Arrow's theorem in judgment aggregation

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4/2005, revised 9/2005

In response to recent work on the aggregation of individual judgments on logically connected propositions into collective judgments, it is often asked whether judgment aggregation is a special case of Arrowian preference aggregation. We argue the opposite. After proving a general impossibility result on judgment aggregation, we construct an embedding of preference aggregation into judgment aggregation and prove Arrow's theorem as a corollary of our result. Although we provide a new proof of Arrow's theorem, our main aim is to identify the analogue of Arrow's theorem in judgment aggregation, to clarify the relation between judgment and preference aggregation and to illustrate the generality of the judgment aggregation model.

JEL Classification: D70, D71

1 Introduction

A new aggregation problem has recently received much attention. How can the judgments of several individuals on logically connected propositions be aggregated into corresponding collective judgments? To illustrate the difficulties involved in judgment aggregation, suppose a three-member committee has to make collective judgments on three connected propositions:

a: "Carbon dioxide emissions are above the threshold x."

$a \to b$: "If carbon dioxide emissions are above the threshold x, then there will be global warming."

b: "There will be global warming."

	a	$a \to b$	b
Individual 1	True	True	True
Individual 2	True	False	False
Individual 3	False	True	False
Majority	True	True	False

Table 1: The discursive dilemma

As shown in Table 1, the first committee member accepts all three propositions; the second accepts a but rejects $a \to b$ and b; the third accepts $a \to b$ but rejects a and b. Then the judgments of each committee member are individually consistent, and yet the majority judgments on the propositions are inconsistent: a majority accepts a, a majority accepts $a \to b$, but a majority rejects b.

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This problem – the so-called *discursive dilemma* (Pettit 2001) – has led to a growing literature on the possibility of consistent judgment aggregation under various conditions. List and Pettit (2002, 2004) have developed a first model of judgment aggregation based on propositional logic and proved an impossibility result, followed by several stronger impossibility results (Pauly and van Hees 2004; Dietrich 2004a; Gärdenfors 2004; van Hees 2004; Nehring and Puppe 2005; Dietrich and List 2005; Dokow and Holzman 2005) and possibility results (Bovens and Rabinowicz 2004; Dietrich 2004a; List 2003, 2004, 2005; Pigozzi 2004). Generalizing the propositional logic framework, Dietrich (2004b) has developed a model of judgment aggregation in general logics, also used in the present paper, which allows the representation of a larger class of aggregation problems.

Although there are obvious differences between judgment aggregation and the more familiar problem of preference aggregation, the recent results resemble earlier results in social choice theory. The discursive dilemma resembles Condorcet's paradox of cyclical majority preferences, and the various impossibility theorems on judgment aggregation resemble Arrow's theorem on preference aggregation. This has led critics to ask whether the recent work on judgment aggregation is a reinvention of the wheel.

In response, it can be argued that the work on judgment aggregation is not a reinvention, but a generalization. The preference aggregation model of Condorcet and Arrow can be embedded into a judgment aggregation model by representing preference orderings as sets of binary ranking judgments (List and Pettit 2004; List 2003, fn. 4), whereas the converse embedding cannot easily be achieved. (This embedding claim applies only to the ordinal preference-relation-based strand of Arrowian social choice theory, but not to the cardinal welfare-function-based strand.)

In this paper, we reinforce this argument. After introducing the judgment aggregation model in general logics and proving a general impossibility result (stated in terms of two theorems), we construct an explicit embedding of preference aggregation into judgment aggregation and prove Arrow's theorem (for strict preferences) as a direct corollary of our result on judgment aggregation. Our aim is not primarily to provide a new proof of Arrow's theorem, but to identify the analogue of Arrow's theorem in judgment aggregation, to clarify the logical relation between judgment and preference aggregation and to highlight the logical structure underlying Arrow's result.

Related results were given by List and Pettit (2004), who derived a simple impossibility theorem on preference aggregation from an earlier result on judgment aggregation, and Nehring (2003), who proved an Arrow-like theorem in the related framework of property spaces. But neither result fully matches Arrow's theorem. List and Pettit's result requires additional neutrality and anonymity conditions, but no Pareto principle. Nehring's result requires an additional monotonicity condition.

2 The judgment aggregation model

We consider a group of individuals $1, 2, \dots, n$ ($n \geq 2$). The group has to make collective judgments on logically connected propositions.

