A Note on Transitivity, Completeness and Suzumura Consistency

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1 Preference and Choice

What is required of the agent who makes her choices on the basis of her preferences? What can be inferred about an agent's preferences from the choices she makes? In considering these questions, it is worth distinguishing between the choices that are *permissable* given her preferences, those that are *mandatory* and those that she *actually* makes. Rationality does not generally require that agents have strict preferences over all alternatives, so it is to be expected that these sets of choices will not coincide. For instance if she is indifferent between two alternatives or is unable to compare them it might be permissable for her to choose both of them, not mandatory to pick either, while in fact choosing only one of them.

There are two implications of this point. Firstly, preference-based explanations are necessarily limited in scope. Invoking someone's preferences will suffice to explain why some choices were not made (i.e. in terms of rational impermissibility) but not typically why some particular choice was made. To take up the slack, explanations must draw on factors other than preference: pyschological one such as the framing of the choice problem or the saliency of particular options, or sociological ones such as the existence of norms or conventions governing choices of the relevant kind.

Secondly, observations of actual choices will only partially constrain preference attribution. For instance, that someone chooses a banana when an apple is available does not allow one to conclude that the choice of an apple was ruled out by her preferences, only that her preferences ruled the banana in. In this simple observation lies a serious obstacle to the ambition of Revealed Preference theory to give conditions on observed choices sufficient for the existence of a preference relation that rationalises them. For the usual practice of inferring the completeness of the agent's preferences from the fact that she always makes a choice when required to is clearly illegitimate if more than one choice is permitted by her preferences.

The upshot is that the usual focus on the case where an agent has complete preferences is quite unjustified. The aim of this note is therefore to explore the two opening questions without assuming completeness, building on the work of

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Sen [8], Richter [6] and especially the recent work of Bossert and Suzumura [3][4]. I argue that when incompleteness of preference is reasonable then rationality does not require full transitivity of preferences. Instead it requires it that they be Suzumura consistent - roughly that there be no cycles of weak preference containing a strong preference. In a similar vein I argue for a choice rule - Strong Maximality - that is roughly intermediate between optimisation and maximisation and show that Suzumura consistency of preference is sufficient to ensure that this choice rule picks a non-empty set of alternatives from any given set of them. Finally, I investigate the rationalisability of choice functions in terms of Suzumura consistent preferences and strong maximal choice.

1.1 Preference

In the usual fashion we introduce a reflexive binary relation $\succeq$ (called the weak preference relation) on a finite set of alternatives X, with symmetric part $\approx$ (indifference) and anti-symmetric part $\succ$ (strict preference). In contrast to the way these terms are often used, we do not assume that in general any two alternatives are comparable under these preference relations. Instead we define a comparability relation $\bowtie$ on alternatives by: $\alpha \bowtie \beta$ iff $\alpha \succeq \beta$ or $\alpha \succeq \beta$. When all alternatives are comparable the preference relation is said to be complete. (Hence it is incomplete if there are alternatives α and β in its domain such that $\alpha \not\bowtie \beta$.)

1.1.1 Transitivity

A number of different forms of transitivity-like properties of preference relations will be of interest. We say that $\succeq$ is:

1. *Transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succeq \beta$ and $\beta \succeq \gamma$ implies that $\alpha \succeq \gamma$ (and intransitive otherwise)
2. *Incompletely transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succeq \beta$ and $\beta \succeq \gamma$ implies that $\gamma \not\succ \alpha$
3. *PI-transitive* iff for all $\alpha, \beta, \gamma \in X$, $\alpha \succ \beta$ and $\beta \approx \gamma$ implies that $\alpha \succ \gamma$
4. *Quasi-transitive* iff $\succ$ is transitive

Transitivity implies incomplete transitivity, PI-transitivity and quasi-transitivity. On the other hand, a reflexive relation is transitive if it is either both complete and incompletely transitivity or both PI-transitive and quasi-transitive [7, Theorem I.6]. But in general a relation can be incompletely transitive without being PI-transitive or quasi-transitive, and vice versa: they constitute alternative weakenings of transitivity.

The view taken here is that completeness is not a rationality requirement on preference. This is not in itself very controversial. Much more so is something that follows rather naturally from this view, namely that transitivity is too strong a requirement to impose on preferences. The problem is that transitivity imposes comparability even when it is not appropriate to do so. The following example serves to illustrate this point.

Example 1 *Suppose that Ann, Bob and Carol are have interval scores in Maths and English as follows:*

- *(Ann) Maths: 80-90, English: 60-70*
- *(Bob) Maths: 56-65, English: 66-75*
- *(Carol) Maths: 75-85, English: 55-65*

The teacher decides to rank them in each subject using the heuristic that two students with overlapping intervals scores in a subject should be regarded as on a par in that subject, but one is ranked higher than the other if the lower bound of their interval score is greater than the upper bound of the interval score of the other. So Ann ranks higher than Bob because she is definitely better at Maths and not comparably worse at English, Bob and Carol are ranked the same because each is better at one of the subjects and Ann and Carol are unranked relative to each other because neither is comparably better than the other in either subject.

The teacher's ranking of her students does not satisfy transitivity, but it is not obvious that her ranking is irrational given her inability to discriminate between Ann and Carol on the basis of their performances. It is not that the teacher should not infer that Ann is better than Carol, but rather that she is not rationally *compelled* to do so. This suggest that in situations in which a preference relation is not complete then the requirements of rationality (with regard to preferences between pairs of a triple of alternatives) are more appropriately expressed by the condition of incomplete transitivity, than by full transitivity.

1.1.2 Consistency

In addition to the basic conditions on preference listed above, which are defined in terms of pairs or triples of alternatives, we are also interested in a number of derived consistency properties of the preference relation that can be defined in terms of these basic ones.

