

# Measurement-Theoretic Foundations of Time Discounting in Economics

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## Abstract

In economics, the concept of time discounting introduces weights on future goods to make these less valuable. Yet, both the conceptual motivation for time discounting and its specific functional form remain contested. To address these problems, this paper provides a measurement-theoretic framework of representation for time discounting. The representation theorem characterises time discounting factors by representations of time differences. This general result can be interpreted with existing theories of time discounting to clarify their formal and conceptual assumptions. It also provides a conceptually neutral framework for comparing the descriptive and normative merits of those theories.

## 1 Introduction

It is standard practice in economics to introduce weightings that evaluate the temporal dimension of a prospect. Such weightings are performed by time discounting factors that make goods in the far future less valuable than those in the near future. Famously, time discounting is a heavily contested concept, both normatively and descriptively (Loewenstein and Elster, 1992). Normatively, it is often questioned whether time discounting is justified at all. Despite this fact, in the spirit of Ramsey (1928), time discounting is deeply entrenched in economic modelling: “It is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible . . . we shall, however, . . . include such a rate of discount in some of our investigations.” Descriptively, there is no consensus on the correct conceptual motivation and functional form of discounting factors (Frederick et al, 2002). The normative and descriptive problems concerning time discounting have generated much disagreement, as reviewed by Loewenstein and Read (2003). This fact renders scientific and policy debates about intertemporal decisions, such as those related to pension systems, public investment and climate change, deeply challenging.

This paper embarks from the supposition that the underlying problem of discussing time discounting lies in the absence of a general representational framework that allows one to compare the different proposals. Here, time discounting is discussed within the framework of the representational theory of measurement (Krantz et al, 1971; Suppes, 2002). It is used to develop a general representational framework of discounting factors by representations of *time difference*. This makes transparent the fact that any time discounting theory needs to endorse a numerical representation of some qualitative evaluation of time difference. In the general measurement-theoretic framework, the qualitative concept of time difference is a formal notion that can accommodate the range of substantial interpretations used in existing theories of time discounting. Those interpretations determine the exact nature of the time difference evaluation that the discounting factor represents in the given theory. Hence, the representational strategy pursued in this paper renders transparent the formal and conceptual assumption common to theories of time discounting.

The paper proceeds as follows. Section 2 gives an overview of the concept of time discounting. Section 3 contains the general measurement-theoretic representation of time differences. Section 4 discusses substantial interpretations of time difference. Section 5 provides time discounting functions with the representation thus obtained. Section 6 concludes.

## 2 Time Discounting

Theories of time discounting usually formulate time discounting functions. Those functions assign numerical values to points in time which are then used to weight the goodness of consequences occurring at those times. Formally, a general time discounting function can be described as follows.

**Definition 1 (Time discounting function)** *A time discounting function  $D$  is a decreasing mapping  $D : T \rightarrow (0, 1]$ , from a set of time points  $T$  to the real interval  $(0, 1]$  such that  $D(0) = 1$ .*

Accordingly, values between 0 and 1 are assigned to time points such that the later the point in time, the lower the value that is assigned to it. This definition is not intended to rule out the possibility of endorsing a more general discounting function such as  $D : T \rightarrow \mathbb{R}$ . Rather, it reflects common properties of the discounting functions endorsed by many time discounting theories. The different approaches to time discounting formulate time discounting functions that are by and large special cases of Definition 1, offering more specific restrictions on what values time discounting factors can take and an interpretation as to why it should be used to discount for time.

For instance, time preference theories of time discounting calculate the discounting factor for each point in time as follows:  $D_e(t) = \delta^t$ . The dis-

counting factor  $\delta$  is given by a constant discount rate  $r$ , i.e.  $\delta = 1/(1 + r)$ , which reflects the weight that is attached to  $t + 1$  in  $t$ . The discount rate is determined by the degree to which an agent is time impatient, his so-called time preferences. The impatience captured by time preferences is taken to be an important psychological fact common for many agents.