Formal logic. Propositions are represented in an appropriate formal logic. A *logic (with negation symbol $\neg$)* is a pair $(\mathbf{L}, \vDash)$ such that

- $\mathbf{L}$ is a non-empty set of *propositions*, where $p \in \mathbf{L}$ implies $\neg p \in \mathbf{L}$,
- $\vDash$ is an *entailment relation* ($\subseteq \mathcal{P}(\mathbf{L}) \times \mathbf{L}$), where, for each set of propositions $A \subseteq \mathbf{L}$ and each proposition $p \in \mathbf{L}$, $A \vDash p$ is read as "A entails p" (we write $p \vDash q$ as an abbreviaton for $\{p\} \vDash q$).

A set $A \subseteq \mathbf{L}$ is *inconsistent* if $A \vDash p$ and $A \vDash \neg p$ for some $p \in \mathbf{L}$, and *consistent* otherwise. A set $A \subseteq \mathbf{L}$ is *minimal inconsistent* if it is inconsistent and every proper subset $B \subsetneq A$ is consistent. A proposition $p \in \mathbf{L}$ is *contingent* if $\{p\}$ and $\{\neg p\}$ are consistent. We require our logic to satisfy the following minimal conditions:

(L1) For all $p \in \mathbf{L}$, $p \vDash p$.
(L2) For all $p \in \mathbf{L}$ and $A \subseteq B \subseteq \mathbf{L}$, if $A \vDash p$ then $B \vDash p$.
(L3) $\emptyset$ is consistent, and each consistent set $A \subseteq \mathbf{L}$ has a consistent superset $B \subseteq \mathbf{L}$ containing a member of each pair $p, \neg p \in \mathbf{L}$.

Many different logics satisfy conditions L1 to L3, including standard propositional logic, standard modal and conditional logics and, for the present purposes, predicate logic, as defined below.

The agenda. The *agenda* is a non-empty subset $X \subseteq \mathbf{L}$, interpreted as the set of propositions on which judgments are to be made, where X is a union of proposition-negation pairs $\{p, \neg p\}$ (with p not itself a negated proposition). For simplicity, we assume that double negations cancel each other out, i.e. $\neg\neg p$ stands for p.[2] In the example above, the agenda is $X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$ in standard propositional logic (or alternatively in a simple conditional logic). We consider agendas X with different types of interconnections. For any $p, q \in X$, we write $p \vDash^* q$ if $\{p, \neg q\} \cup Y$ is inconsistent for some $Y \subseteq X$ consistent with p and with $\neg q$.[3]

- X is *minimally connected* if
  (i) there exists a minimal inconsistent set $Y \subseteq X$ with $|Y| \geq 3$, and
  (ii) there exists a minimal inconsistent set $Y \subseteq X$ such that $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ is consistent for some subset $Z \subseteq Y$ of even size.
- X is *path-connected* if, for every contingent $p, q \in X$, there exist p_1, $p_2, ..., p_k \in X$ (with $p = p_1$ and $q = p_k$) such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$.

[2] When we use the negation symbol $\neg$ hereafter, we mean a modified negation symbol $\sim$, where $\sim p := \neg p$ if p is unnegated and $\sim p := q$ if $p = \neg q$ for some q.

[3] For non-paraconsistent logics (in the sense of L4 in Dietrich 2004b), $\{p, \neg q\} \cup Y$ is inconsistent if and only if $\{p\} \cup Y \vDash q$.

Dokow and Holzman (2005) have recently introduced an algebraic condition on finite agendas, which is equivalent to part (ii) of minimal connectedness, as Ron Holzman has indicated to us. If the logic is compact, path-connectedness is equivalent to Nehring and Puppe's (2005) *total blockedness*; in the general case, path-connectedness is weaker.

The agenda of our example above is minimally connected, but not path-connected. Below we show that preference aggregation problems can be represented by agendas that are both minimally connected and path-connected.

Individual judgment sets. Each individual i's *judgment set* is a subset $A_i \subseteq X$, where $p \in A_i$ means that individual i accepts proposition p. A judgment set A_i is *consistent* if it is a consistent set as defined above; A_i is *complete* if, for every proposition $p \in X$, $p \in A_i$ or $\neg p \in A_i$. A *profile (of individual judgment sets)* is an n-tuple $(A_1, \ldots, A_n)$.

Aggregation rules. A *(judgment) aggregation rule* is a function F that assigns to each admissible profile $(A_1, \ldots, A_n)$ a single collective judgment set $F(A_1, \ldots, A_n) = A \subseteq X$, where $p \in A$ means that the group accepts proposition p. The set of admissible profiles is called the *domain* of F, denoted $Domain(F)$. Examples of aggregation rules are the following.

