On Defining Climate and Climate Change

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Abstract

The aim of the paper is to provide a clear and thorough conceptual analysis of the main candidates for a definition of climate and climate change. Five desiderata on a definition of climate are presented: it should be empirically applicable, it should correctly classify different climates, it should not depend on our knowledge, is should be applicable to the past, present and future and it should be mathematically well-defined. Then five definitions are discussed: climate as distribution over time for constant external conditions, climate as distribution over time when the external conditions vary as in reality, climate as distribution over time relative to regimes of varying external conditions, climate as the ensemble distribution for constant external conditions, and climate as the ensemble distribution when the external conditions vary as in reality. The third definition is novel and is introduced as a response to problems with existing definitions. The conclusion is that most definitions encounter serious problems and that the third definition is most promising.

Contents

1 Introduction

How to *define climate and climate change* is nontrivial and contentious, as also expressed by Todorov (1986, 259):

> The question of climatic change is perhaps the most complex and controversial in the entire science of meteorology. No strict criteria exist on how many dry years should occur to justify the use of the words "climatic change". There is no unanimous opinion and agreement among climatologists on the definition of the term climate, let alone climatic change, climatic trend or fluctuation.

In both public and scientific discourse the notions of climate and climate change are often loosely employed, and it remains unclear what exactly is understood by them.[1] This unclarity is problematic because it may lead to considerable confusion regarding the existence and extent of global warming, for example. How to define climate and climate change is conceptually interesting, but choosing good definitions is also important for being able to make true statements about our climate system. As we will see, adopting definitions with serious problems may imply that the climate has nothing to do with the actual properties of the climate system, that different climates are not correctly classified or that there is no relation to observational records such as past mean surface temperature values.

Of course, different definitions of climate and climate change are discussed in the climate science literature. However, what is *missing* is a *clear and thorough conceptual analysis* of the different definitions and their benefits and problems.[2] This paper aims to *contribute to filling this gap.* After introducing climate variables and a simple climate model (Section 2), five main desiderata on a definition of climate will be presented (Section 3). Then five main candidates for a definition of climate (and the derivative definitions of climate change) will be discussed. By referring to the five desiderata, their benefits and problems will be analysed (Sections 4-5). Problems with existing definitions lead me to propose a *novel* definition of climate (Definition 3) which has not been discussed in the literature before (the other four definitions are among the most commonly endorsed definitions of climate). Finally, the conclusion will summarise the discussion (Section 6).

[1] An example for this is Stott and Kettleborough (2002).

[2] The most detailed discussion so far seems to be Lorenz (1995). Still, a more thorough analysis is needed. For instance, Lorenz's discussion does not take into account recent developments in defining climate and in using time-dependent dynamical systems theory (where the external conditions are allowed to vary) to define climate.

2 Climate Variables and a Simple Climate Model

When talking about the climate of a certain region one is interested in the distribution of *certain variables (called the climate variables)* of that region.[3] These include the *dynamic meteorological variables*, i.e., the variables that describe the state of the atmosphere, such as the surface air temperature or the surface pressure. In the literature one often finds the statement that climate is the expected weather (e.g. Allen 2003; Lorenz 1995), which suggests that the *only* variables of interest are the dynamic meteorological variables. However, this is not clear. When scientists talk about climate they often refer not only to dynamic meteorological variables but to a more extended set of variables. These usually include the variables describing the state of the *ocean* (such as the sub-surface ocean temperature) and sometimes also other variables such as those describing glaciers and ice sheets. In general, when talking about climate, one is definitely interested in the dynamic meteorological variables, and there are some other variables one is definitely *not* interested in, such as those which describe the flora and fauna on Earth in all its details. Apart from this, there is a middle ground of other variables such as those describing glaciers and ice sheets, which one might only include in the list of climate variables in certain contexts. It seems best to me to accept this middle ground. Depending on the purpose at hand, it is better to include more or fewer variables – as climate scientists do in practice.[4]

When illustrating the different definitions of climate, in order for the discussion to be as accessible as possible, *it will be assumed that the following simple climate model is the true model of the evolution of the climate variables*: The only climate variable is the *temperature* with possible values in $[0, 30]$. It evolves according to the deterministic evolution equation

$$x_{t+1} = f(x_t) = \begin{cases} a_t x_t & \text{for } 0 \leq x_t \leq 15 \\ a_t(30 - x_t) & \text{for } 15 < x_t \leq 30, \end{cases} \tag{1}$$

where x_t denotes the temperature at day t. For this assumed true simple climate model the *external conditions* (i.e. the phenomena not described by the climate variables) consist just of a_t. a_t represents the solar energy reaching the Earth at day

[3]Here 'region' is broadly understood. For instance, the region could be London (site-specific climate) or the Earth (spatially aggregated global climate). These spatial aspects of defining climate present further conceptual challenges, which are beyond the scope of this paper.

[4]The climate of a region is generated by the *climate system*, which is an interactive system consisting of the atmosphere, the hydrosphere, the cryosphere, the land surface and the biosphere, forced or influenced by various external forcing mechanisms such as solar radiation (Solomon et al. 2007). Note that many variables figuring in the climate system are *not* climate variables. For instance, the climate system includes a detailed description of the flora and fauna on Earth, but these details are not of concern when talking about the climate.

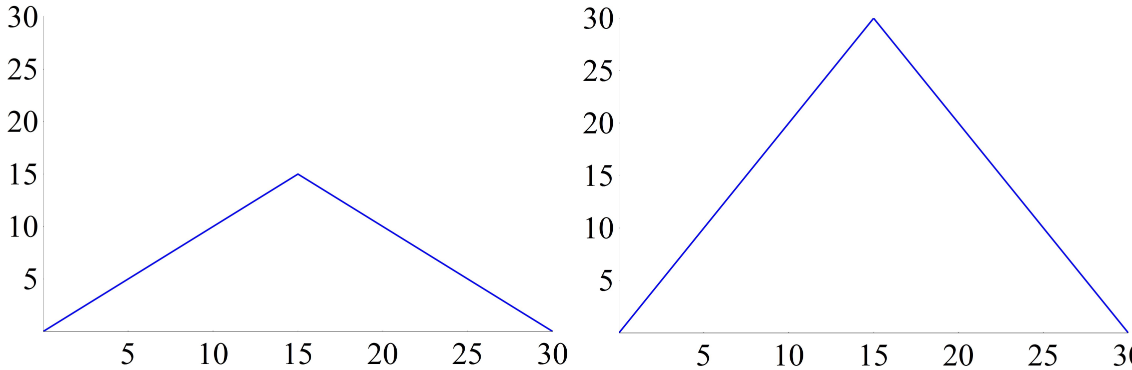

Figure 1: The assumed true simple evolution of the temperature when a_t=1 (left) and a_t=2. (right)

t and is assumed to be periodically fluctuating between the values 1 and 2 (and $a_t = 1$ on 1 July 1983). Figure 1 shows the evolution equation when $a_t = 1$ (left) and $a_t = 2$ (right). Realistic and state-of-the-art climate models are of course much more complex deterministic models (Parker 2006; see Appendix A for a general mathematical definition of deterministic models).[5] They usually arise from ordinary or partial differential equations, implying that the time parameter t varies continuously (and not in discrete steps as for the simple model). For the purposes of this paper these differences are irrelevant as all the points made for the simple model carry over to the more complex continuous-time climate models.[6]

3 Desiderata on a Definition of Climate

For many policy decisions and scientific questions what matters are *not the specific values* of the climate variables at a certain point of time but the *distribution* over the climate variables arising for a certain configuration of the climate system. For instance, when policy advisors ask whether events of extreme heat under our current greenhouse gas emissions will be roughly the same as under the greenhouse gas emissions projected towards the end of the 21th century, they have in mind a comparison of two temperature distributions. They are not concerned with the temperature at certain days (such as whether there will be extreme heat on 1 July 2099).

[5] For a discussion of how climate models are confirmed, see Steele and Werndl (2013).