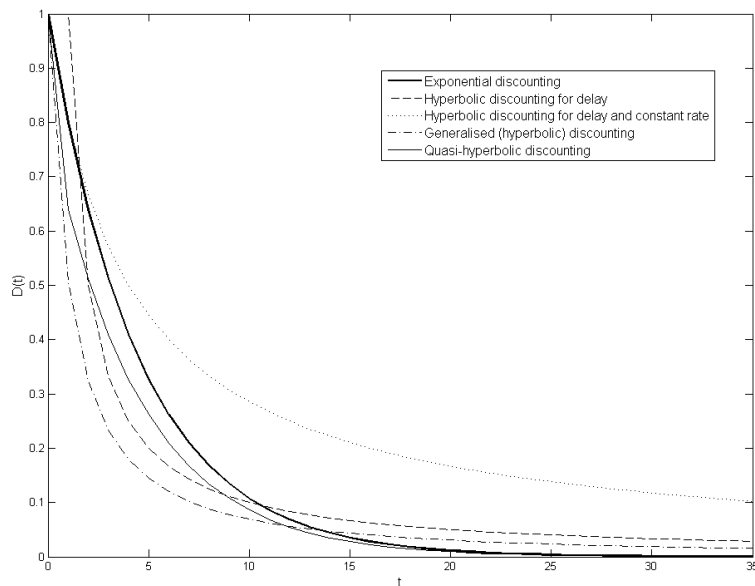


Figure 1: Discounting Functions

More recently, theories of hyperbolic discounting have been proposed, intended to reflect empirical evidence of the ‘myopia’ of real-world agents (Angeletos et al, 2001; Frederick et al, 2002). In those theories, the near future is more heavily discounted than in exponential discounting and there is similar or less time discounting for the far future. Hyperbolic discounting theories motivate time discounting in a variety of ways, for instance, as arising due to the way individuals perceive of delays, as reflecting fundamental risk and uncertainty, or as reflecting changes in the agent’s preferences. Figure 1 displays the graphs of exponential and hyperbolic discounting functions.<sup>1</sup>

<sup>1</sup>Figure 1 displays the graphs of the following functions: Exponential discounting:  $D(t) = \delta^t$ ; Hyperbolic discounting for delay:  $D(t) = 1/t$ ; Hyperbolic discounting for delay and discount rate:  $D(t) = 1/(1 + rt)$ ; Generalised (hyperbolic) discounting:  $D(t) = 1/(1 + \alpha t)^{\gamma/\alpha}$ ; Quasi-hyperbolic discounting:  $D(t) = \begin{cases} 1 & \text{if } t = 0, \\ \beta\delta^t & \text{if } t > 0. \end{cases}$  Parameters:  $\alpha = .7$ ,  $\beta = .8$ ,  $\gamma = .9$ ,  $\delta = .8$  (i.e. given by  $1/(1 + r)$ , where  $r = .25$ ).

The question which of these aforementioned theories of time discounting is the correct one, both normatively and descriptively, has not been resolved (overviews of the debate can be found in Frederick et al, 2002; Loewenstein and Read, 2003). This is due to the fact that these theories motivate discounting for time by different concepts.

From a foundational perspective, such disputes should be resolved by analysing the theories according to their frameworks of representation and measurement. Indeed, for many other contentious concepts in the foundations of economics, descriptive and normative debates can be explored systematically by going back to underlying theories of measurement and representation. Consider as an example expected utility theory, where normative and empirical problems have been debated with regards to the structure of the representation of expected utility (for instance, whether certain axioms are normatively justifiable, formally avoidable and empirically verifiable). Yet, competing theories of time discounting operate in representational frameworks that are radically different from each other. This creates the need for a framework which can facilitate a comparison of the kinds of qualitative properties that are represented numerically by a discounting factor in the different theories.