- *Propositionwise majority voting.* For each $(A_1, ..., A_n)$, $F(A_1, ..., A_n)$ $= \{p \in X :$ more individuals i have $p \in A_i$ than $p \notin A_i\}$.
- *Dictatorship of individual i.* For each $(A_1, ..., A_n)$, $F(A_1, ..., A_n) = A_i$.
- *Inverse dictatorship of individual i.* For each $(A_1, ..., A_n)$, $F(A_1, ..., A_n)$ $= \{\neg p : p \in A_i\}$.

Regularity conditions on aggregation rules. We impose the following conditions on the inputs and outputs of aggregation rules.

Universal domain. The domain of F is the set of all possible profiles of complete and consistent individual judgment sets.

Collective rationality. F generates complete and consistent collective judgment sets.

Propositionwise majority voting, dictatorships and inverse dictatorships satisfy universal domain, but only dictatorships generally satisfy collective rationality. As the discursive dilemma example of Table 1 shows, propositionwise majority voting sometimes generates inconsistent collective judgment sets. Inverse dictatorships satisfy collective rationality only in special cases (i.e. when the agenda is *symmetrical*: for every consistent $Z \subseteq X$, $\{\neg p : p \in Z\}$ is also consistent).

3 An impossibility result on judgment aggregation

Are there any non-dictatorial judgment aggregation rules satisfying universal domain and collective rationality? The following conditions are frequently used in the literature.

Independence. For any proposition $p \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all individuals i, $p \in A_i$ if and only if $p \in A_i^*$] then [$p \in F(A_1, \ldots, A_n)$ if and only if $p \in F(A_1^*, \ldots, A_n^*)$].

Systematicity. For any propositions $p, q \in X$ and profiles $(A_1, \ldots, A_n)$, $(A_1^*, \ldots, A_n^*) \in Domain(F)$, if [for all individuals i, $p \in A_i$ if and only if $q \in A_i^*$] then [$p \in F(A_1, \ldots, A_n)$ if and only if $q \in F(A_1^*, \ldots, A_n^*)$].

Unanimity principle. For any profile $(A_1, \ldots, A_n) \in Domain(F)$ and any proposition $p \in X$, if $p \in A_i$ for all individuals i, then $p \in F(A_1, \ldots, A_n)$.

Independence requires that the collective judgment on each proposition should depend only on individual judgments on that proposition. Systematicity strengthens independence by requiring in addition that the same pattern of dependence should hold for all propositions (a neutrality condition). The unanimity principle requires that if all individuals accept a proposition then this proposition should also be collectively accepted. The following result holds.

Proposition 1. For a minimally connected agenda X, an aggregation rule F satisfies universal domain, collective rationality, systematicity and the unanimity principle if and only if it is a dictatorship of some individual.

Proof. All proofs are given in the appendix. ■

Proposition 1 is related to an earlier result by Dietrich (2004b), which requires an additional assumption on the agenda X but no unanimity principle (the additional assumption is that X is also *asymmetrical*: for some inconsistent $Z \subseteq X$, $\{\neg p : p \in Z\}$ is consistent). This result, in turn, generalizes an earlier result on systematicity by Pauly and van Hees (2004).

From Proposition 1, we can derive two new results of interest. The first is a generalization of List and Pettit's (2002) theorem on the non-existence of an aggregation rule satisfying universal domain, collective rationality, systematicity and anonymity (i.e. invariance of the collective judgment set under permutations of the given profile of individual judgment sets). Our result extends the earlier impossibility result to any minimally connected agenda and weakens anonymity to the requirement that there is no dictator or inverse dictator.

Theorem 1. For a minimally connected agenda X, an aggregation rule F satisfies universal domain, collective rationality and systematicity if and only if it is a (possibly inverse) dictatorship of some individual.

Moreover, the agenda assumption of Theorem 1 is maximally weak if $n \geq 3$ and the agenda is finite or the logic is compact (and X contains at least one contingent proposition), i.e. minimal connectedness is necessary for characterizing (possibly inverse) dictatorships by the conditions of Theorem 2.[4]

The second result we can derive from Proposition 1 is the analogue of Arrow's theorem in judgment aggregation, from which we subsequently derive Arrow's theorem on preference aggregation as a corollary. Note the following lemma.

Lemma 1. For a path-connected agenda X, an aggregation rule F satisfying universal domain, collective rationality, independence and the unanimity principle also satisfies systematicity.

Let us call an agenda *strongly connected* if it is both minimally connected and path-connected. Using Lemma 1, Proposition 1 now implies the following impossibility result.

Theorem 2. For a strongly connected agenda X, an aggregation rule F satisfies universal domain, collective rationality, independence and the unanimity principle if and only if it is a dictatorship of some individual.