The weak preference relation $\succeq$ will be said to be:

1. *Strongly consistent* iff $\succeq$ is transitive
2. *Suzumura consistent* iff for all $\alpha_1, \alpha_2, ..., \alpha_n \in X$, $\alpha_1 \succeq \alpha_2, \alpha_2 \succeq \alpha_3, ..., \alpha_{n-1} \succeq \alpha_n$ implies that $\alpha_n \not\succ \alpha_1$
3. *Weakly consistent* iff $\succ$ is acyclic iff for all $\alpha_1, \alpha_2, ..., \alpha_n \in X$, $\alpha_1 \succ \alpha_2, \alpha_2 \succ \alpha_3, ..., \alpha_{n-1} \succ \alpha_n$ implies that $\alpha_n \not\succ \alpha_1$

These properties are in descending order of strength: strong consistency implies Suzumura consistency which implies weak consistency. Suzumura consistency strengthens incomplete transitivity, by extending it to arbitrary sets of alternatives. As Bossert and Suzumura [4] point out there are three notable characteristics of Suzumura consistency. Firstly it rules out cycles with at least one strict preference and so preferences that satisfy it are not vulnerable to money pumps. Secondly, Suzumura consistency is necessary and sufficient for the existence of a representation of the preference relation by a family of utility functions. And thirdly, any preference relation that is both Suzumura consistent and complete is strongly consistent. So there is good reason to think of Suzumura consistency as being the appropriate consistency condition for incomplete preferences.

2 Preference-based Choice

Let $\mathcal{C}$ be a choice function on $\wp(X)$: a mapping from subsets $A \subseteq X$ to subsets $\mathcal{C}(A) \subseteq A$. Intuitively $\mathcal{C}(A)$ is the set of objects from the set A that *could* be chosen: could permissibly be so in normative interpretations, could factually be so in descriptive ones. When its range is restricted to singleton sets, the choice function is said to be *element valued*. When its range is restricted to non-empty sets it is said to be *decisive*. (Decisiveness is often built into the definition of a choice function, but it will prove more convenient here to make it a separate assumption.)

We are especially interested in the case when a choice function $\mathcal{C}$ can be said to be based on or determined by a preference relation. A natural condition for this being the case is that an object is chosen from a set if and only if no other object in the set is strictly preferred to it. Formally:

SPBC: (Strict Preference Based Choice)

$$\alpha \succ \beta \Rightarrow \forall(A : \alpha \in A), \beta \notin \mathcal{C}(A)$$

SPBC certainly seems necessary for preference-based choice. But is it sufficient? I think not. A further requirement is that two alternatives that are regarded indifferently should always either both be chosen or both not chosen. Formally:

IBC: (Indifference Based Choice)

$$\alpha \approx \beta \Rightarrow \forall(A : \alpha, \beta \in A), \alpha \in \mathcal{C}(A) \Leftrightarrow \beta \in \mathcal{C}(A)$$

SPBC and IBC are not generally sufficient to determine choice because they don't settle the question of how to handle incomparability. Let us therefore consider three possible preference-based rules of choice that do fully determine what may be chosen and consider how they relate to these conditions. To do so it is useful to consider the transitive closure of $\approx$ on X, denoted $\approx^*$ and defined by (1) $\alpha \approx^* \alpha$, and (2) if $\alpha \approx^* \beta$ and $\gamma \approx \beta$ then $\alpha \approx^* \beta$. Note that if $\alpha \approx^* \beta$ then there exists a sequence of elements in X, α_1, α_2, ..., α_n linking α and β in the sense that $\alpha \approx \alpha_1$, $\alpha_1 \approx \alpha_2$, ..., and $\alpha_n \approx \beta$. It follows that $\approx^*$ is transitive and symmetric. Hence $\approx^*$ is an equivalence relation on X. We call the set of $\beta \in A$ such that $\alpha \approx^* \beta$, the indifference class of α in A.

The three rules are as follows:

Optimality: An object is chosen from a set if and only if it is weakly preferred to all others in the set. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \forall(\beta \in A), \alpha \succeq \beta$$

Maximality: An object is chosen from a set if and only if no alternative in the set is strictly preferred to it. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \neg\exists(\beta \in A : \beta \succ \alpha)$$

Strong Maximality: An object is chosen from a set if and only if no alternative in its indifference class is strictly dispreferred to some alternative in the set. Formally, for all A such that $\alpha \in A$:

$$\alpha \in \mathcal{C}(A) \Leftrightarrow \neg\exists(\beta, \gamma \in A : \alpha \approx^* \gamma \text{ and } \beta \succ \gamma)$$

Of these three rules, Optimality is the one that is most commonly taken to express rational preference-based choice (see, for instance, Arrow [1] and Sen [8]). But although Optimality satisfies both SPBC and IBC, it is clearly too strong a condition on *permissible* choice. This is because it implies that if $\alpha \not\bowtie \beta$ then $\mathcal{C}(\{\alpha, \beta\}) = \varnothing$. But even if there are situations in which no choice is permissible (contrary to the usual assumption of decisiveness), this is not a consequence of incomparability. If two alternatives are incomparable it should normally be permissible to choose either of them.

For this reason Maximality is often seen as the more appropriate rule of rational choice when the possibility of incomparability is not ruled out (see Sen [9]). But Maximality is also not quite right, as the following schematic version of our earlier example shows. Suppose that $\alpha > \beta$ and $\beta \approx \gamma$ but $\alpha \not\bowtie \gamma$. Then it would not be unreasonable for $\mathcal{C}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ because the two alternatives are incomparable and $\mathcal{C}(\{\beta, \gamma\}) = \{\beta, \gamma\}$ because the two alternatives are equally preferred, but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha\}$ because β should not be chosen when a strictly preferred alternative - α - is available and γ should not be chosen if β is not, given that $\gamma \approx \beta$. But these choices are inconsistent with Maximality which requires that if $\gamma \notin \mathcal{C}(\{\alpha, \beta, \gamma\})$ then either $\gamma \notin \mathcal{C}(\{\alpha, \gamma\})$ or $\gamma \notin \mathcal{C}(\{\alpha, \beta\})$.

The problem with Maximality is that it leads to violations of IBC. For by Maximality, $\mathcal{C}(\{\beta, \gamma\}) = \{\beta, \gamma\}$ but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha, \gamma\}$. So β is not chosen whenever γ is, even though $\beta \approx \gamma$. So just as admitting the possibility of incompleteness required a shift from Optimality to Maximality, so too recognition of the rational permissibility of incompletely transitive preferences requires a shift from Maximality to Strong Maximality.