The remainder of this paper provides a general measurement-theoretic representation of time discounting. The strategy of representation is as follows: first, a measurement-theoretic representation of time difference is offered. Second, it is discussed how the different theories of time discounting interpret time difference. Third, it is shown how time discounting factors can be given by using the representation of time difference. This representation can serve as a neutral framework for comparing both the descriptive and normative merit of the competing theories.

### 3 Representing Time Difference

The representation of time difference developed here rests on a formal framework of measurement and representation developed in Krantz et al (1971). In Krantz et al (1971, 170), absolute-difference structures are introduced to measure differences along a single dimension between *pairs* of elements in a set. That is, it is asked whether the absolute difference between a pair of elements  $qr \in Q$  is larger than the absolute difference between the pair  $st \in Q$  on some dimension of comparison. For instance, if those elements are food items, the pairs could be compared with regards to their difference in sweetness. Upon those comparisons satisfying the following definition, the absolute differences between pairs of elements can be represented numerically, unique up to an affine transformation.

**Definition 2 (Absolute-difference structure. Krantz et al, 1971, 172)**  
*Suppose  $Q$  is a set with at least two elements and  $\succsim$  is a binary relation on*

$Q \times Q$ . The pair  $\langle Q \times Q, \succsim \rangle$  is an absolute-difference structure iff, for all  $q, r, s, t, q', r', s', t' \in Q$ , and all sequences  $q_1, q_2, \dots, q_i, \dots \in Q$  the following axioms hold:

1. *Weak ordering.*  $\langle Q \times Q, \succsim \rangle$  is a weak order.
2. *Symmetry.* If  $q \neq r$ , then  $qr \sim rq \succ qq \sim rr$ .
3. *Well-Behavedness.*
  - (i) If  $r \neq s$ ,  $qs \succsim qr, rs$  and  $rt \succsim rs, st$ , then  $qt \succsim qs, rt$ .
  - (ii) If  $qs \succsim qr, rs$  and  $qt \succsim qs, st$ , then  $qt \succsim rt$ .
4. *Weak Monotonicity.* Suppose that  $qs \succsim qr, rs$ . If  $qr \succsim q'r'$  and  $rs \succsim r's'$ , then  $qs \succsim q's'$ ; moreover if either  $qr \succ q'r'$  or  $rs \succ r's'$ , then  $qs \succ q's'$ .
5. *Solvability.* If  $qr \succsim st$ , then there exists  $t' \in Q$ , such that  $qr \succsim t'r$  and  $qt' \sim st$ .
6. *Archimedean property.* If  $q_1, q_2, \dots, q_i, \dots$  is a strictly bounded standard sequence (i.e., there exist  $t', t'' \in Q$ , such that for all  $i = 1, 2, \dots$ ,  $t', t'' \succ q_{i+1}q_1 \succsim q_iq_1$  and  $q_{i+1}q_i \sim q_2q_1 \succ q_1q_1$ ), then the sequence is finite.

In this definition, the set  $Q$  can be interpreted as a large collection of consequences, and  $P \subset Q$  can be prospects. For example, take the prospect of a dinner  $P = \{q, r, s, t\}$ , where  $q$  = starter,  $r$  = main,  $s$  = dessert and  $t$  = coffee. According to the above definition, pairs of elements in this prospect can be compared according to their absolute differences on a single dimension. For instance, the consequences of the dinner prospect can be compared with regards to their sweetness. In order to do so, the absolute differences of pairs of elements are ordered according to  $\succsim$ . Take the pairs  $qr$  (starter, main) and  $st$  (dessert, coffee). If the difference in sweetness between main and starter is smaller than that between coffee and dessert, then  $qr \prec st$ . The symmetry condition on this ordering states that the absolute difference between pairs of elements is independent of their ordering, i.e. when comparing the absolute difference in sweetness between starter and main to another pair of consequences, it does not matter whether we write  $qr$  or  $rq$ . Moreover, if there are two consequences that differ in sweetness (that is, we are not looking at the absolute difference in sweetness between, e.g. the dessert and itself), then their absolute difference in sweetness is strictly larger than that of those items with themselves (i.e.  $qr \sim rq \succ qq \sim rr$ ).<sup>2</sup> The other conditions (3.-6.) ensure richness of the ordering  $\succsim$ .