[6] It should be noted that, for several reasons (e.g., incomplete knowledge or the inability to resolve processes numerically at sub-grid level), *stochastic models* are sometimes used to describe the evolution of climate variables (e.g. Hasselmann 1976; Checkroun et al. 2011). Deterministic models are the norm and thus what follows focuses on deterministic models. However, all the definitions of climate discussed in this paper have a stochastic counterpart.

Furthermore, climate scientists tend to think that distributions over climate variables are easier to predict than the actual path taken by the climate variables. Thus both in scientific and policy discourse the notion of the *distribution over the climate variables arising for a certain configuration of the climate system* is needed, and at an intuitive level *the climate simply amounts to this notion.* The following five desiderata present fairly weak requirements on a more rigorous definition of this notion (these desiderata will play a crucial role in the analysis of the five definitions of climate in Sections 4 and 5).

Desideratum 1. A definition of climate should be *empirically applicable.* In particular, (i) it should be possible to estimate the past and present climate from the time series of observations (if good records are available), and (ii) the future climate should be about the future values of the climate variables. A definition that neither fulfills (i) nor (ii) is called *empirically void.*

Desideratum 2. A definition of climate should *correctly classify different climates* for time periods which are uncontroversially regarded as belonging to different climates. For instance, the climate which prevailed in the middle of the last ice age in London is regarded as different from the climate in London in the past few years. Therefore, a definition of climate should classify that these time periods belong to different climates.

Desideratum 3. The climate should *not depend on our knowledge.* When, e.g., policy advisors ask what the past climate was and whether there will be climate change, they do not refer to notions which would differ if we had better or worse knowledge. How well we can predict the climate and climate change, of course, depends on our knowledge, but what the climate is and whether there is climate change is independent of our knowledge.

Desideratum 4. A definition of climate should be applicable to *the past, present and future.* Scientists think that there were several climates in the past, that there is a present climate and that there will be several climates in the future. Thus a definition of climate should be applicable to all these cases. If, e.g., a definition were only about the future climate, it would be regarded as incomplete and a definition that can also be applied to the past and present would be demanded.

Desideratum 5. A definition of climate should be *mathematically well-defined.* In particular, it cannot refer to a limit which does not exist.

As we will see, it will turn out that only one definition discussed in this paper

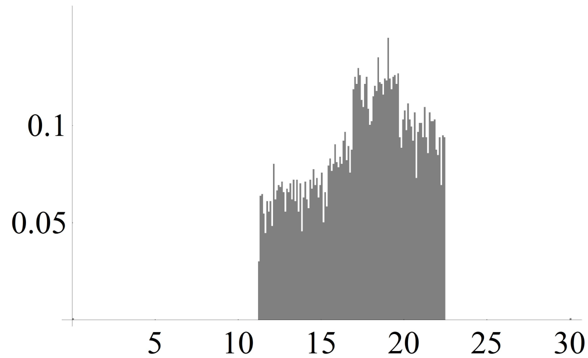

Figure 2: The temperature distribution over time for the simple climate model with constant $c = 1.5$ and with initial value 18.85 from 1 July 1983 to 30 June 2013.

fulfills all the desiderata and hence is a promising definition of climate. Yet this should not be taken to imply that, in principle, there can only be one definition of climate. Different notions of the distribution over the climate variables arising for a certain configuration of the climate system (i.e. what climate intuitively amounts to) might be important for different purposes. Thus it is not excluded that there could be several definitions of climate.

Let us now turn to the five definitions of climate (and the derivative definitions of climate change).

4 Climate as Distribution Over Time

4.1 Definition 1. Distribution Over Time for Constant External Conditions

In reality the external conditions (such as the amount of solar energy a_t for the assumed true simple evolution of the temperature) always fluctuate and thus are not constant. Suppose that the external conditions take the form of small fluctuations around a mean value c over a certain time period. Then, according to the first definition, the climate over this time period is defined by the *finite distribution over time which arises for the true climate model under constant external conditions c*. Different distributions over time correspond to different climates. There is climate change when there are different distributions for two successive time periods. This can be because of different external conditions (external climate change) or because of different initial values under constant external conditions (internal climate change). This definition is widely endorsed (e.g. Dymnikov and Gritsoun 2001; Lorenz 1995).

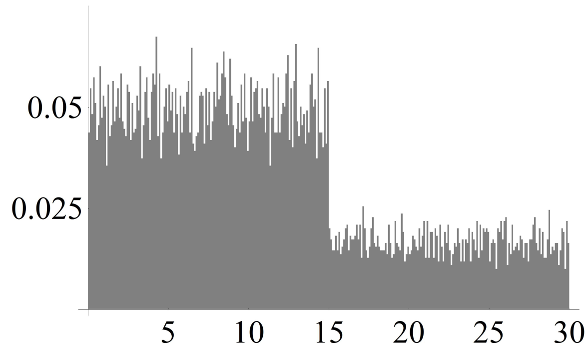

Figure 3: The actual temperature distribution over time for the assumed true simple climate model with initial value 18.85 from 1 July 1983 to 30 June 2013.

Let me illustrate this definition with the assumed true simple evolution of the temperature (cf. Section 2). Here the amount of solar energy reaching the Earth fluctuated around the mean value $c = 1.5$ for the thirty years period from 1 July 1983 to 30 June 2013. Suppose that the initial temperature on 1 July 1983 was 18.85 °C. Then the climate over this period is the distribution over time which arises when the model (1) with $a_t = c = 1.5$ (for all t) and with initial value 18.85 is evolved over thirty years. That is, the value assigned to a set A in [0, 30] is:[7]

$$\frac{\text{The number of days under constant external conditions } c \text{ with a temperature in } A}{\text{The total number of modelled days } (=10950\ (30*365))}. \tag{2}$$

The distribution arising in this way is shown in Figure 2. For instance, the value assigned to [15, 20] (that the temperature was between 15 °C and 20 °C) is 0.517.

There is a fundamental distinction between climate as a distribution or average of actual properties of the climate system versus climate as a model-immanent notion (a property of the true climate model). Because the external conditions are assumed to be constant but vary in reality, Definition 1 is a model-immanent notion and *not* a distribution of the actual properties of the climate system. This would not represent a problem if this definition were empirically applicable in the following sense: When the external conditions are small fluctuations around a mean value c, then the distributions over time of the actual climate system are approximately equal

[7]The question arises how long the time period should be over which the climate distribution is defined (cf. Subsection 4.3). Suppose that the relevant time period is thirty years. Then even if the external conditions fluctuate around c for less than thirty years, say three years, then climate over this time period is still defined as the distribution under constant external conditions over thirty years. This makes intuitive sense: if, e.g., there are high temperatures over three years because of the El Niño, then this is just a warm period under a certain climate.

to the distributions over time under constant external conditions c. However, *there are doubts about this.* Let me illustrate this with the assumed true simple climate model. The actual distribution (where the solar energy a_t fluctuated between 1 and 2 with $a_t = 1$ on 1 July 1983) is given by the actual evolution of the temperature over the thirty years period from 1 July 1983 to 30 June 2013 (with initial temperature 18.85). That is, the value assigned to a set A in [0, 30] is:

$$\frac{\text{The number of days of the actual climate system with a temperature in } A}{\text{The total number of days (=10950 (30*365))}}. \qquad (3)$$

Figure 3 shows the distribution arising in this way. It is clear that the temperature distribution over time for constant external conditions $c = 1.5$ (Figure 2) and the actual temperature distribution over time (Figure 3) are completely different. For instance, the value assigned to [15, 20] (that the temperature is between 15 °C and 20 °C) is 0.517 for the former distribution but 0.082 for the latter distribution.