Let us consider a reformulation of Strong Maximality that will make its implications clearer. Let $\mathbf{X} = \{\mathbf{A}, \mathbf{B}, ...\}$ be the set of equivalence classes in X induced by the relation $\approx^*$. Define a weak preference relation $\geq$ on $\mathbf{X}$ by:

$$\mathbf{A} \geq \mathbf{B} \Leftrightarrow \exists(\alpha \in \mathbf{A}, \beta \in \mathbf{B} : \alpha \succeq \beta)$$

Then choosing from any $A \subseteq X$ in accordance with Strong Maximality is equivalent to choosing the $\geq$-maximal element of the set $\mathbf{A}$ of equivalence classes in A induced by the equivalence relation $\approx^*$.

Now it might be objected that adopting Strong Maximality as a principle of rational choice is tantamount to smuggling transitivity back in. But this is not correct. Suppose $\alpha > \beta$, $\beta \approx \gamma$ and $\gamma \approx \delta$. Then Strong Maximality requires that $\mathcal{C}(\{\alpha, \beta, \delta\}) = \{\alpha, \delta\}$. But if transitivity were imposed then it would be required that $\mathcal{C}(\{\alpha, \beta, \delta\}) = \{\alpha\}$. Nonetheless it might seem that at very least an agent must in effect apply transitivity of indifference in order to determine equivalence classes under $\approx^*$. But there is another way for formulating the rule which might serve to alleviate this worry. Let $\bar{\mathcal{C}}(A)$ be the set of impermissible alternatives: the alternatives that may not be chosen. Now we define $\bar{\mathcal{C}}(A)$ recursively as follows:

1. If $\exists\beta \in A$ such that $\beta \succ \alpha$ then $\alpha \in \bar{\mathcal{C}}(A)$
2. If $\exists\beta \in \bar{\mathcal{C}}(A)$ such that $\beta \approx \alpha$ then $\alpha \in \bar{\mathcal{C}}(A)$

Then Strong Maximality is equivalent to the rule:

Non-elimination: $\alpha \in \mathcal{C}(A) \Leftrightarrow \alpha \notin \bar{\mathcal{C}}(A)$

To apply the rule it suffices that the agent iteratively eliminates alternatives from her choice set by removing any dominated alternatives; then checking if any alternatives that are left are indifferent to any eliminated ones and, if so, removing them as well; then checking if any alternatives that are left are indifferent to any eliminated ones, and so on.

2.1 Properties of Preference-based Choice

Each of the three choice rules under examination expresses a view on the relationship between preference and choice. To examine what these are and how they differ for the three choice rules, let us denote the choice function determined by the weak preference relation $\succeq$ together with Maximality, Optimality or Strong Maximality by $\mathcal{C}^{Max}_{\succeq}$, $\mathcal{C}^{Op}_{\succeq}$ and $\mathcal{C}^{SM}_{\succeq}$ respectively, where these are defined as follows. For any $A \subseteq X$:

$$\begin{aligned}
\mathcal{C}^{Op}_{\succeq}(A) &= \{\alpha \in A : \forall\beta \in A, \alpha \succeq \beta\} \\
\mathcal{C}^{Max}_{\succeq}(A) &= \{\alpha \in A : \forall\beta \in A, \beta \not\succ \alpha\} \\
\mathcal{C}^{SM}_{\succeq}(A) &= \{\alpha \in A : \forall\gamma \in \approx^* (\alpha, A), \neg\exists(\beta \in A : \beta \succ \gamma)\}
\end{aligned}$$

The first thing to note is that the set of permissible choice according to $\mathcal{C}^{Max}_{\succeq}$ is always at least as large as those determined by $\mathcal{C}^{Op}_{\succeq}$ or $\mathcal{C}^{SM}_{\succeq}$. Furthermore when the preference relation is Suzumura consistent then the set of choices that are permissible according to Strong Maximality contain those that are permissible according to Optimality (as well as being contained by those determined by Maximality). On the other hand when the preference relation is complete $\mathcal{C}^{Op}_{\succeq}$ coincides with $\mathcal{C}^{Max}_{\succeq}$ and when it is transitive, $\mathcal{C}^{SM}_{\succeq}$ coincides with $\mathcal{C}^{Max}_{\succeq}$. These claims are established below as Theorem 2.

It is well known that $\mathcal{C}^{Op}_{\succeq}$ is decisive iff $\succeq$ is complete and weakly consistent and that if $\mathcal{C}^{Max}_{\succeq}$ is decisive iff $\succeq$ is weakly consistent. Theorem 3 below establishes a corresponding result for choices that are strongly maximal, namely that a choice function based on Strong Maximality is decisive iff the underlying preference relation is Suzumura consistent. The main significance of this result for our argument is that Suzumura consistency is thereby shown to be both necessary and sufficient for decisive, strongly maximal choice.

Theorem 2 *1.* $\mathcal{C}^{Op}_{\succeq} \subseteq \mathcal{C}^{Max}_{\succeq}$ *and* $\mathcal{C}^{SM}_{\succeq} \subseteq \mathcal{C}^{Max}_{\succeq}$

2. If $\succeq$ *is complete then* $\mathcal{C}^{Op}_{\succeq} = \mathcal{C}^{Max}_{\succeq}$

3. If $\succeq$ *is transitive, then* $\mathcal{C}^{SM}_{\succeq} = \mathcal{C}^{Max}_{\succeq}$

4. If $\succeq$ *is Suzumura consistent, then* $\mathcal{C}^{Op}_{\succeq} \subseteq \mathcal{C}^{SM}_{\succeq}$.

5. If $\succeq$ is complete and transitive then $\mathcal{C}^{SM}_{\succeq} = \mathcal{C}^{Op}_{\succeq} = \mathcal{C}^{Max}_{\succeq}$

Proof. (1) Suppose $\alpha \in \mathcal{C}^{Op}_{\succeq}(A)$. Then $\forall \beta \in A$, $\alpha \succeq \beta$. But then $\forall \beta \in A$, $\beta \not\succ \alpha$. So $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$. Similarly suppose $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$. Now by the symmetry of indifference $\alpha \approx \alpha$, so it follows that $\neg\exists\beta \in A$ such that $\beta \succ \alpha$). So $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$.

(2) Suppose that $\succeq$ is complete. Then for any $\beta \in A$ if $\beta \not\succ \alpha$ then $\alpha \succeq \beta$. Hence if $\alpha \in \mathcal{C}^{Op}_{\succeq}(A)$ then $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$. So $\mathcal{C}^{Op}_{\succeq} = \mathcal{C}^{Max}_{\succeq}$.

