroup $M \subseteq N$ with $|M| \geq 3$, defined by $F(A_1, ..., A_n) = \{p \in X : |\{i \in M : p \in A_i\}| \text{ is odd}\}$, has all properties: it is obviously propositionwise, non-oligarchic and (by oddness of $|M|$) unanimity-preserving, and it generates values in $\mathbf{U}$, as first shown by Dokow and Holzman (2010a).[13]

Case 2: X is trivial. Define $F : \mathbf{U}^n \to \mathcal{P}(X)$ as majority voting among a fixed subgroup $M \subseteq N$ of odd size with $|M| \geq 3$. F is obviously non-oligarchic, propositionwise and unanimity-preserving. Finally, as all minimal inconsistent sets $Y \subseteq X$ have size 2 by triviality, F generates sets in $\mathbf{U}$, as the following classic argument shows. For any $(A_1, ..., A_n) \in \mathbf{U}^n$, the set $A := F(A_1, ..., A_n)$ contains a member of each pair $p, \neg p \in X$ (as M is odd). If A were inconsistent, it would have a minimal inconsistent subset $Y \subseteq A$. We have $|Y| = 2$. So, as each $p \in Y$ is majority-accepted within M and as two majorities within M must overlap, some individual $i \in M$ has $A_i \subseteq Y$, contradicting A_i's consistency. ■

[13] More precisely, Dokow and Holzman show this not for even-number negatability but for an equivalent ('non-affineness') condition. For the proof with even-number negatability, see Dietrich (2007).

A.2 Proof of Theorem 3 on anonymous aggregation

Proof. Let n be even.

First, suppose the agenda is blocked. For a contradiction, let F be an aggregation rule with the required properties. By blockedness, there is a $p \in X$ such that $p \vdash\vdash^* \neg p$ and $\neg p \vdash\vdash^* p$. By Lemma 2, $\mathcal{C}_p = \mathcal{C}_{\neg p}$; call this set $\mathcal{C}$. As n is even, there is a $C \subseteq N$ with $|C| = |N \backslash C|$. Consider a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ in which p is accepted by all $i \in C$ and $\neg p$ by all $i \in N \backslash C$. Since by anonymity $C \in \mathcal{C} \Leftrightarrow N \backslash C \in \mathcal{C}$, either both or none of $p, \neg p$ are in $F(A_1, ..., A_n)$, a contradiction as $F(A_1, ..., A_n) \in \mathbf{U}$.

Conversely, if the agenda is not blocked, there exists an aggregation rule with the stated properties (and even with *monotonicity*), as shown by Nehring and Puppe (2002) who construct a particular (asymmetric) unanimity rule, i.e., an oligarchy with maximal set of oligarchs N. (The main part of their proof is to establish that there exists a judgment set $A \in \mathbf{U}$ with at most one element in common with any minimal inconsistent set $Y \subseteq X$; this set A serves as the default of the oligarchy.) ■

A.3 Proof of Theorem 4 on aggregation without individual veto power and of the tightness claims about inequalities

Proof of Theorem 4. Parts of the argument are adapted from Nehring and Puppe's (2002) proof of their veto power result.[14]

1. First, suppose X is minimally blocked. For a contradiction, suppose $F : \mathbf{U}^n \to \mathbf{U}$ is propositionwise and without individual veto power. By minimal blockedness, there are propositions $p_1, ..., p_k$, not all pairwise logically equivalent, such that $p_1 \vdash^* p_2 \vdash^* ... \vdash^* p_k \vdash^* p_1$. Among these conditional entailments there is one, say $r \vdash^* s$, that is not an unconditional entailment, i.e., such that $r \nvdash s$ (otherwise $p_1, ..., p_k$ would be pairwise logically equivalent). By $r \vdash^* s$ there is a $Y \subseteq X$ such that $Y \cup \{r, \neg s\}$ is inconsistent but $Y \cup \{r\}$ and $Y \cup \{\neg s\}$ are consistent. Hence each of $Y \cup \{r, s\}$ and $Y \cup \{\neg r, \neg s\}$ is also consistent. By $p_1 \vdash^* p_2 \vdash^* ... \vdash^* p_k \vdash^* p_1$ and Lemma 2, $\mathcal{C}_r = \mathcal{C}_s$. This set of winning coalitions – call it $\mathcal{C}$ – need not be closed under taking supersets (as F need not be monotonic), but it certainly contains all coalitions of size at least $n-1$ as F is without veto power. In particular, $\mathcal{C}$ is non-empty, hence contains a minimal member C (with respect to set inclusion). By $N \backslash C \notin \mathcal{C}_{\neg r}$ and $N \in \mathcal{C}_{\neg r}$ we have $C \neq \emptyset$. So there is an $i \in C$. Consider a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ in which