Similar results can be found for several climate models. For instance, Daron (2012) numerically investigated the Lorenz equations where one parameter is subject to fluctuations around a mean value. He found that the finite distributions over time can differ significantly from the distributions arising when the parameters are held fixed. A resonance effect, which can also arise for small fluctuations, is responsible for these different distributions. Also, there is a body of work indicating that the seasonal cycle of the sun leads to different distributions over time. It was found in Goswami et al. (2006) for a model of the monsoon, Jin et al. (1994) for a model of the El Niño, Lorenz (1990) for a simple general circulation model and Kurgansky et al. (1996) for a baroclinic low-order model of the atmosphere that the finite distributions over time are different when the seasonal cycle of the sun is included. It could certainly be that similar results hold for the true climate model. Thus the first definition of climate may be *empirically void (thereby violating Desideratum 1).* Note also that there is the problem that when the external conditions are not small fluctuations around a mean value c but vary considerably, then this definition is not applicable. All this shows a need to take the varying external conditions of the climate system into account. Thus the question arises whether a similar definition can be found where the external conditions are allowed to vary.

4.2 Definition 2. Distribution Over Time when the External Conditions Vary as in Reality

Most directly this can be achieved by defining the climate over a certain time period as the *finite distribution over time for the actual evolution of the climate variables (i.e. when the external conditions vary as in reality).* Again, different climates correspond to different distributions. There is climate change when there are different

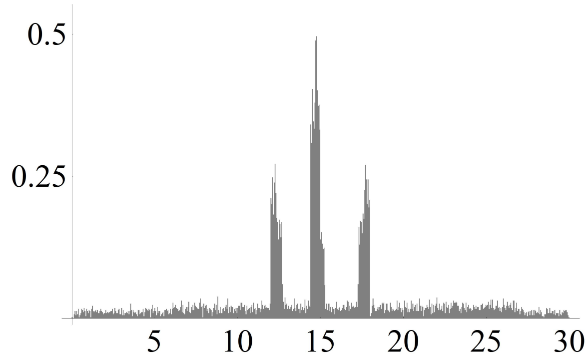

Figure 4: The actual temperature distribution over time for the meteor scenario from 1 July 1000 to 30 June 1030.

distributions for succeeding time periods (and there can be *external* climate change as well as *internal* climate change due to different initial values for the same external conditions). To come back to the assumed true simple evolution of the temperature (cf. Section 2): here the climate for the time period from 1 July 1983 to 30 June 2013 is the temperature distribution (3) as shown in Figure 3.

This definition simply refers to a *distribution of actual properties of the climate system* (so, unlike Definition 1, it is *not* a model-immanent notion). Because of this, arguably, there could be no definition of climate that is easier to estimate from the observations. For this reason, it is very popular and implicitly employed in many climate studies (cf. Lorenz 1995). The World Meteorological Organisation publishes *standard climate normals* that are statistics taken over a period of thirty years (most recently from 1961 to 1990). Thus the climate is commonly identified with the actual distribution of the atmosphere over a period of thirty years (cf. Hulme et al. 2009).

However, there is a serious problem with this definition, which can best be illustrated with a simple hypothetical scenario. Suppose that the time period from 1 July 1000 to 30 June 1030 was marked by two different regimes because on 1 July 1015 the Earth was hit by a meteor and, as a consequence, became a colder place with less temperature variation. More specifically, suppose again that the only climate variable is the temperature and that before the hit of the meteor the evolution of the temperature was given by equation (1) where a_t fluctuates periodically between 1.8 and 2 (with $a_t = 1.8$ and initial temperature 23.249 on 1 July 1000). After the hit of the meteor the evolution was given by equation (1) where a_t fluctuates periodically between 1 and 1.2 (with $a_t = 1$ on 1 July 1015).

According to this second definition, climate is defined as the actual temperature

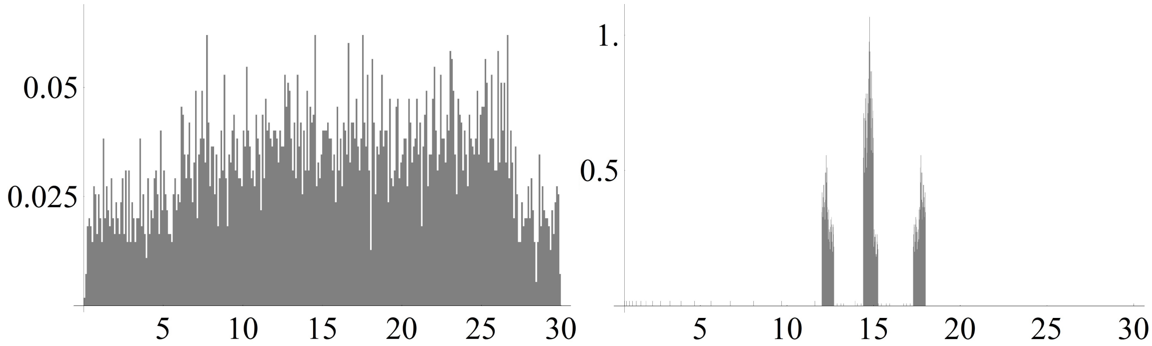

Figure 5: The actual temperature distribution over time before (from 1 July 1000 to 30 June 1015, left) and after the hit of the meteor (from 1 July 1015 to 30 June 1030, right).

distribution over the thirty years period from 1 July 1000 to 30 June 1030 as shown in Figure 4. However, Figure 5 reveals that this distribution is composed of two distributions: the actual temperature distribution over time before the hit of the meteor (from 1 July 1000 to 30 June 1015) and after the hit of the meteor (from 1 July 1015 to 30 June 1030). These two distributions are very different, e.g., the value assigned to $[25, 30]$ (that the temperature was between $25\,°\mathrm{C}$ and $30\,°\mathrm{C}$) is 0.142 for the former but zero for latter distribution. Because a change in external conditions is responsible for these different distributions, one would like to say that the climate before and after the hit of the meteor differ. Yet, the second definition does *not* imply this because the climate is simply the actual distribution over the thirty years period. To put it differently, the external conditions may well change drastically over the time period for which the climate distribution is defined. Hence the second definition *suffers from the shortcoming that it does not correctly classify different climates* for time periods which are uncontroversially regarded as belonging to different climates (*thereby violating Desideratum 2*).

4.3 Definition 3. Distribution Over Time for Regimes of Varying External Conditions

To avoid this problem, a promising possibility seems to be a slight modification of the second definition. Namely, suppose that the actual external conditions over a time period are subject to a certain regime of varying external conditions. Then the climate over this time period is defined as the *finite distribution over time which arises under the regime of varying external conditions*. Some thoughts need to be given on what counts as a regime of varying external conditions. For instance, a reasonable requirement is that the mean of the external conditions should at least be approximately constant. To my knowledge, this definition is *novel* and has not been

explicitly endorsed in the climate literature. Again, different climates correspond to different distributions. There is climate change when there are different climates for two successive time periods, and there can be external climate change as well as internal climate change (due to different initial values).

Let me illustrate this definition with the assumed true simple evolution of the temperature (cf. Section 2). A regime of varying external conditions is when the solar energy fluctuates periodically between 1 and 2 (starting with 1), and the solar energy was subject to this regime from 1 July 1983 to 30 June 2013. Thus the climate over this period is the distribution over time under this regime of varying external conditions (with initial temperature 18.85).[8] That is, a set A in [0, 30] is assigned the value:

$$\frac{\text{The number of days under the external conditions regime with a temperature in } A}{\text{The total number of modelled days (=10950 (30*365))}}. \tag{4}$$

Since the regime of varying external conditions coincides with the actual path taken by the external conditions from 1 July 1983 to 30 June 2013, the third definition of climate is simply given by the distribution shown in Figure 3.

This definition avoids the shortcoming of the second definition of not being able to correctly classify different climates. For instance, the external conditions before and after the hit of the meteor correspond to two different regimes of varying external conditions (namely, the solar energy reaching the Earth fluctuating periodically between 1.8 and 2 starting with 1.8 (first regime), and the solar energy fluctuating periodically between 1 and 1.2, starting with 1 (second regime)). Thus, as desired, the climate before and after the hit of the meteor differ, i.e. the two distributions shown in Figure 5 correspond to different climates.

