Revealed Preference or revealed Choice?

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Let X , $2<\left|X\right|$, denote a finite set of **mutually exclusive alternatives**; let $K(S)=2^X/\varnothing$ for all subsets *S* of *X*.

Let $Q\subseteq X\times X$ denote a binary relation on *X*; it is called a preference if it is a **Weak Order** (complete, reflexive and transitive) and a **Linear Order** is a preference that is anti-symmetric. Its asymmetric (strict preference) and symmetric (indifference) parts are written $P(Q)$ and I(Q) respectively. As usual, for all $x,y\in X$, $(x,y)\in Q$ will be written xQy. $G(S,Q)=\{x\in S:(\forall y\in S)xQy\}$ is the subset of Q-greatest alternatives in $S\in K(X)$ and $L(S,Q)=\{x\in S:(\forall y\in S)yQx\}$ is the subset of Q-least alternatives in $S\in K(X)$. Let $\mathscr{W}$ denote the set of all Weak Orders on *X* and let $\mathscr{L}$ denote the set of all Linear Orders on *X*. $Q'\in\mathscr{L}$ is a **linear refinement** of $Q\in\mathscr{W}$ if and only if $Q'\subseteq Q$. $Q\in\mathscr{W}$ is **upper linear** (ULIN) if and only if, for all distinct $x,y\in X$: $xP(Q)y$ or $yP(Q)x$ for all distinct $x,y\in X\setminus L(X,Q)$ and all $x\in X\setminus L(X,Q)$ and $y\in L(X,Q)$. $Q\in\mathscr{W}$ is **minimally upper linear** (MULI) if and only if it is ULIN and $\left|L(X,Q)\right|\le 2$.

A **choice function** is a function $C:K(S)\to K(S)$ that assigns $C(S)\in K(S)$ to *S*, for all $S\in K(S)$. $\mathscr{C}$ denotes the set of all choice functions on *X*. $C\in\mathscr{C}$ is **imperative** if and only if $\left|C(S)\right|=1$ for all $S\in K(X)$. Let $\mathscr{C}^{IMP}$ denote the set of all imperative choice functions. $C'\in\mathscr{C}$ is a **refinement** of $C\in\mathscr{C}$ if and only if $C'(S)\subseteq C(S)$ for all $S\in K(X)$, written $C'\subseteq C$. $Q\in W$ **rationalises** $C\in\mathscr{C}$ if and only if $C(S)=G(S,Q)$ for all $S\in K(X)$ and $C\in\mathscr{C}$ is **rationalisable** if and only if it is rationalised by some $Q\in W$.

$C\in\mathscr{C}$ has: The **strong contraction property** (SCON) if and only if, for all $T\in K(X)$ and $S\in K(T)$ such that $S\cap C(T)\neq\varnothing$, $S\cap C(T)=C(S)$; the **contraction property** (CON) if and only if, for all $T\in K(X)$ and $S\in K(T)$ such that $S\cap C(T)\neq\varnothing$, $(S\cap C(T))\subseteq C(S)$; and the **weak contraction property** (WCON) if and only if, for all $T\in K(X)$ and $S\in K(T)$ such that $S\cap C(T)\neq\varnothing$, $(S\cap C(T))\cap C(S)\neq\varnothing$. SCON characterises rationalisable choice functions.

$C'\in\mathscr{C}$ **reveals** $C\in\mathscr{C}$ if and only if $C'\in\mathscr{C}^{IMP}$ and $C\subseteq C'$. A choice function is **weakly rationalisable** if and only if it reveals at least one rationalisable choice function. A Choice function is **strongly rationalisable** if and only if all its revealed choice functions are rationalisable.

Theorem 1: *A choice function is weakly rationalisable if and only if it has the WCON property.*

Theorem 2: *A choice function is strongly rationalisable if and only if it is rationalisable by a MULI preference.*

Proposition: *If Q' rationalises C' and reveals C and C is rationalised by Q, then $Q\subseteq Q'$.*

Proof of theorem 1: Let choice function C' have the WCON property and write, $X_1^C = X$. From *WCON*, there is an alternative $x_1 \in X_1^C$ such that $x_1 \in C'(S_1)$ for all $S_1 \in K(X_1^C)$. Let $C(S_1) = \{x_1\}$ for all $S_1 \in K(X_1^C)$. Let $X_2^C = X_1^C \setminus C(X_1^C)$. From *WCON*, there is an alternative $x_2 \in X_2^C$ such that $x_2 \in C'(S_2)$ for all $S_2 \in K(X_2^C)$. Let $C(S_2) = \{x_2\}$ for all $S_2 \in K(X_2^C)$. More generally, for all $i \in \{2, \ldots, |X|\}$, let $X_i^C = X_i^C \setminus \bigcup_{j=0}^{i-1} C(X_j^C)$ and since *WCON* implies that there is an alternative $x_i \in X_i^C$ such that $x_i \in C'(S_i)$ for all $S_i \in K(X_i^C)$, let $C(S_i) = x_i$. Clearly, $\{K(X_1^C, \ldots, X_{|X|}^C\}$ is a partition of $K(X)$. Now consider an arbitrary $S \in K(X)$. $S \in K(X_i^C)$ for some $i \in \{1, \ldots, |X|\}$ and $C(S) = \{x_i\}$. Consider $S' \in K(S)$. It follows that $S' \cup \{x_i\} \in K(S)$, and $C(S' \cup \{x_i\}) = \{x_i\}$ then follows from *WCON*. Thus, C is imperative, has the SCON property, and is a refinement of C', all by construction. Therefore C is rationalisable and revealed by C'. That is, C' is weakly rationalisable.