- individual i accepts all propositions in $\{r, \neg s\}$ (a consistent set by $r \nvdash s$),
- all individuals in $C \backslash \{i\}$ accept all propositions in $\{r, s\} \cup Y$,
- all individuals in $N \backslash C$ accept all propositions in $\{\neg r, \neg s\} \cup Y$.

Now $F(A_1, ..., A_n)$ contains r (as $C \in \mathcal{C}$), each $y \in Y$ (as coalitions of size at least $n-1$ are in $\mathcal{C}$), but not s (as $C \backslash \{i\} \notin \mathcal{C}$ by C's minimality). Hence, $\{r, \neg s\} \cup Y \subseteq F(A_1, ..., A_n)$, a contradiction as $\{r, \neg s\} \cup Y$ is inconsistent.

2. Next, suppose $n \leq k_X$. We show that there is no propositionwise $F : \mathbf{U}^n \to \mathbf{U}$ without individual veto power. For a contradiction, let F be such an aggregation rule. Consider a minimal inconsistent set $Y \subseteq X$ of maximal size. Then $|Y| \geq n$, and so Y has n pairwise distinct elements $p_1, ..., p_n$. By Y's minimal inconsistency, each set

[14] In particular, the aggregation rule constructed in case B of part 3 is a complicated variant of Nehring and Puppe's aggregation rule (which we could not have used here).

$Y_{\neg\{p_i\}}$ is consistent, and hence there is a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ such that $Y_{\neg\{p_i\}} \subseteq A_i$ for each $i \in N$. Now $F(A_1, ..., A_n)$ contains each $p \in Y$, since at least $n-1$ individuals accept p and F is without individual veto power. So $F(A_1, ..., A_n)$ is inconsistent, a contradiction.

3. Now suppose X is not minimally blocked and $n \geq 2^{K-1}$, where $K := |X|/2$. We construct an aggregation rule with the required properties. We may assume without loss of generality that X does not contain distinct but logically equivalent propositions.[15] As X is not minimally blocked and no two propositions are logically equivalent, $\vdash\vdash^*$ is an anti-symmetric relation on X. As $\vdash\vdash^*$ is also transitive, it is a partial order, hence can be extended to a linear order $\leq$ on X that satisfies

(*) $p \leq q \Leftrightarrow \neg q \leq \neg p$ for all $p, q \in X$,

by a standard type of argument (e.g., Duggan 1999): the set of partial orders extending $\vdash\vdash^*$ and satisfying (*) is non-empty (it contains $\vdash\vdash^*$) and closed under taking the union of any chain, hence by Zorn's Lemma contains a maximal element $\leq$, which can be shown to be complete, hence is a linear order. We partition X into the sets $X_<$ and $X_>$ containing the K lowest resp. K highest elements of X, and denote the members of $X_<$ by $p_1, ..., p_K$ in increasing order. We have

(**) $p_1 < ... < p_K < \neg p_K < ... < \neg p_1$ (hence $X_> = \{\neg p : p \in X_<\}$),

as can easily be derived from (*).

We distinguish two cases, A and B.

Case A: $X_<$ *is minimal inconsistent.* We begin by proving a claim.

Claim A1. $X_<$ is the only minimal inconsistent subset of X other than the trivial ones $\{p, \neg p\} \subseteq X$.