This definition of climate is a model-immanent notion and *not* a distribution of actual properties of the climate system. This does not represent a problem because the definition is empirically applicable: when the actual climate system is subject to a certain regime of varying external conditions over a long-enough time period, then the climate of this time period coincides with the distribution over time of the actual evolution of the climate variables. Hence this definition is attractive because there is

[8]The question arises how long the time period should be over which the climate distribution is defined (cf. the comments at the end of this subsection). Suppose that the relevant time period is thirty years. Then even if the external conditions are subject to a certain regime of varying external conditions for less than thirty years, say three years, then climate over this time period is still defined as the distribution under the regime over thirty years (e.g., if there are high temperatures over three years because of the El Niño, then this is just a warm period under a certain climate).

an immediate link to the observations.

Since this definition is promising, let me comment on a few issues (which also arise for Definition 1 and 2). First, there is the question over which finite time period the distributions should be taken. It seems promising to adopt a pragmatic approach as outlined by Lorenz (1995): The purpose of research will influence the choice of the time interval, e.g., if one is interested in inter-glacial climate the time period will be relatively long. Also, the time period should be long enough so that no specific predictions can be made (e.g., longer than the predictability horizon given by the El Niño) but short enough to ensure that changes which are conceived as climatic are subsumed under different climates. What is important is to provide some motivation for the chosen time period (often thirty years are chosen without any clear motivation, which is problematic). Second, it should be mentioned that there will nearly always be climate change. Yet this does not constitute a problem: pragmatically one might say that of interest is only more major climate change with significantly different distributions. Third, the climate depends on the initial value of the climate variables, e.g., for the very simple model on the initial temperature. This does not speak against this definition but implies that predicting the climate would be difficult if small changes in the initial values led to very different distributions.[9]

To sum up the discussion of climate as a finite distribution over time: Because of the assumption of constant external conditions, Definition 1 may be empirically void (thereby violating Desideratum 1). Definition 2 does not correctly classify different climates and thus fails to meet Desideratum 2. In contrast, Definition 3 is promising because it meets all the desiderata.

4.4 Infinite Versions

There are also infinite versions of the three definitions of climate as distribution over time discussed above. According to these infinite versions, the climate is defined by the infinite distribution over time when the time period in equation (2) (for Definition 1), equation (3) (for Definition 2) or equation (4) (for Definition 3) goes to infinity. According to the infinite version of Definition 2, the climate is the actual distribution of the climate system as time goes to infinity, which has the consequence that there cannot be climate change. Because of this, it is hardly ever adopted. The infinite versions of Definition 1 and Definition 3, however, are widely endorsed (Dymnikov and Gritsoun 2001; Lorenz 1995; Palmer 1999).

[9] Note also that the climate of the true model may differ significantly from the climate of models which are close to the true model (Frigg et al. 2013). While this does not speak against this definition, this would make predicting the climate very difficult.

The main advantage of these infinite distributions is that they are *mathematically more tractable*, implying that the tools of dynamical systems theory can be used to analyse them. For instance, recall that climate as distribution over time is defined relative to the initial values of the climate variables. For the infinite version of Definition 1 results of dynamical systems theory show that under certain conditions the climate is the same for almost all initial values (no corresponding results are known for Definition 2 and Definition 3). Namely, this is the case when the dynamics is ergodic or when there is a physical measure (see Appendix B for a mathematical statement of these results). These results are of limited relevance, however, because it is unclear whether the true climate model under constant external conditions is ergodic or has a physical measure. What is more, as I will now argue, *the infinite versions of Definitions 1, 2 and 3 suffer from serious problems* (in addition to the problems already outlined in the Subsections 4.1-4.3). Therefore, for defining the climate, *finite distributions over time are preferable* to infinite distributions over time (and this is the reason why the presentation above focused on finite distributions).

The first problem with these infinite versions is that *the relevant infinite limits may not exist*. More specifically, while infinite limits of equation (2) (Definition 1) usually exist (Petersen 1983), the infinite limits of equation (3) (Definition 2) and equation (4) (Definition 3) often do not exist (Mancho et al. 2013). To the best of my knowledge, it is an open question whether they exist for the true climate model. Hence there is the problem that *the climate may not be mathematically well-defined (thereby violating Desideratum 5).*[10]

Second, to make sure that the infinite distributions relate to the actual finite distributions of the climate system, one has to assume that the distributions over finite time periods of interest are approximated by the infinite distributions. However, *there are doubts about this*. For many climate models the convergence to the infinite distributions is very slow. In particular, Lorenz (1968) introduced the notion of *almost intransitivity* to describe systems where distributions taken over very long but finite time intervals differ from one interval to each other and thus from the infinite distribution. Lorenz (1968, 1970, 1976, 1995) gave examples of simple models that

[10]Furthermore, when the external conditions vary, i.e., for equation (3) (Definition 2) and equation (4) (Definition 3), there are two infinite distributions, which usually differ. More specifically, when the finite distributions range from time point t_0 to time point t_1, one infinite distribution arises for $t_0 \to -\infty$ and the other for $t_1 \to \infty$ (Kloeden and Rasmussen 2011). Consequently, there is *the problem which of the infinite distributions should be identified with the climate.* For the past one can check which infinite distribution is approximated by the observed distributions. However, if the concern is the future, it is unclear how to choose between the two distributions. If the wrong distribution is chosen, *the climate will be empirically void (thereby violating Desideratum 1).*

show almost intransitive behaviour. He concluded that the true climate model may well be almost intransitive, particularly when the climate variables include the ocean variables. Furthermore, there are several climate models that incorporate some realism about the evolution of the climate variables where the distributions taken over long finite time intervals differ considerably from the infinite distributions. For instance, Bhattacharya et al. (1982) for an energy-balance model, Daron (2012) and Daron and Stainforth (2013) for a coupled ocean-atmosphere model and Sempf et al. (2007) for a model of the wintertime atmospheric circulation over the northern hemisphere found considerable differences between distributions taken over long finite intervals and distributions in the infinite limit. It is an open question whether similar results hold for the true climate model, but there remains the *worry that the infinite distributions* may not relate to the actual distributions of the climate system and thus are *empirically void (thereby violating Desideratum 1).*

To sum up: climate as an infinite distribution over time is easier to analyse mathematically. However, there are the additional problems that the relevant limits may not exist (thereby violating Desideratum 5) and that the definitions may be empirically void (thereby violating Desideratum 1).

5 Climate as Ensemble Distribution

5.1 Definition 4. Ensemble Distribution for Constant External Conditions

The first three definitions of climate are distributions *over time*. In stark contrast to this, the remaining two definitions are *ensemble distributions* of the possible values of the climate variables. Suppose that the concern is to make predictions at time t_1 in the future and that from the present t_0 until t_1 the external conditions take the form of small fluctuations around a mean value c. Then, according to the fourth definition, *the climate at time t_1 is the distribution of the possible values of the climate variables at t_1 under constant external conditions c conditional on our uncertainty in the initial values at t_0*. As before, different distributions correspond to different climates. This is a common definition (e.g., Lorenz 1995; Stone and Knutti 2010).

Let me illustrate this definition with the assumed true simple evolution of the temperature (cf. Section 2). Suppose that the temperature is measured to be between 29.77 °C and 30.00 °C degrees today, i.e. at $t_0 = 1$ July 2013. Let p_{t_0} be the uniform probability density over [29.77, 30.00] representing this uncertainty in the initial temperature. Further, suppose that the concern is to predict the temperature at time $t_1 = 1$ July 2040. The amount of solar energy reaching the Earth will fluctuate

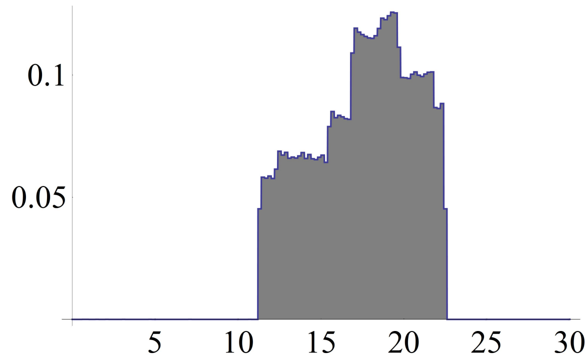

Figure 6: The temperature ensemble distribution on 1 July 2040 for the simple climate model under constant $c = 1.5$ given the uncertainty about the initial temperature on 1 July 2013.

around the mean value $c = 1.5$ from 1 July 2013 to 1 July 2040. Hence the climate on 1 July 2040 is given by the distribution where the value of a set A in $[0, 30]$ is

$$P^c_{t_0,t_1}(A), \tag{5}$$

where $P^c_{t_0,t_1}(A)$ is the value assigned to A by the density that arises when p_{t_0} is evolved forward to 30 June 2040 under equation (1) with $a_t = c = 1.5$ (for all t). Figure 6 shows the distribution arising in this way. For instance, the value assigned to [20, 25] (that the temperature will be between 20 °C and 25 °C) is 0.241.