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<sup>2</sup>In order to deal with different consequences that are equivalent on the dimension of comparison, it has to be assumed that elements in  $Q$  can be interpreted as denoting equivalence classes of elements that are similar under the dimension of comparison.

The above measurement procedure can be interpreted in a variety of ways. In order to do so, one has to define what the elements in  $Q$  are taken to be, what their single dimension of comparison is and then interpret the above conditions on the ordering  $\succsim$  of the difference between pairs. In the example of the dinner, one could for instance change the single domain of comparison to that of time difference. Then,  $\succsim$  reflects the absolute difference in time between all elements of the dinner. For instance, the time difference between starter and main  $qr$  could be larger than that between dessert and coffee  $st$ , due to slow service in the restaurant, such that  $qr \succsim st$ . The next section will discuss how to interpret this notion of time difference to capture the conceptions of time underlying the different theories of time discounting.

Note that absolute differences ordered accordingly do not need to correspond to any supposedly objective standard that is externally given, such as sweetness defined in terms of sugar content in food items or time difference as it is given by a clock. What is crucial is the fact that one can identify one dimension on which absolute differences between pairs of elements are compared, and upon the comparison satisfying the conditions in Definition 2, the ordering  $\succsim$  it yields can be represented numerically in the following way.

**Theorem 3 (Interval representation. Krantz et al, 1971, 173)** *If  $\langle Q \times Q, \succsim \rangle$  is an absolute-difference structure, then there exists a function  $\varphi : Q \rightarrow \mathbb{R}$  such that for all  $q, r, s, t \in Q$ ,*

$$qr \succsim st \text{ iff } |\varphi(q) - \varphi(r)| \geq |\varphi(s) - \varphi(t)|$$

*If  $\varphi'$  is another function with the same property, then  $\varphi' = \alpha\varphi + \beta$ , where  $\alpha, \beta \in \mathbb{R}, \alpha \neq 0$ .*

According to the above theorem, it is possible to numerically represent the qualitative differences between pairs of elements in a set  $Q$ . That is, in the context of measuring absolute differences between consequences on a single dimension, a number  $\varphi \in \mathbb{R}$  can be assigned to any consequence  $q, r, s, t \in Q$  such that for any two consequences, the absolute difference of the numbers assigned to them adequately reflects their absolute difference when compared to any other pair of elements. Moreover, the assigned numbers are unique up to affine transformation.

For some dimensions of comparison, it is possible to identify a specific element  $p \in Q$  to which the above representation can be normalised to. Generally, such a normalisation is permissible if  $p$  an absolute (or true) zero point.

**Corollary 4 (Normalisation)** *Let  $\varphi$  be a representation of  $\langle Q \times Q, \succsim \rangle$ . Then  $\varphi^*$  is a normalisation of  $\varphi$  iff  $\varphi^* = \varphi + \beta$  and  $\varphi^*(p) = 0$ . If  $\varphi'^*$  is*

another function with the same property, then  $\varphi'^* = \alpha\varphi^*$ , where  $\alpha \in \mathbb{R}, \alpha > 0$ .

*Proof.* Immediate from the properties of the interval representation in Theorem 3.  $\square$

The above statement asserts that the interval scale given in Theorem 3 can be normalised to an absolute zero which gives a ratio scale on which only multiplicative transformations are allowed.

The representation and its normalisation establish the ratio-scale measurement of absolute differences between consequences in  $Q$  with respect to a single dimension. In the context of the dinner example and the sweetness dimension of comparison, this means that all consequences in the dinner are assigned a real number  $\varphi \in \mathbb{R}$  that reflects the ordering in absolute difference of sweetness between all possible pairings. For instance, take the following sweetness ordering of the dinner consequences:  $st \succ sr \succ qt \sim qs \succ rt \succ rq$ . According to Theorem 3, this can be represented numerically, for instance:  $\varphi(q) = 55, \varphi(r) = 50, \varphi(s) = 105, \varphi(t) = 5$ . Furthermore, if there is an element that has maximal or minimal sweetness, then the numerical representation can be normalised. Suppose an agent has an espresso with no sugar for coffee and that this is minimally sweet. Then, according to Corollary 4,  $\varphi^* = \varphi - 5$  such that  $\varphi^*(t) = 0$ .