(3) Suppose that $\succeq$ is transitive but that contrary to hypothesis there exists $\alpha \in A$ such that $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$ but $\alpha \notin \mathcal{C}^{SM}_{\succeq}(A)$. Now if $\alpha \notin \mathcal{C}^{SM}_{\succeq}(A)$ then there exists $\beta, \gamma \in A$ such that $\alpha \approx^* \gamma$ and $\beta \succ \gamma$. By transitivity, if $\alpha \approx^* \gamma$ then $\alpha \approx \gamma$ and so by transitivity again, $\beta \succeq \alpha$. But if $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$ then $\beta \not\succ \alpha$. So $\beta \approx \alpha$ and by transitivity, $\beta \approx \gamma$. Hence, contrary to assumption, $\beta \not\succ \gamma$. It follows that if $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$ then $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$ and hence that $\mathcal{C}^{Op}_{\succeq} = \mathcal{C}^{Max}_{\succeq}$.

(4) Suppose that $\succeq$ is Suzumura consistent and that $\alpha \in \mathcal{C}^{Op}_{\succeq}(A)$. Then $\forall \beta \in A$, $\alpha \succeq \beta$. Let $\gamma \in A$ be such that $\alpha \approx^* \gamma$. Then there exists a sequence of elements in A, α_1, α_2, ..., α_n linking γ, α and β in the sense that $\gamma \approx \alpha_1$, $\alpha_1 \approx \alpha_2$, ..., $\alpha_n \approx \alpha$ and $\alpha \succeq \beta$. Hence by Suzumura consistency $\beta \not\succ \gamma$. it follows that $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$.

(5) Follows from 2 and 3. ■

Theorem 3 *1. $\mathcal{C}^{Op}_{\succeq}$ is decisive iff $\succeq$ is complete and weakly consistent*

2. $\mathcal{C}^{Max}_{\succeq}$ is decisive iff $\succeq$ is weakly consistent

3. $\mathcal{C}^{SM}_{\succeq}$ is decisive iff $\succeq$ is Suzumura consistent

Proof. (2) Suppose $\succeq$ is not weakly consistent. Then there exists $A = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subseteq X$, such that $\alpha_0 \succeq \alpha_1, \alpha_1 \succeq \alpha_2, ..., \alpha_{n-1} \succeq \alpha_n$ and $\alpha_n \succ \alpha_0$. But then for $n \geq i \geq 1$, $\alpha_i \notin \mathcal{C}^{Max}_{\succeq}(A)$ because $\alpha_{i-1} \succ \alpha_i$. And $\alpha_0 \notin \mathcal{C}^{Max}_{\succeq}(A)$ because $\alpha_n \succ \alpha_1$. So $\mathcal{C}^{Max}_{\succeq}(A) \neq \varnothing$. Hence $\mathcal{C}^{Max}_{\succeq}$ is not decisive. For the converse see Kreps [5].

(1) Suppose $\succeq$ is either not weakly consistent or incomplete. Suppose it is not weakly consistent. Then since by Theorem 2(1), $\mathcal{C}^{Op}_{\succeq} \subseteq \mathcal{C}^{Max}_{\succeq}$ it follows from 2. that $\mathcal{C}^{Op}_{\succeq}(A) \neq \varnothing$. Now suppose that $\succeq$ is incomplete. Then there exists $\alpha, \beta \in X$ such that $\alpha \not\succeq \beta$ and $\alpha \not\succeq \beta$. Then $\mathcal{C}^{Op}_{\succeq}(\{\alpha, \beta\}) \neq \varnothing$. So $\mathcal{C}^{Op}_{\succeq}$ is not decisive. The converse follows from 2. and Theorem 2 (2).

(3) Suppose that $\mathcal{C}^{SM}_{\succeq}$ is decisive but that, contrary to hypothesis, $\succeq$ is not Suzumura consistent, i.e. for some set $A = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ it is the case that $\alpha_1 \succeq \alpha_2, \alpha_2 \succeq \alpha_3, ..., \alpha_{n-1} \succeq \alpha_n$ but that $\alpha_n \succ \alpha_1$. We prove by induction on i that it then follows that for all $\alpha_i \in A$, $\alpha_i \notin \mathcal{C}_{\succeq}(A)$ and hence that $\mathcal{C}_{\succeq}(A) = \varnothing$. First Strong Maximality implies that $\alpha_1 \notin \mathcal{C}_{\succeq}(A)$ because $\alpha_n \succ \alpha_1$. Now assume that for some $k > 1$, $\alpha_k \notin \mathcal{C}_{\succeq}(A)$. Then there exists some α_j and $\alpha_{j'}$ such that $\alpha_k \approx^* \alpha_j$ but $\alpha_{j'} \succ \alpha_j$. Now consider α_{k+1}. Either $\alpha_k \succ \alpha_{k+1}$ in which case it follows by Strong Maximality that $\alpha_{k+1} \notin \mathcal{C}_{\succeq}(A)$. Or $\alpha_k \approx \alpha_{k+1}$, in which case $\alpha_k \approx^* \alpha_{k+1}$. But then $\alpha_{k+1} \approx^* \alpha_j$ and so by Strong Maximality $\alpha_{k+1} \notin \mathcal{C}_{\succeq}(A)$. But this implies that $\mathcal{C}_{\succeq}(A) = \varnothing$ in contradiction to the assumption of decisiveness. So $\succeq$ must be Suzumura consistent.