Let Y be a non-trivial minimal inconsistent subset. First, we have $|Y \cap X_>| \leq 1$, because if $Y \cap X_>$ had distinct members, say $\neg p_k, \neg p_l$, then $\neg p_l < p_k$ (by $\neg p_l \vdash^* p_k$) but $p_k < \neg p_l$ (as $p_k \in X_<$ and $\neg p_l \in X_>$), a contradiction. In fact, $Y \cap X_> = \emptyset$, by the following argument. Suppose the contrary. Then $Y \cap X_>$ is a singleton, say $\{\neg p_k\}$. The minimal inconsistent set Y does not equal $\{p_k, \neg p_k\}$ (by non-triviality of Y), hence does not contain p_k, hence is a subset of $(X_< \backslash \{p_k\}) \cup \{\neg p_k\}$, a contradiction since the latter set is consistent (by $X_<$'s minimal inconsistency). By $Y \cap X_> = \emptyset$ we have $Y \subseteq X_<$, hence $Y = X_<$ as $X_<$ is (like Y) minimal inconsistent. This completes the proof of Claim A1.

Define a family of thresholds $(m_p)_{p \in X}$ by

$$m_p = \begin{cases} n-1 & \text{if } p \in X_< \\ 2 & \text{if } p \in X_> \end{cases}$$

and consider the aggregation rule $F : \mathbf{U}^n \to \mathcal{P}(X)$ (a *quota rule*) given by

$$F(A_1, ..., A_n) := \{p \in X : |\{i : p \in A_i\}| \geq m_p\}.$$

[15] To see why, suppose the existence proof is done for such agendas X, and now let X be arbitrary. Call two proposition-negation pairs $\{p, \neg p\}, \{q, \neg q\} \subseteq X$ equivalent if p is equivalent to q (hence $\neg p$ to $\neg q$) or p is equivalent to $\neg q$ (hence $\neg p$ to q). This defines an equivalence relation. Consider a (sub)agenda $\tilde{X} \subseteq X$ that includes exactly one pair $\{p, \neg p\}$ from each equivalence class. Clearly, $\tilde{X}$ contains no distinct but logically equivalent propositions, so that there exists an aggregation rule $\tilde{F} : \tilde{\mathbf{U}}^n \to \tilde{\mathbf{U}}$ for $\tilde{X}$ of the required form. $\tilde{F}$ induces an aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$ for X by identifying each $\tilde{A} \in \tilde{\mathbf{U}}$ with the unique $A \in \mathbf{U}$ satisfying $A \supseteq \tilde{A}$. As the reader can check, F inherits from F the required properties, namely propositionwise independence and no individual veto power.

As F is obviously propositionwise and without individual veto power, it remains to prove the following claim.

Claim A2. F generates complete and consistent judgment sets.

Completeness holds because $m_p + m_{\neg p} \leq n+1$ for all $p \in X$ (in fact, with equality). Consistency is equivalent to the system of inequalities

$$\sum_{y \in Y} m_y > n(|Y|-1) \text{ for every minimal inconsistent set } Y \subseteq X, \tag{1}$$

by (the anonymous case of) Nehring and Puppe's (2002) 'intersection property' result.[16] By Claim A1, the system (1) reduces to the single inequality $\sum_{p \in X_<}(n-1) > n(K-1)$, hence to $K(n-1) > n(K-1)$, i.e., to $n > K$. If $K \leq 2$ the latter holds because $n \geq 3$. If $K \geq 3$ it holds by $n \geq 2^{K-1} > K$. This completes the proof of Claim A2.

Case B: $X_<$ *is not minimal inconsistent.* Redefine the family of thresholds $(m_p)_{p \in X}$ as

$$\begin{aligned} m_{p_k} &= \begin{cases} n-1 & \text{for } k=1 \\ n-2^{k-2} & \text{for all } k \in \{2,...,K\}, \end{cases} \\ m_{\neg p_k} &= n+1-m_{p_k} \text{ for all } k \in \{1,...,K\}, \end{aligned}$$

which generates a quota rule $F : \mathbf{U}^n \to \mathcal{P}(X)$ defined by

$$F(A_1,...,A_n) = \{p \in X : |\{i : p \in A_i\}| \geq m_p\}.$$

As F is obviously propositionwise, the proof is completed by proving the following two claims.

Claim B1. F is without individual veto power.