Interpreting the above framework with other dimensions of difference comparisons, similar orderings and representations can be derived. In the context of time difference, it is indeed also possible to normalise the representation to a ratio scale, taking the present as an absolute zero, as it is the natural viewpoint from which prospects and courses of actions are assessed. Indeed, the normalisation also captures the idea of tenses in the formalism, since all consequences in the past take a value smaller than zero (i.e.  $\varphi^* \in \mathbb{R}^-$  for all  $l, m, n, o, \dots \in Q < p$ ) and all consequences in the future take a value larger than zero (i.e.  $\varphi^* \in \mathbb{R}^+$  for all  $q, r, s, t, \dots \in Q > p$ ). It is also possible to normalise to any other time point in the past or future, which is plausible when there is a specific point in time from which temporally extended prospects are analysed. For most applications and indeed for representing time discounting, normalising to the present is the most plausible option.

This also yields the possibility to specifically analyse time differences between  $p$  and other elements in the set  $Q$ . Notably, from the normalisation the following statement follows immediately:  $qp \preccurlyeq rp$  iff  $|\varphi(q)| \leq |\varphi(r)|$ . Hence, consequences can be analysed directly with regards to their time difference to the present. This simplified comparison of time difference will be used to consider substantial interpretations of time difference that is inherent in the different theories of time discounting.

## 4 Interpreting Time Difference

Time difference can be interpreted in a variety of ways. Firstly, it could be interpreted as equivalent to clock-time. Indeed, the six axioms on the binary relation  $\succsim$  would follow immediately from the idea that clock-time can be discretely represented by a succession of integers and that each consequence corresponds to exactly one of those integers. The purpose of the framework introduced here is however not to capture measurement of time as clock-time, but to develop a framework that can compare what different theories of time discounting take to be relevant about time difference.

The formal notion of time difference can be interpreted by the conceptions of time that are endorsed by the different theories of time discounting. For all interpretations,  $Q$  is a set of consequences and  $\succsim$  orders pairs of those according to their absolute time difference. However, the approaches differ widely in how exactly that difference is interpreted and how rich the description of the consequences needs to be.

**Time preference.** Time preference theories of discounting evaluate time differences according to the degree of impatience they induce in the agent. Indeed, at the heart of these theories lies the idea that time impatience of agents is both psychologically plausible and plays a major role in intertemporal evaluations, as pointed out by the precursors of time preference theories, such as Boehm-Bawerk, Fisher, Jevon and Pigou (Frederick et al, 2002). In those theories,  $qp \prec rp$  iff a higher degree of impatience is associated with  $r$  than with  $q$ , as no time impatience is associated with  $p$ . With the additional assumption that  $r$  and  $p$  take a positive value under a desirability evaluation, this captures time preferences as used in the representations of Samuelson (1937) and Koopmans (1960).

**Risk and uncertainty.** Risk and uncertainty theories of time discounting evaluate time differences according to the degree of fundamental risk or uncertainty they induce. These theories, for instance Weitzman (2001), Gollier (2002) and Halevy (2008), use time-indexed probability functions and risk evaluations to motivate time discounting. Hence,  $qp \prec rp$  iff more fundamental uncertainty is associated with  $r$  than with  $q$ , as no fundamental uncertainty is associated with  $p$ . Additional assumptions on how the risk and uncertainty evaluation of time differences is delineated from risk and uncertainty in goodness evaluations are needed to employ time discounting functions thus motivated.