Now let C' be weakly rationalisable. C' therefore reveals a choice function C that is rationalisable or, equivalently, has the SCON property. Consider $T \in K(X)$ and $S \in K(T)$ such that $x \in S$ and assume $C(T) = \{x\}$. From SCON, given that C is imperative, it follows that $C(S) = \{x\}$. Since $C \subseteq C'$, it follows that $x \in C'(T)$ and $x \in C'(S)$. Thus, C' has the SCON property and is therefore rationalisable.

Proof of Theorem 2: Let C' be rationalised by a MULI preference Q' and consider arbitrary $S, T \in K(X)$ such that $S \in K(T)$ and an arbitrary $x \in C'(T)$. It follows that, $S \cup \{x\} \in K(T)$. Since C' is rationalisable, and therefore has the SCON property, $x \in C'(S \cup \{x\})$. Either $x \notin L(X, Q')$ or $x \in L(X, Q')$. In case $x \notin L(X, Q')$, $C'(T) = \{x\}$ and $C'(S \cup \{x\}) = \{x\}$, since C' is rationalised by a MULI preference. Thus, for all $C \in \mathscr{C}$ revealed by C', $C(T) = \{x\}$ and $C(S \cup \{x\}) = \{x\}$ since C is imperative and a refinement of C'. For this case therefore, the requirements of SCON for C are satisfied. If $x \in L(X, Q')$, $L(X, Q') = \{x, y\}$ since Q' is a MULI preference. If $x = y$ then $T = \{x\} = S \cup \{x\}$. Then $C'(T) = \{x\}$ and $C'(S \cup \{x\}) = \{x\}$. Since all C revealed by C' are imperative refinements of C', it follows that $C(T) = \{x\}$ and $C(S \cup \{x\}) = \{x\}$, and the requirements of SCON are satisfied by C. If $x \neq y$ then, for all $S \in K(\{x, y\}, T)$, $C'(\{x, y\}) = \{x, y\}$ and for all imperative refinements C revealed by C', either $C(\{x, y\}) = \{x\}$ or $C(\{x, y\}) = \{y\}$. If $C(\{x, y\}) = \{x\}$ then either $S \cup \{x\} = \{x, y\}$ or $S \cup \{x\} = \{x\}$. In the former case, $C(S \cup \{x\}) = C(\{x, y\}) = \{x\}$ and in the latter case $C(S \cup \{x\}) = C(\{x\}) = \{x\}$. In either case, the requirement of SCON is satisfied by C. In case $C(\{x, y\}) = \{y\}$ a similar argument shows that the requirement of SCON is satisfied by C. Since this exhausts all possible cases, all C revealed by C' have the SCON property and are therefore rationalisable. This completes this part of the proof.

To complete the proof, consider $C' \in \mathscr{C}$ such that all the choice functions it reveals are rationalisable. Let $C \in \mathscr{C}$ denote one of these revealed choice function. Assume to the contrary that $C' \in \mathscr{C}$ is not rationalised by a MULI preference. There are two possibilities; either $C' \in \mathscr{C}$ is not rationalisable or it is rationalisable by a preference that is not MULI.

If C' is not rationalisable then it does not have the SCON property. Therefore, for some $T \in K(X)$, some $S \in K(T)$ and some $x \in S$ such that $S \cap C'(T) \neq \varnothing$, either $x \in (S \cap C'(T)) \setminus C'(S)$ or $x \in C'(S) \setminus (S \cap C'(T))$. If $x \in (S \cap C'(T)) \setminus C'(S)$, consider a choice function C revealed by C' such that $C(T) = \{x\}$. Now $x \notin C'(S)$ implies $x \in C(S)$, since $C \subseteq C'$. Thus, C does not have the SCON property or, equivalently, C is not rationalisable and this is a contradiction. Therefore assume that $x \in C'(S) \setminus (S \cap C'(T))$ for some $x \in X$. This implies that $x \notin C'(T)$ and therefore, $C(S) = \{x\}$ and $C(T) = \{y\}$, $y \neq x$, for some $y \in S \cap C'(T)$ and some C revealed by C'. It follows that C does not have the SCON property or, equivalently, C is not rationalisable. Since this is a contradiction, C' must be rationalisable.

Assume therefore that C' is rationalised by a non MULI preference Q'. If Q' is not MULI then, for distinct $x, y, z \in X$, either $xI(Q')y$, $xP(Q')z$ and $yP(Q')z$ or $L(X, Q') = \{x, y, z\}$. If $xI(Q')y$, $xP(Q')z$ and $yP(Q')z$ then $C'(\{x, y, z\}) = \{x, y\}$ and $C'(\{x, y\}) = \{x, y\}$. For some C revealed by C', $C(\{x, y, z\}) = \{x\}$ and $C'(\{x, y\}) = \{y\}$. That is, C does not have the SCON property or, equivalently, C is not rationalisable. Therefore, assume that $L(X, Q') = \{x, y, z\}$. This implies that $C(\{x, y, z\}) = \{x, y, z\}$ and $C'(\{x, y\}) = \{x, y\}$. Exactly the same argument just used again shows that C does not have the SCON property or, equivalently, C is not rationalisable and this is a contradiction. Therefore C is rationalisable by a MULI preference.

Conclusions

Rationalisability in the standard theory of rational choice has two major roles. First, it is said to provide *completeness* to the theory in the following sense. A choice function obtained by maximising a preference will have the SCON property. But could there be choice functions that cannot be obtained from maximising a preference that have the SCON property? Arrow's (1959) results completes rational choice theory for the choice functions considered in this paper, by showing that all choice functions with the SCON property are rationalisable. The results in this paper show that if *observability* is insisted upon, as revealed preference approaches usually do, then this completeness fails.

The second major role of rationalisability in the theory of rational choice is to provide the basis of welfare judgements. Again, apart from very restricted cases, this fails if *observability* is required.