It obviously suffices to show that $m_p \leq n-1$ for all $p \in X$. There are three kinds of propositions to consider:

- For each $k \in \{1,...,K\}$, obviously $m_{p_k} \leq n-1$.
- For each $k \in \{1,2\}$, $m_{\neg p_k} = n+1-m_{p_k} = 2$, which is at most $n-1$ by $n \geq 3$.
- For each $k \in \{3,...,K\}$, $m_{\neg p_k} = n+1-m_{p_k} = 2^{k-2}+1$, which is at most $n-1$ because, by $n \geq 2^{K-1} \geq 2^{k-1} \geq 4$, we have $n-1 \geq n/2+1 \geq 2^{K-2}+1 \geq 2^{k-2}+1$.

This completes the proof of Claim B1.

Claim B2. F generates complete and consistent judgment sets.

As in the proof of Claim A2, completeness is equivalent to the system of inequalities $m_p + m_{\neg p} \leq n+1$, $p \in X$, which is satisfied (with equality), and consistency is equivalent to the system (1) (using the fact that by Claim B1 the thresholds $(m_p)_{p \in X}$ belong to $\{1,...,n\}$, in fact to $\{2,...,n-1\}$). Consider any minimal inconsistent set $Y \subseteq X$. There are four cases.

- Let $Y \subseteq X_< = \{p_1,....,p_K\}$ with $p_1 \notin Y$. Then

$$\sum_{y \in Y} m_y = n|Y| - \sum_{p_k \in Y} 2^{k-2},$$

[16] We use this result in the variant presented in Dietrich and List (2007e), valid for thresholds in the grid $\{1,...,n\}$.

in which

$$\sum_{p_k \in Y} 2^{k-2} \leq \sum_{k=2}^{K} 2^{k-2} = 2^{K-1} - 1 < 2^{K-1} \leq n.$$

So, $\sum_{y \in Y} m_y > n(|Y| - 1)$.

- Let $Y \subseteq X_< = \{p_1, ..., p_K\}$ with $p_1 \in Y$. Then

$$\sum_{y \in Y} m_y = m_{p_1} + \sum_{p_k \in Y \backslash \{p_1\}} m_{p_k} = (n-1) + n(|Y| - 1) - \sum_{p_k \in Y \backslash \{p_1\}} 2^{k-2}.$$

As $Y \neq X_<$ (by case-B assumption), we have $Y \subsetneq X_<$, hence $Y \backslash \{p_1\} \subsetneq \{p_2, ..., p_K\}$. So, as 2^{k-2} is increasing in k,

$$\sum_{p_k \in Y \backslash \{p_1\}} 2^{k-2} \leq \sum_{p_k \in \{p_3, ..., p_K\}} 2^{k-2} = \sum_{k=3}^{K} 2^{k-2} = 2^{K-1} - 2 < n - 1.$$

Hence, again $\sum_{y \in Y} m_y > n(|Y| - 1)$.

- Let $Y \cap X_> \neq \emptyset$ with $p_1 \notin Y$. We have $|Y \cap X_>| \leq 1$ by the argument in the proof of Claim A1. Let $\neg p_l$ be the unique member of $Y \cap X_>$. We also have $Y \cap X_< \neq \emptyset$, since otherwise $Y = \{\neg p_l\}$, which is impossible as Y is inconsistent and we have excluded contradictions from the agenda. Further, $Y \backslash \{\neg p_l\} \subseteq \{p_2, ..., p_{l-1}\}$ (as for each $p_k \in Y \backslash \{\neg p_l\}$ we have $p_k \vdash^* p_l$, hence $p_k < p_l$, and so $k < l$). This implies that $l \neq 1$ (as $Y \backslash \{\neg p_l\} \neq \emptyset$), so that $m_{p_l} = n - 2^{l-2}$, and hence $m_{\neg p_l} = n + 1 - m_{p_l} = 2^{l-2} + 1$. We have

$$\sum_{y \in Y} m_y = m_{\neg p_l} + \sum_{p_k \in Y \backslash \{\neg p_l\}} m_{p_k} = (2^{l-2} + 1) + n(|Y| - 1) - \sum_{p_k \in Y \backslash \{\neg p_l\}} 2^{k-2},$$

in which, by $Y \backslash \{\neg p_l\} \subseteq \{p_2, ..., p_{l-1}\}$,

$$\sum_{p_k \in Y \backslash \{\neg p_l\}} 2^{k-2} \leq \sum_{k=2}^{l-1} 2^{k-2} = 2^{l-2} - 1 < 2^{l-2} + 1.$$

So, again $\sum_{y \in Y} m_y > n(|Y| - 1)$.