For the other direction, suppose that $\succeq$ is Suzumura consistent, but that for some set A, $\mathcal{C}^{SM}_{\succeq}(A) = \varnothing$. If $\mathcal{C}^{SM}_{\succeq}(A) = \varnothing$ then $\mathcal{C}^{Max}_{\geq}(\mathbf{A}) = \varnothing$ and so by (2), $>$

is cyclic i.e. there exists subsets of X such that $\mathbf{A}_1 > \mathbf{A}_2, \mathbf{A}_2 > \mathbf{A}_3 ..., \mathbf{A}_{n-1} > \mathbf{A}_n$,and $\mathbf{A}_n > \mathbf{A}_1$. So by definition there exists $\alpha_1, \alpha'_1, \alpha_2, ..., \alpha'_n, \alpha_n \in A$ such that $\alpha_1 \approx^* \alpha'_1, ..., \alpha'_n \approx^* \alpha_n$ and $\alpha'_1 \succ \alpha_2$, $\alpha'_2 \succ \alpha_3$,..., $\alpha'_{n-1} \succ \alpha_n$, but $\alpha'_n \succ \alpha_1$. But by Suzumura consistency, if $\alpha_1 \approx^* \alpha'_1, \alpha'_1 \succ \alpha_2$, $\alpha_2 \approx^* \alpha'_2, \alpha'_2 \succ \alpha_3$,..., $\alpha'_{n-1} \succ \alpha_n$ and $\alpha_n \approx^* \alpha'_n$ then $\alpha'_n \not\succ \alpha_1$. So $\mathcal{C}^{Max}_{\geq}(\mathbf{A}) \neq \varnothing$. Hence $\mathcal{C}^{SM}_{\succeq}$ is decisive. ■

2.2 Properties of Choice Functions

What properties of choice functions are induced by our choice rules? The following properties of choice functions - Sen's alpha, beta and gamma conditions - have figured prominently in the existing literature. Let $\alpha \in A$, $\beta \in B$, and $\gamma \in C$; then:

Set Contraction: If $\alpha \in \mathcal{C}(B)$ and $A \subseteq B$ then $\alpha \in \mathcal{C}(A)$

Set Expansion: If $\alpha, \beta \in \mathcal{C}(B)$, $B \subseteq A$ and $\beta \in \mathcal{C}(A)$, then $\alpha \in \mathcal{C}(A)$

Set Union: If $\alpha \in \mathcal{C}(A)$ and $\alpha \in \mathcal{C}(B)$, then $\alpha \in \mathcal{C}(A \cup B)$

It is well known that Optimality-based choice will satisfy both Set Contraction and Set Expansion so long as the underlying weak preference relation is weakly consistent (see Sen [8]). In fact choice based on weakly consistent preferences will satisfy Set Contraction given any one of the three choice rules under examination. Set Expansion on the other hand need not be be satisfied by maximal or strongly maximal choice. This is as it should be: when preferences are not complete, then choices should not satisfy Set Expansion. Suppose, for example, that the agent cannot compare α and β, but that no alternative in B is preferred to either. So both are permissible choices. Now suppose that $A = B \cup \{\gamma\}$ and that $\gamma \succ \alpha$ but $\gamma \not\bowtie \beta$. Then β is a still permissible choice but not α.

More interesting perhaps is that $\mathcal{C}^{SM}_{\succeq}$, unlike the other two rules, does not satisfy Set Union, for Sen has shown that satisfaction of this condition along with Set Contraction is essentially equivalent to the choice function being binary in composition (Sen [8]). On the other hand, as we establish below, it is both necessary and sufficient that strongly maximal choices be based on PI-transitive preferences for $\mathcal{C}^{SM}_{\succeq}$ to satisfy Set Union.

Theorem 4 *1. $\mathcal{C}^{Op}_{\succeq}/\mathcal{C}^{Max}_{\succeq}/\mathcal{C}^{SM}_{\succeq}$ all satisfy Set Contraction*

2. $\mathcal{C}^{Op}_{\succeq}$ and $\mathcal{C}^{Max}_{\succeq}$ satisfy Set Union, but $\mathcal{C}^{SM}_{\succeq}$ need not.

3. If $\mathcal{C}^{SM}_{\succeq}$ is decisive then $\mathcal{C}^{SM}_{\succeq}$ satisfies Set Union iff $\succeq$ is PI-transitive

Proof. (1) Suppose $B \subseteq A$ and $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$. Then $\forall \beta, \gamma \in A$ such that $\alpha \approx^* \gamma$, $\beta \not\succ \gamma$. Hence $\forall \beta, \gamma \in B$ such that $\alpha \approx^* \gamma$, $\beta \not\succ \gamma$. So $\alpha \in \mathcal{C}^{SM}_{\succeq}(B)$. Similarly for $\mathcal{C}^{Op}_{\succeq}$ and $\mathcal{C}^{Max}_{\succeq}$.

(2) Suppose that $\alpha \in \mathcal{C}^{Max}_{\succeq}(A)$ and $\alpha \in \mathcal{C}^{Max}_{\succeq}(B)$. Then $\forall \beta \in A$, $\beta \not\succ \alpha$ and $\forall \beta \in B$, $\beta \not\succ \alpha$. Hence $\forall \beta \in A \cup B$, $\beta \not\succ \alpha$. It follows that $\alpha \in \mathcal{C}^{Max}_{\succeq}(A \cup B)$. Similarly for $\mathcal{C}^{Op}_{\succeq}$. However consider a case in which $\alpha \not\bowtie \beta$, $\alpha \approx \gamma$ and $\beta \succ \gamma$.

Then $\mathcal{C}^{SM}_{\succeq}(\{\alpha, \beta\}) = \{\alpha, \beta\}$, $\mathcal{C}^{SM}_{\succeq}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ but $\alpha \notin \mathcal{C}^{SM}_{\succeq}(\{\alpha, \beta, \gamma\})$ because $\alpha \approx^* \gamma$ and $\beta \succ \gamma$.

(3) Suppose $\succeq$ is PI-transitive, that $\alpha \in \mathcal{C}^{SM}_{\succeq}(A)$ and that $\alpha \in \mathcal{C}^{SM}_{\succeq}(B)$, but contrary to hypothesis $\alpha \notin \mathcal{C}^{SM}_{\succeq}(A \cup B)$. Then there exists $\beta, \gamma \in A \cup B$ such that $\alpha \approx^* \gamma$ and $\beta \succ \gamma$. But then by repeated applications of PI-transitivity it follows that $\beta \succ \alpha$. Hence $\alpha \notin \mathcal{C}^{SM}_{\succeq}(A)$ or $\alpha \notin \mathcal{C}^{SM}_{\succeq}(B)$, depending on whether $\beta \in A$ or $\beta \in B$. Now suppose that $\succeq$ is not PI-transitive. Then there exists $\alpha, \beta, \gamma \in X$ such that $\alpha \succ \beta$ and $\beta \approx \gamma$ but $\alpha \not\succ \gamma$. Then either $\gamma \succ \alpha$, $\alpha \approx \gamma$ or $\alpha \not\bowtie \gamma$. Suppose $\gamma \succ \alpha$ or $\alpha \approx \gamma$. Then $\mathcal{C}^{SM}_{\succeq}(\{\alpha, \beta, \gamma\}) = \varnothing$, contrary to the assumption that $\mathcal{C}^{SM}_{\succeq}$ is decisive. So suppose that $\alpha \not\bowtie \gamma$. Then $\gamma \in \mathcal{C}^{SM}_{\succeq}(\{\alpha, \gamma\})$ and $\gamma \in \mathcal{C}^{SM}_{\succeq}(\{\beta, \gamma\})$, but $\gamma \notin \mathcal{C}^{SM}_{\succeq}(\{\alpha, \beta, \gamma\})$ because $\alpha \succ \beta$ and $\gamma \approx \beta$. So Set Union is violated. ■