Clearly, this definition is a model-immanent notion since it refers to an ensemble of initial conditions that is evolved forward in time but actual initial conditions are unique. Proponents of this definition think that this model-immanent notion is predictively useful in the following sense: What one is interested in is the actual ensemble distribution at time t_1, i.e. the distribution over the possible values of the climate variables given the uncertainty in the initial values and the *actual* path of the external conditions taken by the climate system. So the assumption is made that when the external conditions take the form of small fluctuations around a mean value c from t_0 to t_1, then the ensemble distribution at time t_1 under constant external conditions c is approximately the same as the actual ensemble distribution at time t_1.

However, *there are doubts about this assumption*. Let me illustrate this with the assumed true simple climate model. The actual temperature ensemble distribution is shown in Figure 7. It is the distribution where the value of a set A in $[0, 30]$ is

$$P_{t_0,t_1}(A), \tag{6}$$

where $P_{t_0,t_1}(A)$ is the value assigned to A by the density that arises when p_{t_0} is evolved forward to 1 July 2040 under equation (1) given the periodic fluctuations of

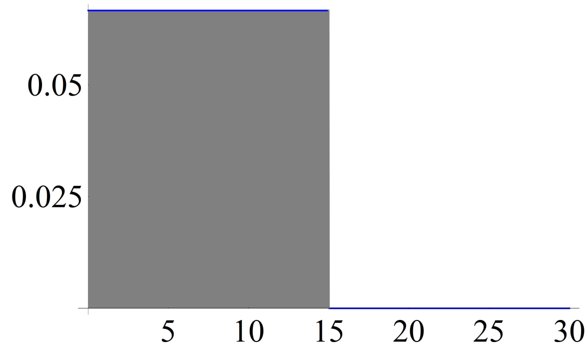

Figure 7: The actual temperature ensemble distribution on 1 July 2040 for the simple climate model given the uncertainty about the initial temperature on 1 July 2013.

the solar energy a_t between 1 and 2 (with $a_t = 1$ on 1 July 2013). It is clear that the ensemble distribution for constant external conditions $c = 1.5$ (Figure 6) and the actual ensemble distribution (Figure 7) are very different. For instance, the value assigned to [20, 25] (that the temperature will be between 20 °C and 25 °C) is 0.241 for the former but 0 for the latter distribution.[11]

Similar results hold for several climate models. For instance, Daron (2012) numerically investigated the Lorenz equations where one parameter is subject to fluctuations around a mean value. He found that the ensemble distributions can differ significantly from the ensemble distributions when the parameters are held fixed. The mechanism responsible for the different distributions is a resonance effect, which can also arise for small fluctuations. Also, there is growing evidence that the inclusion of the seasonal cycle of the sun may lead to different ensemble distributions: it was found in Gowsami et al. (2006) for model of the monsoon, in Jin et al. (1994) for a model of the El Niño and in Lorenz (1990) for a very simple general circulation model that the ensemble distributions differ when the seasonal cycle of the sun is included. It is certainly possible that similar results hold for the true climate model. Hence there is the *problem that this definition of climate may be empirically void (thereby violating Desideratum 1)*. There is also the additional problem that when the external conditions are not small fluctuations around a mean value, then the definition is not applicable. All this shows the need to take the varying external conditions into account.

[11] Note that the actual ensemble distribution on 30 June 2040 is given by the uniform distribution over $[0, 30]$. Thus the ensemble distribution under constant external conditions $c = 1.5$ (Figure 6) is *not* the average of the ensemble distribution on 1 July 2040 (where $a_t = 2$) and the ensemble distribution on 30 June 2040 (where $a_t = 1$). That is, it is not the average of the ensemble distributions for the periodically fluctuating external conditions.

5.2 Definition 5. Ensemble Distribution when the External Conditions Vary as in Reality

Most directly this can be achieved by defining the climate as the actual ensemble distribution. More specifically, suppose again that the concern is to make predictions at time t_1 in the future. According to the fifth definition, *the climate at time t_1 is the distribution of the possible values of the climate variables at t_1 given the actual path taken by the external conditions and conditional on our uncertainty in the initial values at t_0*. As before, different distributions correspond to different climates. This definition is widely endorsed (Daron 2012; Daron and Stainforth 2013; Smith 2002). To illustrate this definition with the assumed true simple evolution of the temperature (cf. Section 2): The climate on 1 July 2040 conditional on our knowledge that the temperature was between 29.77 °C and 30.00 °C on 1 July 2013 is the ensemble distribution (6) as shown in Figure 7.

Definition 5 is again a model-immanent notion since it refers to an ensemble of initial conditions that is evolved forward in time but actual initial conditions are unique. This model-immanent notion is *predictively very useful* because it quantifies the likelihood of the future possible properties of the actual climate system given our present uncertainty in the initial values.

However, there are several problems (which also arise for Definition 4). First of all, climate is defined relative to our present uncertainty about the initial values. This implies that *the climate and the derivative notion of climate change are dependent on our knowledge and that Desideratum 3 is not met.*

Second, this definition is always presented as defining the *future* climate. So the question arises what the *past* and *present* climate amount to (which are also needed to define climate change). While I have not found anything in print about this, it seems natural to say that the climate on 1 July 2013 is the distribution that scientists on, say, 1 July 1983 would have predicted as the climate of 1 July 2013 based on their uncertainty about the initial values on 1 July 1983 and the actual path taken by the external conditions.[12] Climate change is then defined as the change between the present and the future climate. Next to external climate change there is also internal climate change (because of different uncertainties in the initial values or different prediction lead-times). In the example the climate of 1 July 2013 is defined by choosing the reference point thirty years in the past (i.e. a prediction lead-time of thirty years), but this choice seems arbitrary. One could argue that pragmatic

[12]This has also been suggested by the climate scientist David Stainforth (personal communication).

considerations as discussed in Subsection 4.3 will fix a suitable prediction lead-time, say 30 years. Yet, then the same prediction lead-time should be used when defining the future climate. However, this is undesirable because for the future climate the prediction lead-time should only be determined by how many years scientists want to predict in the future (and should not be constrained to, say, 30 years). So it is difficult to see how the present and past climate could be defined. *Hence Definition 4 is not applicable to the past, present and future and Desideratum 4 is violated.*

Third, there is the problem that climate, thus defined, does not have anything to do with distributions over time of actual properties of the climate system (cf. Schneider and Dickinson 2000). This leaves us in the awkward situation that *the observational records of past temperatures etc. do not tell us anything about climate and climate change. Therefore, Desideratum 1 is not met.*

To sum up the discussion of climate as future ensemble distribution: Because of the assumption of constant external conditions, Definition 4 may be empirically void (thereby violating Desideratum 1). While Definition 5 avoids this shortcoming, there are several other problems (which also arise for Definition 4). Namely, the climate depends on our knowledge about the initial values (thereby violating Desideratum 3). Also, it is unclear how the past and present climate could be defined, and thus Desideratum 4 is not met. Finally, there is no relation to the past observations of the climate system, and thus Desideratum 1 is violated. Given all these problems, it seems better to say that Definition 5 is not about the climate but just refers to the expected future distribution of the climate variables (a distribution which is useful to consider for predictive purposes).[13]

5.3 Infinite Versions

There are also infinite versions of the two ensemble distributions of climate just discussed. According to these infinite versions, the climate is defined as the infinite distribution which arises when the prediction lead-time in equation (5) (for Definition 4) or equation (6) (for Definition 5) goes to infinity. These infinite versions are popular definitions (e.g. Checkroun et al. 2011; Stone et al. 2009).