- Let $Y \cap X_> \neq \emptyset$ with $p_1 \in Y$. By arguments like in the previous case, one can show that $Y \cap X_>$ has a unique member, say $\neg p_l$, that $Y \backslash \{\neg p_l\} \subseteq \{p_1, ..., p_{l-1}\}$, and that $m_{\neg p_l} = 2^{l-2} + 1$. So,

$$\begin{aligned} \sum_{y \in Y} m_y &= m_{\neg p_l} + m_{p_1} + \sum_{p_k \in Y \backslash \{p_1, \neg p_l\}} m_{p_k} \\ &= (2^{l-2} + 1) + (n-1) + n(|Y| - 2) - \sum_{p_k \in Y \backslash \{p_1, \neg p_l\}} 2^{k-2} \\ &= 2^{l-2} + n(|Y| - 1) - \sum_{p_k \in Y \backslash \{p_1, \neg p_l\}} 2^{k-2}, \end{aligned}$$

in which, by $Y \backslash \{p_1, \neg p_l\} \subseteq \{p_2, ..., p_{l-1}\}$,

$$\sum_{p_k \in Y \backslash \{p_1, \neg p_l\}} 2^{k-2} \leq \sum_{k=2}^{l-1} 2^{k-2} = 2^{l-2} - 1 < 2^{l-2}.$$

So, again $\sum_{y \in Y} m_y > n(|Y| - 1)$. This completes the proof of Claim B2. ■

Proof that the bound k_X in Theorem 4 is tight. Consider any $K \in \{2, 3, ...\}$. We have to specify an agenda X with $k_X = K$ such that for all $n > K$ there is 'possibility'. Let X be an agenda $X = \{p_1, \neg p_1, ..., p_K, \neg p_K\}$ (containing K pairs) whose only minimal inconsistent set (apart from the trivial ones $\{p, \neg p\} \subseteq X$) is $Y = \{p_1, ..., p_K\}$. (Such agendas exist of course, except in very 'poor' logics.) Obviously, $k_X = |Y| = K$. Fix a group size $n > K$. Define thresholds m_p, $p \in X$, as $n-1$ for $p \in Y$ and as 2 for $p \in X \backslash Y$. The 'quota rule' $F : \mathbf{U}^n \to \mathcal{P}(X)$ given by

$$F(A_1, ..., A_n) = \{p \in X : |\{i : p \in A_i\}| \geq m_p\}$$

is trivially propositionwise and without individual veto power, and it generates outputs in $\mathbf{U}$ by an argument analogous to that which shows Claim A2 in the proof of Theorem 4. ■

Proof that the bound $2^{\frac{|X|}{2}-1}$ in Theorem 4 cannot be tightened to a bound without exponential growth. We show that every sequence $(b_K)_{K=1,2,...}$ in $(0, \infty)$ for which Theorem 4 holds with '$2^{|X|/2-1}$' replaced by '$b_{|X|/2}$' grows exponentially (i.e., there is an $a > 1$ such that $b_K \geq a^K$ for all sufficiently large K). Let $(b_K)_{K=1,2,...}$ be such a sequence; we establish exponential growth by showing that $b_K > m_K$ for all K, where $(m_k)_{k=1,2,...}$ denotes the *Fibonacci sequence*, which is defined recursively by $m_1 = m_2 = 1$ and $m_k = m_{k-1} + m_{k-2}$ for all $k \geq 3$ and grows exponentially (with m_k/m_{k-1} converging to the *golden mean*).