**Preference change.** Preference change theories of time discounting evaluate time differences according to the degree of change in the propositional attitudes of agents. In those theories, the future goodness evaluations of agents are discounted with their diminished present credibility due to changes in preferences (Strotz, 1956; Frederick et al, 2002, 389). In those theories,  $qp \prec rp$  iff there is more preference change associated with  $r$  than with  $q$ , when compared to preferences at  $p$ . In order for this interpretation



to hold, richer descriptions of consequences have to be assumed (e.g. the consequence  $t$  is an agent-relative proposition “Agent  $A$  eats a dessert”), such that the description specifically includes a reference to the agent whose preferences change.

**Delay.** Delay theories of time discounting evaluate time differences according to how agents perceive the delay they induce. In those theories, initiated by Ainslie (1992) amongst others, empirical results on how agents perceive of delays are generalised and used to motivate time discounting. Hence, in this interpretation,  $qp \prec rp$  iff an agent perceives a longer delay between  $rp$  than with  $qp$ , and no delay is associated with  $p$ .

In addition to those interpretations, there is a large class of time discounting theories that combine the above interpretations of time difference (overviews are in Frederick et al, 2002; Loewenstein and Read, 2003).

Interpreting time difference with the conceptual content from the different theories of time discounting makes transparent how those theories establish the numerical representation of a qualitative concept. Furthermore, it also makes transparent that the specific interpretation of time difference has to motivate the normalisation of the numerical representation according to Corollary 4. More generally, the framework given in this paper allows to recast the conceptions of time discounting the different theories endorse in terms of their inherent interpretation of time difference. This overcomes the problem of the stark differences between the representational frameworks those theories are stated in. It is now shown how time discounting factors can be given by using the representation of time difference.

## 5 Time Difference Discounting

The numerical representation of absolute differences can be used for describing the consequences with regards to the single dimension of comparison. Here, the representation are used to construct weights. More formally, it is possible to formulate a discounting function which transforms the differences into discounting factors, which can be used as weights assigned to consequences.

**Theorem 5 (Difference discounting)** *There exists a function  $Disc : \mathbb{R} \rightarrow (0, 1]$  such that*

- (i)  $Disc(0) = 1$ ,
- (ii)  $Disc$  is strictly increasing on  $(-\infty, 0]$  and strictly decreasing on  $[0, \infty)$ ,
- (iii)  $Disc(\varphi^*(q)) \geq Disc(\varphi^*(r)) \Leftrightarrow rp \succcurlyeq qp$ , for all  $q, r \in Q$ .

*Proof.* A number of functions fulfill these conditions. Consider the assignment  $x \mapsto 1/(|x| + 1)$  that defines a function  $Disc$  from  $\mathbb{R}$  to  $(0, 1]$ . Clearly,

$Disc(0) = 1$ . This function is also strictly increasing on  $(-\infty, 0]$  and strictly decreasing on  $[0, \infty)$ . To show that  $Disc$  has property (iii), consider any  $q, r \in Q$ . We have

$$\begin{aligned}
rp \succsim qp &\Leftrightarrow |\varphi^*(r) - \varphi^*(p)| \geq |\varphi^*(q) - \varphi^*(p)| \text{ (as } \varphi^* \text{ represents } \succsim \text{)} \\
&\Leftrightarrow |\varphi^*(r)| \geq |\varphi^*(q)| \text{ (by } \varphi^*(p) = 0\text{)} \\
&\Leftrightarrow 1 + |\varphi^*(r)| \geq 1 + |\varphi^*(q)| \\
&\Leftrightarrow \frac{1}{1 + |\varphi^*(r)|} \leq \frac{1}{1 + |\varphi^*(q)|} \\
&\Leftrightarrow Disc \circ \varphi^*(r) \leq Disc \circ \varphi^*(q) \text{ (by definition of } Disc\text{)}. \quad \square
\end{aligned}$$

In this statement, the representation of differences between consequences is used to formulate weights  $Disc(0, 1]$ . The weighting assigns the unit weight to the element to which the difference representation is normalised, resulting in no discounting at all. Farther, the difference discounting function assigns a number in the real interval  $(0, 1)$  to all other differences such that the larger the difference, the lower the weight. This makes it possible to use the function  $Disc \circ \varphi^*$  as a weight for differences of consequences. Such difference discounting can be employed with regards to any single dimension which is representable on a ratio scale according to the above measurement procedure, for instance sweetness as well as time difference.