The main advantage of these infinite versions is that they are *mathematically more tractable*, and, as a consequence, dynamical systems theory can be used to analyse

[13] One could also introduce an ensemble definition for *regimes of varying external conditions* (corresponding to Definition 3). The need for Definition 3 arose because it avoids a serious problem encountered by Definition 2 (that different climates are not classified correctly). No problems are avoided by introducing an ensemble definition for regimes of varying external conditions (compared to Definition 5). Hence there is no need for such a definition.

them. Let me give two examples for this. First, recall that climate as the future ensemble distribution is defined relative to the present uncertainty in the initial values. For the infinite versions conditions are known under which the infinite ensemble distribution is the same for any arbitrary uncertainty in the initial values (then climate does not depend on our uncertainty and Desideratum 3 is not violated). Namely, this is the case when the dynamics is mixing or when there is a strong physical measure (for the infinite version of Definition 4), or when there is a time-dependent strong physical measure (for the infinite version of Definition 5) (see Appendix C for a mathematical statement of these results). As nice as these results are, their relevance is unclear because it is unknown whether the true climate model under constant external conditions is mixing or has a strong physical measure, or whether the true climate dynamics has a time-dependent strong physical measure (cf. Daron 2012).

Second, recall that the climate as the future ensemble distribution does not seem to have anything to do with the time series of past observations. For the infinite version of Definition 4 it can be shown that when the dynamics is mixing or has a strong physical measure, then the infinite ensemble distribution and the infinite distribution over time are the same for almost all initial conditions (note that there are *no* corresponding results for Definition 5[14]). Hence if distributions over finite long time periods approximate the distribution over an infinite time period, the infinite ensemble distribution can be estimated from the observations and Desideratum 1 is not violated (see Appendix D for a mathematical statement of these results). However, again, the relevance of these results is unclear because it is unknown whether the true climate model under constant external conditions is mixing or has a strong physical measure (and whether distributions over finite long time periods approximate the distributions over an infinite time period – cf. Subsection 4.4).

What speaks against the infinite versions of Definitions 4 and 5 is that *they suffer from several shortcomings* (in addition to the ones already outlined in Subsections 5.1-5.2). Consequently, for defining the climate *finite ensemble distributions are preferable* to infinite ensemble distributions, and for this reason the discussion above concentrated on finite distributions. More specifically, there is the problem that *the infinite limits defining the infinite ensemble distributions may not exist, implying that Desideratum 5 may not be met* (Lasota and Mackey 1985; Provatas and Mackey 1991). To my knowledge, it is unknown whether these limits exist for the true climate model.[15]

[14]There *cannot* be any corresponding results for Definition 5: Because the external conditions vary, the infinite ensemble distributions vary with time but the infinite distributions over time do not. Hence the infinite ensemble distributions cannot be equal to the infinite distributions over time.

[15]For Definition 5 where the external conditions vary (i.e. for equation (6)) there is the additional problem that there are two infinite limits: one for $t_0 \to -\infty$ and one for $t_1 \to \infty$ (the latter limits

Furthermore, in order for the infinite ensemble distribution to be predictively useful, one has to assume that the ensemble distribution at time point t_1 (what is of concern in practice) is approximated by the infinite ensemble distribution. However, *there are doubts about this.* For several models that incorporate some realism about the evolution of the climate variables (including models with constant and varying external conditions) the finite ensemble distributions only converge very slowly to the infinite ensemble distributions (cf. Smith 2002). For instance, for the coupled ocean-atmosphere model investigated by Daron (2012) and Daron and Stainforth (2013) there is convergence only after 100 model years. Further, if the true climate model turned out to be almost intransitive, the convergence could be very slow (Bhattacharya et al. 1982; Sempf et al. 2007). Whether for the true climate model the finite ensemble distributions for predictions lead-times of interest are approximated by the infinite ensemble distributions is unknown, but there remains the *worry that the infinite ensemble distribution may be empirically void (thereby violating Desideratum 1).*

To conclude: defining climate as an infinite ensemble distribution makes the mathematical analysis of climate more tractable. However, there are the additional problems that the relevant limits may not exist (implying that Desideratum 5 may not be met) and that the definitions may be empirically void (implying that Desideratum 1 may not be met).

6 Conclusion

The aim of the paper was to provide a clear and thorough analysis of the main candidates for a definition of climate and climate change. Five definitions were discussed: climate as the finite distribution of the climate variables over time for constant external conditions (Definition 1), climate as the finite distribution of the climate variables over time when the external conditions vary as in reality (Definition 2), climate as the finite distribution of the climate variables over time relative to a regime of varying external conditions (Definition 3), climate as the ensemble distribution of the climate variables for constant external conditions (Definition 4), and climate as the ensemble distribution of the climate variables when the external conditions vary as in reality (Definition 5). Definition 3 is a novel contribution of this paper and was proposed as a response to problems with existing definitions. The other four definitions are among the most commonly endorsed definitions of climate.

exist only very rarely, cf. Kloeden and Rasmussen 2011). If both limits exist, they usually differ. Hence the question arises which of them should be identified with the climate, and it is unclear how to answer this question.

A definition of climate should be empirically applicable (Desideratum 1), it should correctly classify different climates (Desideratum 2), it should not depend on our knowledge (Desideratum 3), it should be applicable to the past, present and future (Desideratum 4) and it should be mathematically well-defined (Desideratum 5). The discussion has shown that *only Definition 3 (where climate is the finite distribution over time under a certain regime of varying external conditions) meets all the desiderata* and hence is the most promising definition.

This can also be seen as follows. The Definition 2 is unattractive because it does not classify different climates correctly (i.e., Desideratum 2 is not met). Definitions of climate where the external conditions vary as in reality are preferable to definitions where the external conditions are held constant because the latter may be empirically void (thereby violating Desideratum 1). Distributions over time are preferable to ensemble distributions because for the latter the climate depends on our knowledge (thereby violating Desideratum 3), it is unclear how to define the past and present climate (thereby violating Desideratum 4) and there is no relation to the observational record (thereby violating Desideratum 1). Given this, Definition 3 comes out as the most promising definition.

Infinite versions of Definitions 1-5 were also discussed. They were quickly dismissed since they suffer from the additional problems that they may be empirically void (thereby not meeting Desideratum 1) and that the relevant limits may not exist (thereby not meeting Desideratum 2).

Finally, we are now in the position to look at the characterisation of climate given by the influential report of the IPCC (Solomon et al. 2007, 942):

> Climate in a narrow sense is usually defined as the average weather, or more rigorously, as the statistical description in terms of the mean and variability of relevant quantities over a time period ranging from months to thousands or millions of years. The classical period for averaging these variables is 30 years, as defined by the World Meteorological Organization. The relevant quantities are most often surface variables such as temperature, precipitation and wind. Climate in a wider sense is the state, including a statistical description, of the climate system.

This characterisation is vague and ambiguous. 'Climate in a narrow sense' may be intended to refer to the distributions of a more limited set of climate variables and 'climate in a wider sense' to the distributions of a more extended set of climate variables (cf. Section 2). Apart from this, 'climate in a narrow sense' seems to refer to a distribution *over time*, i.e. to Definition 1, 2 or 3. The most direct interpretation is

that it refers to the finite distribution over time for the actual path of the external conditions (i.e. Definition 2, which, as argued, is unattractive). 'Climate in a wider sense' is even more open to different interpretations and could in principle be interpreted as referring to any of the definitions discussed in this paper.

In any case, the vagueness may well be intended to subsume under one characterisation the various different definitions of climate. Still, I hope that this paper has raised awareness of the importance of choosing a good definition of climate. In this way conceptual confusion and wrong statements about our climate system can hopefully be avoided.