Consider a fixed $K \in \{1, 2, ...\}$. To (ultimately) show that $b_K > m_K$, we consider an agenda $X = \{p_1, \neg p_1, ..., p_K, \neg p_K\}$ whose minimal inconsistent subsets (except the trivial ones of type $\{p_k, \neg p_k\}$) are precisely the sets $Y_{k,l} := \{p_k, p_{k+1}, \neg p_l\}$ with $k, l \in \{1, ..., K\}$ and $k + 1 < l$. Such an agenda does indeed exist, except in 'poor' languages, as we should quickly convince ourselves of. For instance, suppose $\mathbf{L}$ is the language of classical propositional logic with (at least) the connectives $\neg, \vee$ and (at least) the atomic propositions $p_1, ..., p_K$, and let $\mathbf{L}$ be endowed with the following consistency notion (which enforces inconsistency of each set $Y_{k,l}$): a set $A \subseteq \mathbf{L}$ is consistent if and only if $A \cup \{\vee_{p \in Y_{k,l}} \neg p : k, l \in \{1, ..., K\}$ and $k + 1 < l\}$ is classically consistent; in other words, our consistency notion is classical consistency conditional on negating at least one member from each set $Y_{k,l}$. The sets $Y_{k,l}$ are precisely the non-trivial minimal inconsistent subsets of X. To see why, note first that each set $Y_{k,l}$ is obviously non-trivial and minimal inconsistent. Conversely, suppose $Y \subseteq X$ is non-trivial and minimal inconsistent. Then for some k we have $p_k, p_{k+1} \in Y$: otherwise Y would be consistent, as we could extend Y to a (consistent and complete) set $\overline{Y} \in \mathbf{U}$ by adding each $\neg p_j$ for which Y contains none of $p_j, \neg p_j$. Let k be *smallest* such that $p_k, p_{k+1} \in Y$. There exists an $l > k + 1$ such that $\neg p_l \in Y$: otherwise Y could be extended to a consistent and complete set $\overline{Y} \in \mathbf{U}$ by adding

- each $\neg p_j$ for which Y contains none of $p_j, \neg p_j$ and $j < k$,
- each p_j for which Y contains none of $p_j, \neg p_j$ and $j > k$.

Note that $Y \supseteq Y_{k,l}$, so that $Y = Y_{k,l}$ by minimal inconsistency.

The proof that $b_K > m_K$ is completed by establishing the following two claims.

Claim 1. X is not minimally blocked.

Let $\leq$ be the linear order on X defined by $p_1 < p_2 < ... < p_K < \neg p_K < ... < \neg p_1$. Check that, for any distinct $p, q \in X$, if $p \vdash^* q$ then $p < q$. So, as there is no $<$-cycle,

there is no $\vdash^*$-cycle, as required.

Claim 2. If $b_K \leq m_K$ then for some group size $n \geq b_K$ (namely for $n = m_K$) there is no propositionwise aggregation rule $F : \mathbf{U}^n \to \mathbf{U}$ without individual veto power.

Let $n = m_K$, and assume for a contradiction that $F : \mathbf{U}^n \to \mathbf{U}$ is a propositionwise aggregation rule without individual veto power (it need not be monotonic or anonymous). For each integer h, let $\mathcal{C}^h$ be the set of coalitions $C \subseteq N$ of size at least h. We prove by induction that $\mathcal{C}^{n-m_k} \subseteq \mathcal{C}_{p_k}$ for all $k = 1, ..., K$.

First, $\mathcal{C}^{n-m_1} = \mathcal{C}^{n-1} \subseteq \mathcal{C}_{p_1}$ and $\mathcal{C}^{n-m_2} = \mathcal{C}^{n-1} \subseteq \mathcal{C}_{p_2}$, as F is without veto power.

Now let $k \in \{3, ..., K\}$, and suppose $\mathcal{C}^{n-m_{k'}} \subseteq \mathcal{C}_{p_{k'}}$ whenever $k' < k$. Suppose for a contradiction that $\mathcal{C}^{n-m_k} \not\subseteq \mathcal{C}_{p_k}$. Then there is a $C \in \mathcal{C}^{n-m_k}$ such that $C \notin \mathcal{C}_{p_k}$. So, $N \backslash C \in \mathcal{C}_{\neg p_k}$, and by $|N \backslash C| \leq m_k = m_{k-1} + m_{k-2}$ we can partition $N \backslash C$ into coalitions C_1, C_2 of sizes $|C_1| \leq m_{k-1}$ and $|C_2| \leq m_{k-2}$. Hence, $N \backslash C_1 \in \mathcal{C}^{n-m_{k-1}}$ and $N \backslash C_2 \in \mathcal{C}^{n-m_{k-2}}$. So, by induction hypothesis, $N \backslash C_1 \in \mathcal{C}_{p_{k-1}}$ and $N \backslash C_2 \in \mathcal{C}_{p_{k-2}}$. As C, C_1, C_2 form a partition of N and as $\{p_{k-2}, p_{k-1}, \neg p_k\} = Y_{k,k+2}$ is minimal inconsistent, there is a profile $(A_1, ..., A_n) \in \mathbf{U}^n$ in which