Dokow and Holzman (2005) have recently shown that, if $n \geq 3$ and the agenda is finite and contains only contingent propositions, the agenda assumption of Theorem 2 is maximally weak, i.e. strong connectedness is necessary for characterizing dictatorships by the conditions of Theorem 2. In fact, the necessity holds whenever $n \geq 3$ and the agenda is finite or the logic is compact (and X contains at least one contingent proposition). A related result with an additional monotonicity condition has been proved by Nehring and Puppe (2005).

Our impossibility results continue to hold under slightly generalized definitions of minimally connected and strongly connected agendas.[5]

Of course, it is debatable whether and when independence or systematicity are plausible requirements on judgment aggregation. The literature contains extensive discussions of these conditions and their possible relaxations. In our

[4]It can then be shown that, if X is not minimally connected, there exists an aggregation rule that satisfies universal domain, collective rationality and systematicity and is not a (possibly inverse) dictatorship. Let M be a subset of $\{1, ..., n\}$ of odd size at least 3. If part (i) of minimal connectedness is violated, then majority voting among the individuals in M satisfies all requirements. If part (ii) is violated, the aggregation rule F with universal domain defined by $F(A_1, ..., A_n) := \{p \in X : \text{the number of individuals } i \in M \text{ with } p \in A_i \text{ is odd}\}$ satisfies all requirements. The second example is inspired by Dokow and Holzman (2005).

[5]In the definition of minimal connectedness, (i) can be weakened to the following: (i*) there is an inconsistent set $Y \subseteq X$ with pairwise disjoint subsets Z_1, Z_2, Z_3 such that $(Y \backslash Z_j) \cup \{\neg p : p \in Z_j\}$ is consistent for any $j \in \{1, 2, 3\}$. In the definition of strong connectedness (by (i), (ii) and path-connectedness), (i) can be dropped altogether, as path-connectedness implies (i*). In the definitions of minimal connectedness and strong connectedness, (ii) can be weakened to (ii*) in Dietrich (2004b).

view, the importance of Theorems 1 and 2 lies not so much in establishing the impossibility of consistent judgment aggregation, but rather in indicating what conditions must be relaxed in order to make consistent judgment aggregation possible. The theorems describe boundaries of the logical space of possibilities.

4 Arrow's theorem

We now show that Arrow's theorem (for strict preferences) can be stated in the judgment aggregation model, where it is a direct corollary of Theorem 2. We consider a standard Arrowian preference aggregation model, where each individual has a strict preference ordering (asymmtrical, transitive and connected, as defined below) over a set of options $K = \{x, y, z, ...\}$ with $|K| \geq 3$. We embed this model into our judgment aggregation model by representing preference orderings as sets of binary ranking judgments in a simple predicate logic.[6]

A simple predicate logic for representing preferences. We consider a predicate logic with constants $x, y, z, ... \in K$ (representing the options), variables v, w, v_1, v_2 , ..., identity symbol $=$, a two-place predicate P (representing strict preference), logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if-then), and universal quantifier $\forall$. Formally, $\mathbf{L}$ is the smallest set such that

- $\mathbf{L}$ contains all propositions of the forms $\alpha P \beta$ and $\alpha = \beta$, where α and β are constants or variables, and
- whenever $\mathbf{L}$ contains two propositions p and q, then $\mathbf{L}$ also contains $\neg p$, $(p \wedge q)$, $(p \vee q)$, $(p \rightarrow q)$ and $(\forall v)p$, where v is any variable.

Notationally, we drop brackets when there is no ambiguity. The entailment relation $\vDash$ is defined as follows. For any set $A \subseteq \mathbf{L}$ and any proposition $p \in \mathbf{L}$,

$$A \vDash p \text{ if and only if } \begin{array}{l} A \cup Z \text{ entails } p \text{ in the standard} \\ \text{sense of predicate logic,} \end{array}$$

where Z is the set of rationality conditions on strict preferences:

$(\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1)$	("asymmetry");
$(\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3)$	("transitivity");
$(\forall v_1)(\forall v_2)(\neg\, v_1{=}v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1))$	("connectedness").[7]

The agenda. The *preference agenda* is the set X of all propositions of the forms $xPy, \neg xPy \in \mathbf{L}$, where x and y are distinct constants.

Lemma 2. The preference agenda X is strongly connected.

[6] Although we consider strict preferences for simplicity, a similar embedding is also possible for weak preferences.

[7] For technical reasons, Z also contains, for each pair of distinct constants x, y, the condition $\neg\, x{=}y$, reflecting the mutual exclusiveness of the options.