Appendix

A. Deterministic Models

Climate models are *deterministic models* $(X, \Sigma_X, T(x, t_0, t))$ of the evolution of the climate variables. Here the set X represents all possible values of the climate variables, the σ-algebra Σ_X represents all subsets of X of interest, and $T(x, t_0, t) : X \times \mathbb{Z} \times \mathbb{Z} \to X$ (the dynamics) is a measurable function such that $T(x, t_0, t_0) = x$ and $T(x, t_0, t + s) = T(T(x, t_0, t), t, s)$ for all t_0, t, s and x.[16] Intuitively speaking, $T(x, t_0, t)$ gives one the value of the climate variables at time t when x is the value at time t_0. The *solution* when x is the initial value of the climate variables at t_0 is the function $T_{x,t_0}(t) : \mathbb{Z} \to X$, $T_{x,t_0}(t) = T(x, t_0, t)$. These deterministic models are *time-dependent*, and the recently developed *theory of time-dependent dynamical systems* is needed to analyse them (cf. Kloeden and Rasmussen 2011).

If the governing equations do not depend on time, the models are *time-independent* and one is in the realm of classical dynamical systems theory. Then the reference to t_0 can be omitted because all that matters is the evolved time period t between two values of the climate variables. That is, the *solution* through x is the function $T_x(t) = T(x, t)$, where $T(x, t)$ gives the value of the climate variables that started in x after t time steps. Suppose that the functions $T_t : X \to X$, $T_t(x) = T(x, t)$, are bijective for all t and that there is a probability measure μ on X which is *invariant*, i.e.

$$\mu(T(A, t)) = \mu(A) \text{ for all } A \in \Sigma_X \text{ and all } t \in \mathbb{Z}. \tag{7}$$

Then $(X, \Sigma_X, T(x, t), \mu)$ is a *measure-preserving deterministic model* (cf. Petersen 1983).

[16] If $T(x, t_0, t)$ is defined only on $X \times \mathbb{Z} \times \mathbb{Z} \cap [t_0, \infty)$, then the dynamics is *non-invertible* (as opposed to the usual case in climate science when the dynamics is *invertible*, i.e. defined on $X \times \mathbb{Z} \times \mathbb{Z}$). All that is stated in the Appendix can be easily adapted to carry over to a non-invertible dynamics.

B. Independence from the Initial Conditions for the Infinite Version of Definition 1

For the infinite version of Definition 1 there are two cases where the dependence on the initial conditions is very weak (thus many argue that, pragmatically speaking, there is no dependence – cf. Lorenz 1970, 1995). First, a measure-preserving deterministic model $(X, \Sigma_X, T(x,t), \mu)$ is *ergodic* iff for all $A \in \Sigma_X$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)) = \mu(A) \tag{8}$$

for any $x \in B$ with $\mu(B) = 1$ (cf. Petersen 1983).[17] Ergodicity immediately implies that the value assigned to any set A by the infinite version of Definition 1 is the same for almost all x (i.e. those in B).

The second result is about *attractors* Ω which are invariant[18] sets $\Omega \subseteq X \subseteq \mathbb{R}^n$ that attract all initial values in X, i.e., $\lim_{t\to\infty} dist(T_t(x), \Omega) = 0$ for all $x \in X$. A measure μ_Ω on Ω is called a *physical measure* iff for Lebesgue-almost all[19] $x \in X$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)) = \mu_\Omega(A), \tag{9}$$

whenever $\mu_\Omega(\delta A) = 0$ where δA denotes the boundary of A (cf. Eckmann and Ruelle 1985; Ruelle 1976). Hence for a physical measure the value assigned to any $A \in \Sigma_X$ with $\mu_\Omega(\delta A) = 0$ by the infinite version of Definition 1 is the same for Lebesgue-almost all initial values.

C. Independence from the Uncertainty in the Initial Values for the Infinite Versions of Definitions 4 and 5

The independence result for the infinite version of Definition 4 holds under two conditions. First, a measure-preserving deterministic model $(X, \Sigma_X, T(x,t), \mu)$ is *mixing* iff for all probability densities p_{t_0} (relative to μ) and all sets $A \in \Sigma_X$ (Petersen 1983; Werndl 2009a; Werndl 2009b):[20]

$$\lim_{t_1\to\infty} P^c_{t_0,t_1}(A) = \mu(A). \tag{10}$$

[17] $\chi(A)$ is the characteristic function, i.e. $\chi(x) = 1$ for $x \in A$ and 0 otherwise.
[18] That is, $T_t(\Omega) = \Omega$ for all $t \in \mathbb{Z}$.
[19] That is, for all $x \in Y$, $Y \subseteq X$, with $\lambda(X \setminus Y) = 0$, where λ is the Lebesgue measure.
[20] Since all that matters is the evolved time period, the limit could equally be taken for $t_0 \to -\infty$.

Mixing immediately implies that the measure assigned to a set by the infinite version of Definition 4 is the same for all initial densities p_{t_0} and hence there is no dependence on the uncertainty.[21]

Second, a measure μ^{Ω} on the attractor Ω is called *strongly physical* iff for any density p_{t_0} (relative to the Lebesgue measure λ) on X and for any set $A \in \Sigma_X$:[22]

$$\lim_{t_1 \to \infty} P^c_{t_0,t_1}(A) = \mu^{\Omega}(A), \tag{11}$$

whenever $\mu^{\Omega}(\delta A) = 0$ where δA denotes the boundary of A (cf. Ruelle 1976; Tasaki et al. 1998). Here the dependence on the uncertainty in the initial values is negligible in the sense that the measure assigned to any A with $\mu^{\Omega}(\delta A) = 0$ by the infinite version of Definition 4 is the same for all initial uncertainties.

The independence result for the infinite version of Definition 5 holds when $t_1 \to \infty$[23] and when there are time-dependent strong physical measures. To define them, the following definition is needed. A *pullback attractor* $\Omega \subseteq \mathbb{Z} \times \mathbb{R}^n$ is an invariant set[24] where for all initial values $x \in X$

$$\lim_{t_0 \to -\infty} dist(T(x, t_0, t), \Omega(t)) = 0. \tag{12}$$

Time-dependent strong physical measures μ^{Ω}_t defined on $\Omega(t)$, $t \in \mathbb{Z}$, where Ω is a pullback attractor are defined by the condition that for any t and t_0, any initial density p_{t_0} (relative to the Lebesgue measure λ) on X and for any set $A \in \Sigma_X$ (cf. Buzzi 1999):

$$\lim_{t_0 \to -\infty} P_{t_0,t_1}(A) = \mu^{\Omega}_t(A), \tag{13}$$

whenever $\mu^{\Omega}_t(\delta A) = 0$ (δA denotes the boundary of A). Hence for time-dependent strong physical measures the dependence on the uncertainty is negligible because the measure assigned to a set A with $\mu^{\Omega}_t(\delta A) = 0$ by the infinite version of Definition 5 is the same for all initial uncertainties.

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[21] One might ask when the stronger condition holds that any initial probability density p_{t_0} converges to the measure μ in the sense that $\lim_{t_1 \to \infty} \int_X |1 - p^c_{t_0,t_1}| d\mu = 0$, where $p^c_{t_0,t_1}$ is the density that arises when p_{t_0} is evolved forward to t_1. Most climate models are invertible (cf. footnote 16). In this case the stronger condition *cannot* hold because the measure is invariant. For non-invertible models $(X, \Sigma_X, T(x,t), \mu)$ this condition holds iff they are *exact*, i.e. when $\lim_{t \to \infty} \mu(T(A,t)) = 1$ for all $A \in \Sigma_X$ with $\mu(A) > 0$ (Berger 2001; Lasota and Mackey 1985). Exactness is a stronger condition than mixing. It is unclear whether realistic non-invertible climate models are exact.

[22] Since all that matters is the evolved time period, the limit could equally be taken for $t_0 \to -\infty$.

[23] Recall that there are two possible infinite limits for the infinite version of Definition 5 (cf. footnote 15).

[24] That is, $\Omega(t) = T(\Omega(t_0), t_0, t)$ for all $t, t_0 \in \mathbb{Z}$, where $\Omega(t) := \{x \in \mathbb{R}^n \mid (t, x) \in \Omega\}$.