3 Rationalisability

A question that naturally arises is whether, and under what conditions, the choices that are formally represented by a choice function can be rationalised or explained in terms of an underlying preference relation that, together with some choice rule, determines it. To tackle it, let us say that a choice function $\mathcal{C}$ is *rationalisable* by a consistent weak preference relation $\succeq$ iff $\mathcal{C}$ is generated by $\succeq$ together with a given choice rule R, i.e. iff $\mathcal{C} = \mathcal{C}^{R}_{\succeq}$. This definition of rationalisability contains two unspecified parameters: the type of consistency to be required of preference and the type of choice rule to be used in the determination of the choice function. Different kinds of rationalisability will be associated with different choices of values for these parameters and a particular choice function may be rationalisable relative to some combination of consistency property and choice rule but not another. Here we will require that the preference relation be at least weakly consistent in order to speak of rationalisation and differentiate between O-, M- and SM-rationalisations of a choice function in accordance with the choice rule that determines it.

3.1 Revealed Preference

In the literature on revealed preference the question of rationalisability is typically approached by defining the weak preference relation $\succeq_{\mathcal{C}}$ 'revealed' by a choice function $\mathcal{C}$ in the following way:

Revealed Preference: $\alpha \succeq_{\mathcal{C}} \beta \Leftrightarrow \exists A \subseteq X$ such that $\beta \in A$ and $\alpha \in \mathcal{C}(A)$

It is then possible to ask what properties of the revealed preference relation $\succeq_{\mathcal{C}}$ are implied by various assumed properties of the choice function $\mathcal{C}$. It is well known for instance that if $\mathcal{C}$ satisfies both Set Contraction and Set Expansion then $\succeq_{\mathcal{C}}$ so defined is both complete and transitive (see Sen [8, Theorem II]). In this case, as we learnt from Theorem 3 (5), our three choice rules coincide and so it is reasonable to speak without further qualification of the revealed preference relation $\succeq_{\mathcal{C}}$ as rationalising or explaining the choices represented by $\mathcal{C}$. But when either transitivity and completeness is not implied by the properties of the choice function then this neat relationship breaks down. Indeed in the absence of grounds for presuming completeness, the underlying conception of revealed preference becomes much less compelling.

The fundamental problem with the usual definition of the revealed preference relation is that it does not allow for any distinction between an attitude of indifference between two alternatives and an inability to compare them. Indeed the effect of Revealed Preference is to collapse the two since it entails that $\alpha \approx_{\mathcal{C}} \beta \Leftrightarrow \exists A, B \subseteq X$ such that $\alpha, \beta \in A, B$, $\alpha \in \mathcal{C}(A)$ and $\beta \in \mathcal{C}(B)$ and so ascribes to the agent an attitude of indifference between any two alternatives that can permissibly be chosen from some set containing both - in particular to any alternatives α, β such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}-$ irrespective of whether they are comparable or not.

To allow for incomparability we need to build a revealed weak preference relation up from its component revealed strict preference and indifference relations. I suggest that the following definitions encode the correct way to do so.

RSP: $\alpha \succ_{\mathcal{C}} \beta \Leftrightarrow \forall (A : \alpha \in A), \beta \notin \mathcal{C}(A)$

RI: $\alpha \approx_{\mathcal{C}} \beta \Leftrightarrow \forall (A : \alpha, \beta \in A)$, $\alpha \in \mathcal{C}(A) \Leftrightarrow \beta \in \mathcal{C}(A)$

RWP: $\alpha \succeq_{\mathcal{C}} \beta \Leftrightarrow \alpha \succ_{\mathcal{C}} \beta$ or $\alpha \approx_{\mathcal{C}} \beta$

RSP strengthens SPBC into a biconditional that mandates the inference that one prospect is strictly preferred to another iff the latter is never chosen when the former is available. Note that RSP implies that $\beta \notin \mathcal{C}(\{\alpha, \beta\})$ if $\alpha \succ_{\mathcal{C}} \beta$. The converse is only true however if $\mathcal{C}$ satisfies Set Contraction. Put more positively, if a choice function satisfies Set Contraction then the revealed strict preference relation based on it is binary in composition.

RI similarly strengthens IBC into a biconditional, but the inference it mandates is more controversial; namely that two alternatives are indifferent iff they are either both chosen or both not. The intuition underlying RI is that what distinguishes indifference between two alternatives from their incomparability is that in the former case (indifference) no third alternative should be strictly be preferred to one, but not the other, of the pair, while in the latter case (incomparability) such a third alternative could exist. The problem is that in a sufficiently sparse domain such a third alternative might not in fact exist and then RI would mandate an inference of indifference when the case is actually one of incomparability. On the other hand, when the underlying set of alternatives contain for every pair of alternatives, α and β, a third alternative α^+, that is comparably better than α, or alternative β^- that is comparably worse than β, then RI will be applicable.