The correspondence between preference orderings and judgment sets. It is easy to see that each (asymmetrical, transitive and connected) preference ordering over K can be represented by a unique complete and consistent judgment set in X and vice-versa, where individual i strictly prefers x to y if and only if $xPy \in A_i$. For example, if individual i strictly prefers x to y to z, this is uniquely represented by the judgment set $A_i = \{xPy, yPz, xPz, \neg yPx, \neg zPy, \neg zPx\}$.

The correspondence between Arrow's conditions and conditions on judgment aggregation. For the preference agenda, the conditions of universal domain, collective rationality, independence and the unanimity principle ("Pareto"), as stated above, exactly match the standard conditions of Arrow's theorem.

As the preference agenda is strongly connected, Arrow's theorem now follows from Theorem 2.

Corollary 1. (Arrow's theorem) For the preference agenda X, an aggregation rule F satisfies universal domain, collective rationality, independence and the unanimity principle if and only if it is a dictatorship of some individual.

5 Concluding remarks

After proving a general impossibility result on judgment aggregation, stated in terms of Theorems 1 and 2, we have shown that Arrow's theorem (for strict preferences) is a corollary of this result, specifically of Theorem 2 applied to the aggregation of binary ranking judgments in a simple predicate logic.

This finding illustrates the generality of the judgment aggregation model. The model can represent a large class of judgment aggregation problems in general logics – all logics satisfying conditions L1 to L3 – of which a predicate logic for representing preferences is a special case. Other logics to which the model applies are propositional, modal or conditional logics and predicate logics representing relational structures other than preference orderings. Impossibility and possibility results on judgment aggregation, such as Theorems 1 and 2, can apply to aggregation problems in all these logics.

An alternative, very general model of aggregation is the one introduced by Wilson (1975) and recently used by Dokow and Holzman (2005), where a group has to determine its yes/no views on several issues based on the group members' views on these issues (subject to feasibility constraints). Wilson's model can also be represented in our model; here Dokow and Holzman's results apply to a logic satisfying L1 to L3 and a finite agenda.[8]

[8] In Wilson's model, the notion of consistency (feasibility) rather than that of entailment is a primitive. This is slightly less general than our model, as the notion of entailment fully specifies a notion of consistency, while the converse does not hold for all logics satisfying L1 to L3.

Although we have constructed an explicit embedding of preference aggregation into judgment aggregation, we have not proved the impossibility of a converse embedding. We suspect that such an embedding is hard to achieve, as Arrow's standard model cannot easily capture the different informational basis of judgment aggregation. It is unclear what an embedding of judgment aggregation into preference aggregation would look like. In particular, it is unclear how to specify the *options* over which individuals have preferences. The *propositions* in an agenda are not candidates for options, as propositions are usually not mutually exclusive. Natural candidates for options are perhaps entire *judgment sets* (complete and consistent), as these are mutually exclusive and exhaustive. But in a preference aggregation model with options thus defined, individuals would feed into the aggregation rule not a single judgment set (option), but an entire preference ordering over all possible judgment sets (options). This would be a different informational basis from the one in judgment aggregation. In addition, the explicit logical structure within each judgment set would be lost under this approach, as judgment sets in their entirety, not propositions, would be taken as primitives. However, although we are sceptical, the construction of a useful converse embedding remains an open challenge.

6 References

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A Appendix

Proof of Proposition 1. Let X be minimally connected and let F be any aggregation rule. Put $N := \{1, ..., n\}$. If F is dictatorial, F obviously satisfies universal domain, collective rationality, systematicity and the unanimity principle. Now assume F satisfies the latter conditions. Then there is a set $\mathcal{C}$ of ("winning") coalitions $C \subseteq N$ such that, for every $p \in X$ and every $(A_1, ..., A_n) \in \mathit{Domain}(F)$, $F(A_1, ..., A_n) = \{p \in X : \{i : p \in A_i\} \in \mathcal{C}\}$. For every consistent set $Z \subseteq X$, let A_Z be some consistent and complete judgment set such that $Z \subseteq A_Z$.

Claim 1. $N \in \mathcal{C}$, and, for every coalition $C \subseteq N$, $C \in \mathcal{C}$ if and only if $N \backslash C \notin \mathcal{C}$.

The first part of the claim follows from the unanimity principle, and the second part follows from collective rationality together with universal domain.

Claim 2. For any coalitions $C, C^* \subseteq N$, if $C \in \mathcal{C}$ and $C \subseteq C^*$ then $C^* \in \mathcal{C}$.