D. Relation to Infinite Distributions Over Time for the Infinite Version of Definition 4

For the infinite version of Definition 4 two conditions are known which relate infinite ensemble distributions to infinite distributions over time. First, if the measure-preserving deterministic system $(X, \Sigma_X, T(x,t), \mu)$ is *mixing*, equations (10) and (8) imply that for any initial probability density p_{t_0} and any set $A \in \Sigma_X$

$$\lim_{t_1 \to \infty} P^c_{t_0,t_1}(A) = \mu(A) = \lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)) \tag{14}$$

for all initial values $x \in B$ with $\mu(B) = 1$. Hence the measure assigned to a set A by the infinite ensemble distribution and the infinite distribution over time is the same for almost all initial values (i.e. those in B).

Second, for an attractor Ω with a *strong physical measure* μ^Ω equations (11) and (9) imply that for any initial density p_{t_0}, any set A with $\mu^\Omega(\delta A) = 0$ and Lebesgue-almost all initial values $x \in X$:

$$\lim_{t_1 \to \infty} P^c_{t_0,t_1}(A) = \mu^\Omega(A) = \lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \chi_A(T_t(x)). \tag{15}$$

Hence the measure assigned to a set A with $\mu^\Omega(\delta A) = 0$ by the infinite ensemble distribution and the infinite distribution over time is the same for Lebesgue-almost all initial values.

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References

Allen, M. (2003). Liability for climate change. *Nature* 421, 891-892.

Berger, A. (2001). *Chaos and Chance. An Introduction to Stochastic Aspects of Dynamics*. Berlin: Walter de Gruyter.

Bhattacharya, K., Ghil, M. and Vulis, I. L. (1982). Internal variability of an energy-balance model with delayed albedo effects. *Journal of the Atmospheric Sciences* 39, 1747-1773.

Buzzi, J. (1999). Exponential decay of correlations for random Lasota-Yorke maps. *Communications in Mathematical Physics* 208, 25-54.

Checkroun, M. D., Simonnet, E. and Ghil, M. (2011). Stochastic climate dynamics: Random attractors and time-dependent invariant measures. *Physica D*, 240 (21), 1685-1700.

Daron, J. D. (2012). Examining the Decision-Relevance of Climate Model Information for the Insurance Industry. *Doctoral Thesis*. London School of Economics and Political Science. http://etheses.lse.ac.uk/380/

Daron, J. D. and Stainforth, D. (2013). On predicting climate under climate change. *Environmental Research Letters* 8, 1-8.

Dymnikov, V. and Gritsoun, A. (2001). Atmospheric model attractors: chaoticity, quasi-regularity and sensitivity. *Nonlinear Processes in Geophysics* 8, 201-208.

Eckmann, J.-P. and Ruelle, D. (1985). Ergodic theory of chaos and strange attractors. *Reviews of Modern Physics* 1985, 617-656.

Frigg, R., Bradley, S., Du, H. and Smith, L. A. (2013). Laplace's demon and climate change. *Grantham Research Institute on Climate Change and the Environment Working Paper* 103, London School of Economics.

Goswami, B. N. Wu, G. and Yasunari, T. (2006). The annual cycle, intraseasonal oscillations, and roadblock to seasonal predictability of the Asian summer monsoon. *Journal of Climate* 19, 5078-5099.

Hasselmann, K. (1976). Stochastic climate models, Part 1. *Tellus* XXVIII (6), 473-484.

Hulme, M., Dessai, S., Lorenzoni, I. and Nelson, D. (2009). Unstable Climates: exploring the statistical and social constructions of climate. *Geoforum* 40, 197-206.

Jin, F.-F., Neelin, J. D., and Ghil, M. (1994). El Niño on the devil's staircase: annual subharmonic steps to chaos. *Science* 264, 70-72.

Kloeden, P. E. and Rasmussen, M. (2011). *Nonautonomous Dynamical Systems.* Mathematical Surveys and Monographs Volume 176. American Mathematical Society.

Kurgansky, M. V., Dethloff, K., Pisnichenko, I. A., Gernandt, H., Chmielevski, F.-M. and Jansen, W. (1996). Long term climate variability in a simple, nonlinear atmospheric model. *Journal of Geophysical Research* 101 (D2), 4299-4314.

Lasota, A. and MacKey, M. C. (1985). *Probabilistic Properties of Deterministic Systems.* Cambridge: Cambridge University Press.

Lorenz, E. (1963). Deterministic nonperiodic flow. *Journal of Atmospheric Sciences* 20, 130-141.

Lorenz, E. (1968). Climatic determinism. *Meteorological Monographs* 8, 1-3.

Lorenz, E. (1970). Climatic change as a mathematical problem. *Journal of Applied Meteorology* 9 (3), 325-329.

Lorenz, E. (1976). Nondeterministic theories of climatic change. *Quaternary Research* 6 (4), 495-506.

Lorenz, E. (1995). Climate is what you expect. Prepared for publication by *NCAR.* Unpublished, 1-33.

Mancho, A. M., Wiggins, S., Curbelo, J. and Mendoza, C. (2014). Lagrangian descriptors: a method for revealing phase space structures of general time dependent dynamical systems. Forthcoming in: *Communications in Nonlinear Science and Numerical Simulation.*

Palmer, T. (1999). A nonlinear dynamical perspective on climate prediction. *Journal of Climate* 12, 575-591.

Parker, W. (2006). Understanding pluralism in climate modeling. *Foundations of Science* 11, 349-368.

Provotas, N. and Mackey, M. C. (1991). Asymptotic periodicity and banded chaos. *Physica D* 53, 295-318.

Petersen, K. E. (1983). *Ergodic Theory.* Cambridge: Cambridge University Press.

Ruelle, D. (1976). A measure associated with Axiom A attractors. *American Journal of Mathematics* 98, 619-654.

Schneider, S. H. and Dickinson, R. E. (2000). Climate modelling. *Climatic Change* 45, 203-221.

Sempf, M., Dethloff, K., Handorf, D., Kurgansky, M. V. (2007). Toward understanding the dynamical origin of atmospheric regime behavior in a baroclinic model. *Journal of the Atmospheric Sciences* 64, 887-904.

Smith, L. (2002). What might we learn from climate forecasts. *Proceedings of the National Academy of Sciences of the United State of America* 99 (1), 2487-2492.

Solomon, S., Qin, D., Manning, M., Chen, Z., Marquis, M., Averyt, K. B., Tingor, M. and Miller, H. L. (2007). *Climate Change 2007: The Physical Science Basis. Contribution of Working Group I to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change.* Cambridge: Cambridge University Press. (IPCC Report)

Steele, K. and Werndl, C. (2013). Climate Models, calibration and confirmation. *The British Journal for the Philosophy of Science* 64 (3), 609-635.

Stone, D. A. and Knutti, R. (2010). Weather and climate. In: Fung, F., Lopez, A. and New, M. (eds), *Modelling the Impact of Climate Change on Water Resources.* Chichester: John Wiley & Sons.

Stone, D. A., Allen, M. R., Stott, P. A., Pall, P., Min, S.-K., Nozawa, T. and Yukimoto, S. (2009). The detection and attribution of human influence on climate. *The Annual Review of Environment and Resources* 34, 1-16.

Stott, P. A. and Kettleborough, J. A. (2002). Origins and estimates of uncertainty in predictions of twenty-first century temperature rise. *Nature* 416, 723-305.

Tasaki, S., Gilbert, T. and Dorfman, J. R. (1998). An analytical construction of the SRB measures for baker-type maps. *Chaos* 8, 424-444.

Todorov, A. V. (1986). Reply. *Journal of Applied Climate and Meteorology* 25, 258-259.

Werndl, C. (2009a). What are the new implications of chaos for unpredictability? *The British Journal for the Philosophy of Science* 60, 195-220.

Werndl, C. (2009b). Justifying Definitions in Mathematics – Going Beyond Lakatos. *Philosophia Mathematica* 17, 313-340.