RWP defines $\succeq_{\mathcal{C}}$ in terms of the relations of strict preference, $\succ_{\mathcal{C}}$, and indifference, $\approx_{\mathcal{C}}$, that are revealed by the choice function $\mathcal{C}$ in accordance with RSP and RI. So defined $\succeq_{\mathcal{C}}$ is not necessarily complete, since it can be case that there are sets A and B such that $\alpha, \beta \in A, B$ but $\alpha \in \mathcal{C}(A)$ and $\beta \in \mathcal{C}(B)$. This would arise when α and β are incomparable and A and B contain elements that respectively dominate α and β but not the other. Furthermore, although $\approx_{\mathcal{C}}$ must be symmetric and $\succeq_{\mathcal{C}}$ reflexive, in the absence of any further assumptions about $\mathcal{C}$ it is not assured that $\succeq_{\mathcal{C}}$ is a weak preference relation, nor that $\succ_{\mathcal{C}}$ and $\approx_{\mathcal{C}}$ are its symmetric and anti-symmetric parts. For this we must assume that $\mathcal{C}$ is decisive.

Theorem 5 *Suppose that $\mathcal{C}$ is decisive. Then $\succeq_{\mathcal{C}}$ is a weakly consistent weak preference relation with symmetric and anti-symmetric parts $\succ_{\mathcal{C}}$ and $\approx_{\mathcal{C}}$.*

Proof. RI implies the symmetry of $\approx_{\mathcal{C}}$ and, together with RWP, the reflexivity of $\succeq_{\mathcal{C}}$. Note firstly that it is not possible that both $\alpha \succ_{\mathcal{C}} \beta$ and that $\alpha \approx_{\mathcal{C}} \beta$. For if $\alpha \succ_{\mathcal{C}} \beta$, then by RSP $\beta \notin \mathcal{C}(\{\alpha, \beta\})$. So by decisiveness $\alpha \in \mathcal{C}(\{\alpha, \beta\})$ and hence by RI $\alpha \not\approx_{\mathcal{C}} \beta$. Similarly if $\alpha \approx_{\mathcal{C}} \beta$ then by RI and decisiveness $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}$. So by RSP, $\alpha \not\succ_{\mathcal{C}} \beta$. To establish the anti-symmetry of $\succ_{\mathcal{C}}$, let $\Gamma := \{A \subseteq X : \alpha, \beta \in A\}$. Suppose that $\alpha \succ_{\mathcal{C}} \beta$ so that by RSP, $\forall A \in \Gamma$, $\beta \notin \mathcal{C}(A)$. Then $\beta \notin \mathcal{C}(\{\alpha, \beta)\}$ and hence by decisiveness, $\alpha \in \mathcal{C}(\{\alpha, \beta)\}$. So it is not the case that $\forall A \in \Gamma$, $\alpha \notin \mathcal{C}(A)$, i.e. $\beta \not\succ_{\mathcal{C}} \alpha$. Finally suppose that, contrary to hypothesis, $\succeq_{\mathcal{C}}$ is not weakly consistent. Then there exists a sequence of alternatives α_1, α_2, ..., α_n, such that, $\alpha_1 \succ_{\mathcal{C}} \alpha_2$, $\alpha_2 \succ_{\mathcal{C}} \alpha_3$, ..., $\alpha_{n-1} \succ_{\mathcal{C}} \alpha_n$ and $\alpha_n \succ_{\mathcal{C}} \alpha_1$. Then by RSP, $\mathcal{C}(\{\alpha_1, \alpha_2, ..., \alpha_n\}) = \varnothing$ contrary to decisiveness. So $\succeq_{\mathcal{C}}$ must be weakly consistent. ■

3.2 Conditions for Rationalisability

Let us now turn to the question of whether it is possible in general to rationalise an arbitrary choice function $\mathcal{C}$ in terms of the revealed weak preference relation $\succeq_{\mathcal{C}}$ defined by RWP. As is to be expected, without some restrictions on $\mathcal{C}$ and/or its domain, the answer is negative for each of the three types of rationalisability under consideration.

1. *O-rationalisability*: Consider $\mathcal{C}$ and domain $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha, \beta\}$ but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\beta, \gamma\}$. Then by RWP, $\alpha \not\succeq_{\mathcal{C}} \beta$ and $\beta \not\succeq_{\mathcal{C}} \alpha$. So $\mathcal{C}^{Opt}_{\succeq_{\mathcal{C}}}(\{\alpha, \beta\}) = \varnothing \neq \mathcal{C}(\{\alpha, \beta\})$.

2. *M-rationalisability*: Consider $\mathcal{C}$ and domain $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha\}$, $\mathcal{C}(\{\beta, \gamma\}) = \{\beta, \gamma\}$, $\mathcal{C}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$ but $\mathcal{C}(\{\alpha, \beta, \gamma\}) = \{\alpha\}$. Then by RWP, $\alpha \succ_{\mathcal{C}} \beta$, $\beta \approx_{\mathcal{C}} \gamma$ but $\gamma \not\bowtie_{\mathcal{C}} \alpha$. So $\mathcal{C}^{Max}_{\succeq_{\mathcal{C}}}(\{\alpha, \beta, \gamma\}) = \{\alpha, \gamma\} \neq \mathcal{C}(\{\alpha, \beta, \gamma\})$.

3. *SM-rationalisability*: Consider $\mathcal{C}$ and domain $\{\alpha, \beta, \gamma\}$ such that $\mathcal{C}(\{\alpha, \beta\}) = \{\alpha\}$, $\mathcal{C}(\{\beta, \gamma\}) = \{\beta\}$, and $\mathcal{C}(\{\alpha, \gamma\}) = \{\gamma\}$. So by RWP, $\alpha \not\bowtie_{\mathcal{C}} \beta$, $\beta \not\bowtie_{\mathcal{C}} \gamma$ and $\gamma \not\bowtie_{\mathcal{C}} \alpha$. But then $\mathcal{C}^{SM}_{\succeq_{\mathcal{C}}}(\{\alpha, \beta\}) = \{\alpha, \beta\} \neq \mathcal{C}(\{\alpha, \beta\})$.

What conditions on $\mathcal{C}$ are sufficient to ensure rationalisability? Our earlier observation that satisfaction of Set Contraction and Set Expansion is sufficient for O-rationalisability extends to both M- and SM-rationalisability: this is a consequence of Theorem 3 (5). This result is of marginal interest however since these conditions are very restrictive and indeed suffice to ensure the completeness of the revealed preference relation.

It is possible to do better. Below we establish that it is sufficient that a choice function be decisive and satisfy Set Contraction and Set Union, that it have a weakly consistent M-rationalisation. Since both conditions are also implied by Maximality, this theorem provides the required characterisation of consistent maximal choice.