Let $C, C^* \subseteq N$ with $C \in \mathcal{C}$ and $C \subseteq C^*$. Assume for contradiction that $C^* \notin \mathcal{C}$. Then $N \backslash C^* \in \mathcal{C}$. Let Y be as in part (ii) of the definition of minimally connected agendas, and let Z be a *smallest* subset of Y such that $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ is consistent and Z has even size. We have $Z \neq \emptyset$, since otherwise the (inconsistent) set Y would equal the (consistent) set $(Y \backslash Z) \cup \{\neg z : z \in Z\}$. So, as Z has even size, there are two distinct propositions $p, q \in Z$. Since Y is minimal inconsistent, $(Y \backslash \{p\}) \cup \{\neg p\}$ and $(Y \backslash \{q\}) \cup \{\neg q\}$ are each consistent. This and the consistency of $(Y \backslash Z) \cup \{\neg z : z \in Z\}$ allow us to define a profile $(A_1, ..., A_n) \in \mathit{Domain}(F)$ as follows. Putting $C_1 := C^* \backslash C$ and $C_2 := N \backslash C^*$ (note that $\{C, C_1, C_2\}$ is a partition of N), let

$$A_i := \begin{cases} A_{(Y \backslash \{p\}) \cup \{\neg p\}} & \text{if } i \in C \\ A_{(Y \backslash Z) \cup \{\neg z : z \in Z\}} & \text{if } i \in C_1 \\ A_{(Y \backslash \{q\}) \cup \{\neg q\}} & \text{if } i \in C_2. \end{cases} \quad (1)$$

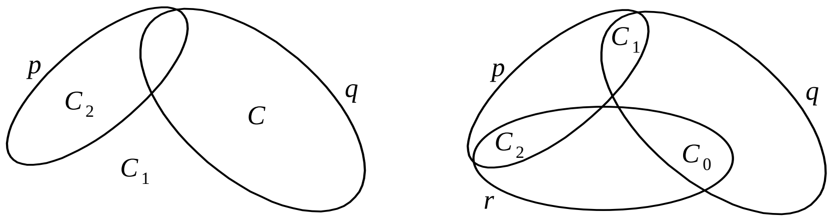

Figure 1: The profiles constructed in the proofs of claims 2 (left) and 3 (right).

By (1), we have $Y\backslash Z \subseteq F(A_1, ..., A_n)$ as $N \in \mathcal{C}$. Also by (1), we have $q \in F(A_1, ..., A_n)$ as $C \in \mathcal{C}$, and $p \in F(A_1, ..., A_n)$ as $C_2 = N\backslash C^* \in \mathcal{C}$. In summary, writing $Z^* := Z\backslash\{p, q\}$, we have (*) $Y\backslash Z^* \subseteq F(A, ..., A_n)$. We distinguish two cases.

Case $C_1 \notin \mathcal{C}$. Then $C \cup C_2 = N\backslash C_1 \in \mathcal{C}$. So $Z^* \subseteq F(A_1, ..., A_n)$ by (1), which together with (*) implies $Y \subseteq F(A_1, ..., A_n)$. But then $F(A_1, ..., A_n)$ is inconsistent, a contradiction.

Case $C_1 \in \mathcal{C}$. So $\{\neg z : z \in Z^*\} \subseteq F(A_1, ..., A_n)$ by (1). This together with (*) implies that $(Y\backslash Z^*) \cup \{\neg z : z \in Z^*\} \subseteq F(A_1, ..., A_n)$. So $(Y\backslash Z^*) \cup \{\neg z : z \in Z^*\}$ is consistent. As Z^* also has even size, the minimality condition in the definition of Z is violated.

Claim 3. For any coalitions $C, C^* \subseteq N$, if $C, C^* \in \mathcal{C}$ then $C \cap C^* \in \mathcal{C}$.

Consider any $C, C^* \in \mathcal{C}$. Let $Y \subseteq X$ be as in part (i) of the definition of minimally connected agendas. As $|Y| \geq 3$, there are pairwise distinct propositions $p, q, r \in Y$. As Y is minimally inconsistent, each of the sets $(Y\backslash\{p\}) \cup \{\neg p\}$, $(Y\backslash\{q\}) \cup \{\neg q\}$ and $(Y\backslash\{r\}) \cup \{\neg r\}$ is consistent. This allows us to defined a profile $(A_1, ..., A_n) \in Domain(F)$ as follows. Putting $C_0 := C \cap C^*$, $C_1 := C^*\backslash C$ and $C_2 := N\backslash C^*$ (note that $\{C_0, C_1, C_2\}$ is a partition of N), let

$$A_i := \begin{cases} A_{(Y\backslash\{p\})\cup\{\neg p\}} & \text{if } i \in C_0 \\ A_{(Y\backslash\{r\})\cup\{\neg r\}} & \text{if } i \in C_1 \\ A_{(Y\backslash\{q\})\cup\{\neg q\}} & \text{if } i \in C_2. \end{cases} \tag{2}$$