- all $i \in C$ have $A_i \supseteq \{p_{k-2}, p_{k-1}, p_k\}$
- all $i \in C_1$ have $A_i \supseteq \{p_{k-2}, \neg p_{k-1}, \neg p_k\}$
- all $i \in C_2$ have $A_i \supseteq \{\neg p_{k-2}, p_{k-1}, \neg p_k\}$.

Then $F(A_1, ..., A_n)$ contains p_{k-2} by $N \backslash C_2 \in \mathcal{C}_{p_{k-2}}$, p_{k-1} by $N \backslash C_1 \in \mathcal{C}_{p_{k-1}}$ and $\neg p_k$ by $N \backslash C \in \mathcal{C}_{\neg p_k}$, a contradiction as $F(A_1, ..., A_n)$ is consistent.

As $n = m_K$, we have in particular $\mathcal{C}^0 \subseteq \mathcal{C}_{p_K}$. By $\mathcal{C}^0 = \mathcal{P}(N)$ it follows that $\mathcal{C}_{p_K} = \mathcal{P}(N)$, whence $\mathcal{C}_{\neg p_K} = \emptyset$, a contradiction as F is without veto power. ■

Inspection of the last proof shows that a *tight* lower bound on n for Theorem 4 would have to be intermediate in strength between the current bound '$n \geq 2^{\frac{|X|}{2}-1}$' and the weakest candidate '$n > m_{|X|/2}$' (where m_K is the K^{th} Fibonacci number). Where in this range the tight bound lies is left as an open question.

A.4 Proof of Theorem 5

To prove the result, we define a binary relation $\sim$ on X.

Definition 7 *For any $p, q \in X$, write $p \sim q$ if there exists a finite sequence $p_1, ..., p_k \in X$ with $p_1 = p$ and $p_k = q$ such that any neighbours p_l, p_{l+1} are neither exclusive nor exhaustive (i.e., $\{p_l, p_{l+1}\}$ and $\{\neg p_l, \neg p_{l+1}\}$ are consistent).*

The following lemma summarizes the main properties of $\sim$. Call an agenda X *nested* if it can be written as $X = \{p_1, \neg p_1, ..., p_K, \neg p_K\}$ such that $p_k \vdash p_{k+1}$ for all $k \in \{1, ..., K-1\}$. Nestedness implies simplicity: as any two members of a nested agenda X are (directly) logically dependent, there exist plenty of minimal inconsistent sets $Y \subseteq X$ but all of them have only size 2.

Lemma 4 *$\sim$ defines an equivalence relation on X, with*

- *a single equivalence class if X is non-nested,*
- *exactly two equivalence classes, each of which contains one member of each pair $p, \neg p \in X$, if X is nested.*

Proof. These properties are shown in Dietrich and List (2007c), albeit in a semantic framework with propositions represented as sets of possible worlds; we leave the simple translation to the reader. ■

An aggregation rule F is called *systematic on* Z ($\subseteq X$) if, for all $p, p' \in Z$ and all admissible profiles $(A_1, ..., A_n), (A'_1, ..., A'_n)$, $[p \in A_i \Leftrightarrow p' \in A'_i$ for all individuals $i]$ implies $p \in F(A_1, ..., A_n) \Leftrightarrow p' \in F(A'_1, ..., A'_n)$. For 'systematic on X' we simply say 'systematic'.