Set Contraction and Set Union conditions are in fact also sufficient for a SM-rationalisation, but in this case it does not give us the characterisation that we seek since Set Union is not necessary for preference-based strongly maximal choice. What is required it turns out is a weaker version of Set Union. To state it let $\tilde{\mathcal{C}}(A, \alpha) := \{\gamma \in A : \forall B \subseteq X,\ \alpha \in \mathcal{C}(B) \Leftrightarrow \gamma \in \mathcal{C}(B)$. Then:

Restricted Union If $\tilde{\mathcal{C}}(B, \alpha) \subseteq \mathcal{C}(A)$ and $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(B)$, then $\alpha \in \mathcal{C}(A \cup B)$

In the second theorem below we show that it is sufficient that the choice function be decisive and satisfy Set Contraction and Restricted Union, that it have a Suzumura consistent SM-rationalisation. Since both conditions are implied by Strong Maximality, this last result gives us the characterisation of consistent, strongly maximal choice that we want.

Theorem 6 *Suppose that $\mathcal{C}$ is a decisive choice function satisfying Set Contraction and Set Union. Then there exists a weakly consistent weak preference relation $\succeq_{\mathcal{C}}$ that M-rationalises C.*

Proof. Let $\mathcal{C}$ be a choice function with domain X and let $\succeq_{\mathcal{C}}$ be defined from it by RWP, RSP and RI. Suppose that $\alpha \notin \mathcal{C}^{M}_{\succeq_{\mathcal{C}}}(A)$. Then by RSP and RI there exists $\beta \in A$ such that $\forall(B : \alpha \in B)$, $\alpha \notin \mathcal{C}(B)$. So in particular, $\alpha \notin \mathcal{C}(A)$. Now suppose that $\alpha \in \mathcal{C}_{\succeq_{\mathcal{C}}}(A)$. Then by RSP there does not exist any $\beta \in A$ such that $\forall(B \subset X : \beta \in B)$, $\alpha \notin \mathcal{C}(B)$. Hence for all $\beta_i \in A$, there exists a set $B_i \subseteq X$ such that $\beta \in B_i$ and $\alpha \in \mathcal{C}(B_i)$. But then by Set Union, $\alpha \in \mathcal{C}(\cup B_i) = \mathcal{C}(A)$. The weak consistency of $\succeq_{\mathcal{C}}$ then follows from Theorem 3 (2). ■

Theorem 7 *Suppose that $\mathcal{C}$ is a decisive choice function satisfying Set Contraction and Restricted Set Union. Then there exists a Suzumura consistent weak preference relation $\succeq_{\mathcal{C}}$ that SM-rationalises C.*

Proof. Let $\mathcal{C}$ be a choice function with domain X and let $\succeq_{\mathcal{C}}$ be defined from it by RWP, RSP and RI. Suppose that $\alpha \notin \mathcal{C}^{SM}_{\succeq_{\mathcal{C}}}(A)$. Then by RSP and RI there exists $\beta, \gamma \in A$ such that $\forall(B : \alpha, \gamma \in B)$, $\alpha \in \mathcal{C}(B) \Leftrightarrow \gamma \in \mathcal{C}(B)$ and $\forall(B : \beta \in B)$, $\gamma \notin \mathcal{C}(B)$. So in particular, since $\beta, \gamma \in A$, it follows that $\gamma \notin \mathcal{C}(A)$ and hence $\alpha \notin \mathcal{C}(A)$. Now suppose that $\alpha \in \mathcal{C}^{SM}_{\succeq_{\mathcal{C}}}(A)$. Then by RSP, for all $\gamma \in \tilde{\mathcal{C}}(A, \alpha)$, there does not exist any $\beta \in A$ such that $\forall(B \subseteq X : \beta \in B)$, $\gamma \notin \mathcal{C}(B)$. Hence for all $\beta \in A$, there exists a set $B \subseteq X$ such that $\beta \in B$ and $\gamma \in \mathcal{C}(B)$. So by But then by Set Contraction, $\gamma \in \mathcal{C}(\gamma, \beta\})$. But then it follows that for all $\beta \in A$, $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(\tilde{\mathcal{C}}(A, \alpha) \cup \{\beta\})$. For if there were some $\gamma \in \tilde{\mathcal{C}}(A, \alpha)$, such that $\gamma \notin \mathcal{C}(\tilde{\mathcal{C}}(A, \alpha) \cup \{\beta\})$, then by Set Contraction, $\gamma \notin \mathcal{C}(\gamma, \beta\})$, contrary to what has been established. Now note that since $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(\tilde{\mathcal{C}}(A, \alpha) \cup \{\beta\})$, it follows by Restricted Union that if, for some $B \subseteq X$, $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(\tilde{\mathcal{C}}(A, \alpha) \cup B)$, then for all $\beta \in A$, $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(\tilde{\mathcal{C}}(A, \alpha) \cup B \cup \{\beta\})$. Hence $\tilde{\mathcal{C}}(A, \alpha) \subseteq \mathcal{C}(A)$. In particular $\alpha \in \mathcal{C}(A)$. The Suzumura consistency of $\succeq_{\mathcal{C}}$ then follows from Theorem 3 (3). ■

References

[1] Arrow, K. J. (1959) "Rational Choice Functions and Orderings", *Economica* 26:121-127

[2] Baigent, N (2007) "Choices, Norms and Preference Revelation", *Analyse & Kritik: Zeitschrift fuer Sozialtheorie* 2, 139 - 145.

[3] Bossert, W., Y. Sprumont and Suzumura, K. (2005) "Consistent Rationalisability". *Economica*, 72: 185-200

[4] Bossert, W. and K. Suzumura (2010) *Consistency, Choice, and Rationality*, Harvard University Press

[5] Kreps, David M. (1988) *Notes on the Theory of Choice*, Boulder and London: Westview Press

[6] Richter, R. K. (1966) "Revealed Preference Theory", *Econometrica* 34: 635-45

[7] Sen, A. K. (1969) "Choice Functions and Revealed Preference", *The Review of Economic Studies* 36 (3): 381-91

[8] Sen, A. K. (1971) "Choice Functions and Revealed Preference", *The Review of Economic Studies* 38 (3): 307-317

[9] Sen, A. K. (1997) "Maximization and the Act of Choice", *Econometrica* 65: 745-780