By (2), $Y\backslash\{p, q, r\} \subseteq F(A_1, ..., A_n)$ as $N \in \mathcal{C}$. Again by (2), we have $q \in F(A_1, ..., A_n)$ as $C_0 \cup C_1 = C^* \in \mathcal{C}$. As $C \in \mathcal{C}$ and $C \subseteq C_0 \cup C_2$, we have $C_0 \cup C_2 \in \mathcal{C}$ by claim 2. So, by (2), $r \in F(A_1, ..., A_n)$. In summary, $Y\backslash\{p\} \subseteq F(A_1, ..., A_n)$. As Y is inconsistent, $p \notin F(A_1, ..., A_n)$, and hence $\neg p \in F(A_1, ..., A_n)$. So, by (2), $C_0 \in \mathcal{C}$.

Claim 4. There is a dictator.

Consider the intersection of all winning coalitions, $\widetilde{C} := \cap_{C \in \mathcal{C}} C$. By claim 3, $\widetilde{C} \in \mathcal{C}$. So $\widetilde{C} \neq \emptyset$, as by claim 1 $\emptyset \notin \mathcal{C}$. Hence there is a $j \in \widetilde{C}$. As j belongs to every winning coalition $C \in \mathcal{C}$, j is a dictator: indeed, for each profile

$(A_1, ..., A_n) \in Domain(F)$ and each $p \in X$, if $p \in A_j$ then $\{i : p \in A_i\} \in \mathcal{C}$, so that $p \in F(A_1, ..., A_n)$; and if $p \notin A_i$ then $\neg p \in A_i$, so that $\{i : \neg p \in A_i\} \in \mathcal{C}$, implying $\neg p \in F(A_1, ..., A_n)$, and hence $p \notin F(A_1, ..., A_n)$. ■

Proof of Theorem 1. Let X be minimally connected, and let F satisfy universal domain, collective rationality and systematicity. If F satisfies the unanimity principle, then, by Proposition 1, F is dictatorial. Now suppose F violates the unanimity principle.

Claim 1. X is symmetrical, i.e. if $A \subseteq X$ is consistent, so is $\{\neg p : p \in A\}$.

Let $A \subseteq X$ be consistent. Then there exists a complete and consistent judgment set B such that $A \subseteq B$. As F violates the unanimity principle (but satisfies systematicity), the set $F(B, ..., B)$ contains no element of B, hence contains no element of A, hence contains all elements of $\{\neg p : p \in A\}$ by collective rationality. So, again by collective rationality, $\{\neg p : p \in A\}$ is consistent.

Claim 2. The aggregation rule $\widehat{F}$ with universal domain defined by $\widehat{F}(A_1, ..., A_n) := \{\neg p : p \in F(A_1, ..., A_n)\}$ is dictatorial.

As F satisfies collective rationality and systematicity, so does $\widehat{F}$, where the consistency of collective judgment sets follows from claim 1. $\widehat{F}$ also satisfies the unanimity principle: for any $p \in X$ and any $(A_1, ..., A_n)$ in the universal domain, where $p \in A_i$ for all i, $p \notin F(A_1, ..., A_n)$, hence $\neg p \in F(A_1, ..., A_n)$, and so $p = \neg\neg p \in \widehat{F}(A_1, ..., A_n)$. Now Proposition 1 applies to $\widehat{F}$, and hence $\widehat{F}$ is dictatorial.

Claim 3. F is inverse dictatorial.

The dictator for $\widehat{F}$ is an inverse dictator for F. ■

Proof of Lemma 1. Let X and F be as specified. To show that F is systematic, consider any $p, q \in X$ and any $(A_1, ..., A_n), (A_1^*, ..., A_n^*) \in Domain(F)$ such that $C := \{i : p \in A_i\} = \{i : q \in A_i^*\}$, and let us prove that $p \in F(A_1, ..., A_n)$ if and only if $q \in F(A_1^*, ..., A_n^*)$. If p and q are both tautologies ($\{\neg p\}$ and $\{\neg q\}$ are inconsistent), the latter holds since (by collective rationality) $p \in F(A_1, ..., A_n)$ and $q \in F(A_1^*, ..., A_n^*)$. If p and q are both contradictions ($\{p\}$ and $\{q\}$ are inconsistent), it holds since (by collective rationality) $p \notin F(A_1, ..., A_n)$ and $q \notin F(A_1^*, ..., A_n^*)$. It is impossible that one of p and q is a tautology and the other a contradiction, because then one of $\{i : p \in A_i\}$ and $\{i : q \in A_i^*\}$ would be N and the other $\emptyset$.