Any of the specific time difference interpretations introduced above can be assumed to obtain a discounting function with a difference representation. As alluded to above, this does not mean that time differences thus understood necessarily correspond to time as commonly understood as clock-time. Hence, a correspondence between a measurement of time difference and clock-time will need to be assumed explicitly in order to obtain a representation of *time discounting* in terms of time differences that satisfies Definition 1. The next definition expresses such a correspondence between some externally given time index and the time differences in the above representation.

**Definition 6 (Correspondence between time difference and time)**

Let  $T = \{0, 1, \dots\}$  be a set of externally given time points,  $\langle Q \times Q, \succsim \rangle$  an absolute difference structure and  $\varphi^*$  a ratio-scale representation of time difference. Correspondence between time difference and time holds iff  $\varphi^* \circ T$  is given by a mapping  $m : t \mapsto \varphi^*$  such that  $m(0) = \varphi^*(p)$  and  $\varphi^*(q) \leq \varphi^*(r)$  iff  $m(s) < m(t)$ , for all  $q, r \in Q$  and for all  $s, t \in T$ .

This definition asserts that each point in time  $t \in T$  is associated with a consequence  $q \in Q$  and that furthermore a mapping from externally given time points  $t \in T$  to numerical representations of time differences of consequences  $\varphi^*$  is monotonic. Note that by Corollary 4,  $\varphi^*(p) = 0$  and hence

corresponds to  $t = 0$  under any time difference interpretation that both allows the normalisation and can be linked to clock-time. The latter is a substantial conceptual assumption: after all, the degree of impatience, fundamental uncertainty, preference change or delay perception that is measured as time difference could be influenced by a number of other factors and, for instance, fluctuate when compared to an externally given time index. The latter could indeed rule out that time difference thus understood is an evaluation suitable for time discounting. However, due one can associate the time difference with clock-time, time discounting as stated in Definition 1 can be obtained.

According to the above definition, time differences are taken into account only for consequences for which  $\varphi^*(q) \geq 0$ . Likewise, the externally given time points  $T = \{0, 1, \dots\}$  are interpreted as present ( $t = 0$ ) or future time points (all  $t > 0$ ). The following general representation would indeed work for an unrestricted  $\varphi^*$ , however, since most time discounting functions are concerned with the present and the future, restricting attention to positive values simplifies the exposition. If there is such a correspondence between a set of time points and the normalised representation of time difference  $\varphi^*$ , then time discounting according to time differences as stated in the above difference discounting theorem is possible via the function  $Disc \circ \varphi^* \circ T$ .

**Theorem 7 (Time discounting with difference representation)** *Let  $\langle Q \times Q, \succ \rangle$  be an absolute time difference structure and  $\varphi^*$  its ratio-scale representation. If there is correspondence between  $\varphi^*$  and a set of externally given time points  $T$ , then the function  $Disc \circ \varphi^* \circ T$  is a time discounting function  $D$ .*

*Proof.* By Theorem 5,  $Disc \circ \varphi^*$  is a decreasing mapping from the ratio-scale representation of time differences  $\varphi^*$  to a real interval  $(0, 1]$ . By Definition 6, there is a monotonic mapping from time points  $T$  to  $\varphi^*$ . Hence,  $Disc \circ \varphi^* \circ T$  is a decreasing mapping from time points  $T$  to a real interval  $(0, 1]$  such that  $Disc(\varphi^*(p)) = 1$  which satisfies Definition 1.  $\square$

The above statement concludes the development of a general representation of time discounting functions by time difference: taking any pair of consequences in  $Q$  and comparing their time difference with other pairs, each consequence can be assigned a number  $\varphi^*$  that indicates their time difference on a ratio scale with the present being assigned the value zero.  $\varphi^*$  is then used to obtain the function  $Disc \circ \varphi^* \circ T$  which behaves like a time discounting factor. Hence, under the assumption of correspondence between time difference and time, there is a time discounting function with a difference representation.