Lemma 5 *A propositionwise and implication-preserving aggregation rule $F : \mathbf{U}^n \rightarrow \mathbf{U}$ is systematic on each $\sim$-equivalence class.*

Proof. Let F be as specified. As F is propositionwise, it suffices to show that $\mathcal{C}_p = \mathcal{C}_q$ for all $p, q \in X$ such that $p \sim q$. In fact, by a straightforward inductive argument it suffices to show that $\mathcal{C}_p = \mathcal{C}_q$ for all $p, q \in X$ for which $\{p, q\}$ and $\{\neg p, \neg q\}$ are each consistent.

Consider any such $p, q \in X$ and any $C \subseteq N$; we show that $C \in \mathcal{C}_p \Leftrightarrow C \in \mathcal{C}_q$. As $\{p, q\}$ and $\{\neg p, \neg q\}$ are consistent, there exist judgment sets $A_1, ..., A_n \in \mathbf{U}$ such that

$$p, q \in A_i \text{ for all } i \in C \text{ and } \neg p, \neg q \in A_i \text{ for all } i \notin C.$$

We have $p \in A_i \Leftrightarrow q \in A_i$ for all i, so that by applying implication preservation in both directions we obtain

$$p \in F(A_1, ..., A_n) \Leftrightarrow q \in F(A_1, ..., A_n).$$

Now $C \in \mathcal{C}_p$ is equivalent to $p \in F(A_1, ..., A_n)$, hence (as just shown) to $q \in F(A_1, ..., A_n)$, and so to $q \in \mathcal{C}_q$. ■

Lemmas 4 and 5 imply the following global systematicity result.

Lemma 6 *If the agenda is non-nested, every propositionwise and implication-preserving aggregation rule $F : \mathbf{U}^n \rightarrow \mathbf{U}$ is systematic.*

While the last systematicity result assumed just a non-nested agenda, the following monotonicity result makes the stronger non-simplicity assumption.

Lemma 7 *For a non-simple agenda, every propositionwise and implication-preserving aggregation rule $F : \mathbf{U}^n \rightarrow \mathbf{U}$ is monotonic.*

Proof. Let X and F be as specified. By Lemma 6, F is systematic. So $\mathcal{C}_p$ is the same for all $p \in X$; call this set $\mathcal{C}$. Let $C \subseteq C' \subseteq N$ with $C \in \mathcal{C}$; we have to show that $C' \in \mathcal{C}$. As X is non-simple, there exists a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$. Choose pairwise distinct $p, q, r \in Y$. As each of $Y_{\neg\{p\}}$, $Y_{\neg\{q\}}$, $Y_{\neg\{r\}}$ is consistent, there are $A_1, ..., A_n \in \mathbf{U}$ such that

- for all $i \in C$, $Y_{\neg\{q\}} \subseteq A_i$,
- for all $i \in C' \backslash C$, $Y_{\neg\{r\}} \subseteq A_i$,
- for all $i \in N \backslash C'$, $Y_{\neg\{p\}} \subseteq A_i$.

As $\neg q \in A_i \Rightarrow p \in A_i$ for all i, we have $\neg q \in F(A_1, ..., A_n) \Rightarrow p \in F(A_1, ..., A_n)$ by implication preservation. So, as $\neg q \in F(A_1, ..., A_n)$ by $C \in \mathcal{C}$, we have $p \in F(A_1, ..., A_n)$, and hence $C' \in \mathcal{C}$ (as F is propositionwise). ■

Proof of Theorem 5. First, let X be non-simple. For a contradiction suppose F is an an aggregation rule with all required properties. By X's non-nestedness and the last two lemmas, F is systematic and monotonic. Hence F is dictatorial by a standard result for non-simple agendas (Nehring and Puppe 2002), a contradiction.

Conversely, let X be simple. As $n \geq 3$, there exists an odd-sized non-singleton subgroup $M \subseteq N$. The aggregation rule $F : \mathbf{U}^n \rightarrow \mathcal{P}(X)$ defined as majority voting among the members of M is implication-preserving (as one can verify), non-dictatorial (by $|M| > 1$) and of course propositionwise, and it generates judgment sets in $\mathbf{U}$ (as $|M|$ is odd and X is simple; see the argument in case 2 of part 2 of the proof of Theorem 2). ■