Now consider the remaining case where both p and q are contingent. We say that C is *winning for* r ($\in X$) if $r \in F(B_1, ..., B_n)$ for some (hence by independence any) profile $(B_1, .., B_n) \in Domain(F)$. with $\{i : r \in B_i\} = C$. We have to show that C is winning for p if and only if C is winning for q. Suppose C is winning for p, and let us show that C is winning for q (the converse implication can be shown analogously). As X is path-connected and p and q are contingent, there are $p = p_1, p_2, ..., p_k = q \in X$ such that $p_1 \vDash^* p_2$, $p_2 \vDash^* p_3$, ..., $p_{k-1} \vDash^* p_k$. We show by induction that C is winning for each of $p_1, p_2, ..., p_k$. If $j = 1$ then C is winning for p_1 by $p_1 = p$. Now let $1 \leq j < k$

and assume C is winning for p_j. We show that C is winning for p_{j+1}. By $p_j \vDash^* p_{j+1}$, there is a set $Y \subseteq X$ such that (i) $\{p_j\} \cup Y$ and $\{\neg p_{j+1}\} \cup Y$ are each consistent, and (ii) $\{p_j, \neg p_{j+1}\} \cup Y$ is inconsistent. Using (i) and (ii), the sets $\{p_j, p_{j+1}\} \cup Y$ and $\{\neg p_j, \neg p_{j+1}\} \cup Y$ are each consistent. So there exists a profile $(B_1, ..., B_n) \in Domain(F)$ such that $\{p_j, p_{j+1}\} \cup Y \subseteq B_i$ for all $i \in C$ and $\{\neg p_j, \neg p_{j+1}\} \cup Y \subseteq B_i$ for all $i \notin C$. Since $Y \subseteq A_i$ for all i, $Y \subseteq F(A_1, ..., A_n)$ by the unanimity principle. Since $\{i : p_j \in A_i\} = C$ is winning for p_j, we have $p_j \in F(A_1, ..., A_n)$. So $\{p_j\} \cup Y \subseteq F(A_1, ..., A_n)$. Hence, using collective rationality and (ii), we have $\neg p_{j+1} \notin F(A_1, ..., A_n)$, and so $p_{j+1} \in F(A_1, ..., A_n)$. Hence, as $\{i : p_{j+1} \in A_i\} = C$, C is winning for p_{j+1}. ■

Proof of Lemma 2. Let X be the preference agenda. X is minimally connected, as, for any pairwise distinct constants x, y, z, the set $Y = \{xPy, yPz, zPx\} \subseteq X$ is minimal inconsistent, where $\{\neg xPy, \neg yPz, zPx\}$ is consistent.

To prove path-connectedness, note that, by the axioms of our predicate logic for representing preferences, (*) $\neg xPy$ and yPx are equivalent (i.e. entail each other) for any distinct $x, y \in K$. Now consider any (contingent) $p, q \in X$, and let us construct a sequence $p = p_1, p_2, ..., p_k = q \in X$ with $p \vDash^* p_2, ..., p_{k-1} \vDash^* q$. By (*), if p is a negated proposition $\neg xPy$, then p is equivalent to the non-negated proposition yPx ; and similarly for q. So we may assume without loss of generality that p and q are non-negated propositions, say p is xPy and q is $x'Py'$. We distinguish three cases, each with subcases.

Case $x = x'$. If $y = y'$, then $xPy \vDash^* xPy = x'Py'$ (take $Y = \emptyset$). If $y \neq y'$, then $xPy \vDash^* xPy' = x'Py'$ (take $Y = \{yPy'\}$).

Case $x = y'$. If $y = x'$, then, taking any $z \in K \backslash \{x, y\}$, we have $xPy \vDash^* xPz$ (take $Y = \{yPz\}$), $xPz \vDash^* yPz$ (take yPx), and $yPz \vDash^* yPx = x'Py'$ (take $Y = \{zPx\}$). If $y \neq x'$, then $xPy \vDash^* x'Py$ (take $Y = \{x'Px\}$) and $x'Py \vDash^* x'Py'$ (take $Y = \{yPy'\}$).

Case $x \neq x', y'$. If $y = x'$, then $xPy \vDash^* xPy'$ (take $Y = \{yPy'\}$) and $xPy' \vDash^* x'Py'$ (take $Y = \{x'Px\}$). If $y = y'$, then $xPy \vDash^* x'Py = x'Py'$ (take $Y = \{x'Px\}$). If $y \neq x', y'$, then $xPy \vDash^* x'Py'$ (take $Y = \{x'Px, yPy'\}$). ■