The general representation theorem makes transparent the requirements any theory of time discounting has to fulfil in a measurement-theoretic

framework: namely, a conceptual interpretation of time difference has to be given that renders plausible both the representation and the normalisation procedure in an absolute-difference structure. Furthermore, to satisfy Definition 1, there has to be correspondence between the normalised representation and clock-time. Note that fulfilling these requirements does not directly imply descriptive or normative plausibility with regards to discounting utility in intertemporal decisions. Rather, the framework developed here makes transparent that additional assumptions are needed to endorse the time discounting of *utility*. The general representational framework developed here relates to intertemporal decision-making as follows.

**Remark 8 (Discounted Utility)** *Let  $P_A, P_B \subset Q$  be two collections of consequences,  $P_A = \{q_A, r_A, \dots\}$  and  $P_B = \{q_B, r_B, \dots\}$ . Supposing the above representation of time discounting, when weighting the goodness evaluations  $u(P_A)$  and  $u(P_B)$  with time discounting functions thus obtained, then*

$$P_A \succsim P_B \text{ iff } \sum_{t=0}^{\infty} \text{Disc}(\varphi^*(q_{At}))u(q_{At}) \geq \sum_{t=0}^{\infty} \text{Disc}(\varphi^*(q_{Bt}))u(q_{Bt}).$$

Accordingly, time difference discounting functions can be integrated in utility calculations as a weight. Note that the time discounting function as introduced above gives weights that are completely independent of the utility representation of preferences in the statement. Nothing more is required than a common domain  $Q$  over which both preferences  $\succsim$  and time differences  $\succsim$  are defined.

Crucially, the descriptive and normative status of the above statement will depend on the time difference interpretation given. This makes transparent the fact that explicit justifications are needed to endorse time discounting of goodness evaluations such as utility. Hence, the degree to which one accepts the normative attractiveness of a particular interpretation of time difference determines the degree of the normative appeal of discounted utility.

This more general representational framework allows one to separate the issue of interpreting time difference and the specific functional form of time discounting. While the time discounting theories commonly endorse specific functional forms, i.e. exponential (time preference) and hyperbolic (risk and uncertainty, preference change, delay) discounting, the framework introduced here allows for their development in the same general measurement-theoretic framework. More specific conditions on the correspondence of time and time difference give specific functional forms of exponential and hyperbolic discounting (e.g. exponential discounting requires a *constant* correspondence between time difference and time that gives a constant factor  $\gamma = |\varphi^*(q) - \varphi^*(r)|$ , for all consequences  $q, r \in Q$  that correspond to subsequent points in time). For hyperbolic discounting, the correspondence has

to be declining. The plausibility of specific functional forms such as exponential and hyperbolic discounting will then depend on how time difference is interpreted, and how well the given interpretation can motivate additional conditions on the correspondence between time difference and clock-time.

## 6 Conclusions

In the measurement-theoretic framework for time discounting provided in this paper, a discounting factor is determined by a ratio-scale representation of time differences between consequences. This renders transparent the formal and conceptual assumption common to theories of time discounting. The general framework developed here has a number of applications in foundational work regarding time discounting. Formally, the framework can be employed to assess and render transparent formal assumptions that those specific accounts of time discounting make. Conceptually, it can be related to a number of interpretations of time differences, including time preference, preference change, delay as well as risk and uncertainty. From an empirical point of view, it can be asked whether existing accounts of descriptive time discounting approaches satisfy the measurement conditions needed to specify their functional form. Concerning a possible justification of time discounting, the framework lends itself to a neutral comparison of the normative appeal of different substantial interpretations of time difference.

## References

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