Symmetric priors *vs* asymmetric expectation (part 2)

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Outline

Asymmetric Benthamite calculations

Idea

Probability assignments must depend both on the "quantity" and on the "quality" of the available information

- ambiguity aversion as a normative rationale for the Ellsberg preferences in the 1 urn, 3 colours problem ("not enough info to determine what the odds are")
- asymmetric view of knowledge and ignorance in the 2 urns problem ("you know what game you are playing")
- Relaxation of the Independence axiom

Ellsberg's violation of the A-A (Ind)

Recall the linear structure of A-A allows us to mix acts, i.e for $f, g, \in F$, $\alpha \in (0,1)$ we can always consider $\alpha f + (1-\alpha)g$. So let

$$\left(\frac{1}{2}R + \frac{1}{2}B\right)(s) = \begin{cases} \frac{1}{2} \text{ if } s \in R \cup B \\ 0 \text{o.w.} \end{cases} = \left(\frac{1}{2}R \cup B + \frac{1}{2}0\right)(s)$$

Similarly

$$\left(\frac{1}{2}Y + \frac{1}{2}B\right)(s) = \left(\frac{1}{2}Y \cup B + \frac{1}{2}0\right)(s)$$

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By (Ind)

$$R \succ Y \Rightarrow \frac{1}{2}R + \frac{1}{2}B \succ \frac{1}{2}Y + \frac{1}{2}B \Leftrightarrow \frac{1}{2}R \cup B \succ \frac{1}{2}Y \cup B$$

.

But

$$\frac{1}{2}Y \cup B \succ \frac{1}{2}R \cup B \Rightarrow \frac{1}{2}Y \cup B \succ \frac{1}{2}R \cup B,$$

contradiction

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Ambiguity, uncertainty aversion and all that

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2. Dropping probability in favour of (point-valued) non-additive belief measures or *capacities*
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Axiomatically

Several relaxations of the A-A Ind condition

Certainty Independence

Certainty Independence (CIn)

For $f, g \in F, h \in X$ and for $\alpha \in (0,1)$

$$\text{if } f \succ g \text{ then } \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h.$$

Idea

Independence (compositionality via mixing) is fine for *objective uncertainty* (i.e. risk) only

- Recall risk preferences $\succ_\Delta \subseteq X^2$ are uniquely induced by act-preferences for constant acts i.e.

$$f \succ g \Leftrightarrow x \succ_\Delta y$$

where $f, g \in F$ are such that $f(s) = x$ and $g(s) = y$ for all $s \in S$

Uncertainty Aversion (UAv)

For $f, g \in F$ and for $\alpha \in (0,1)$

$$\text{if } f \sim g \text{ then } \alpha f + (1-\alpha) g \succsim f.$$

UNCERTAINTY AVERSION (UAV)

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IDEA

Fix $\alpha = 1/2$ and acts Y, B in the Ellsberg 3-colour urn problem. It is reasonable that $Y \sim B$. Hence by UAv,

$$\frac{1}{2}Y + \frac{1}{2}B \succsim Y$$

can be seen as the agent's preference for tossing a fair coin to decide which colour to bet on, rather than taking the "epistemic responsibility" of choosing one themselves.

Theorem (Gilboa and Schmeidler (1989))

The following are equivalent

1. $\succsim \subseteq \mathcal{A}$ *satisfies WeO, Mon, Con, CInd, UAv*
2. $\exists$ *non-constant* $u : C \to \mathbb{R}$ *and a unique closed, convex* $\mathcal{C} \subseteq \Delta(\Sigma)$ *such that for all* $f, g \in F$

$$f \succsim g \Leftrightarrow \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(f) \right\} \geq \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(g) \right\},$$

where as usual

$$E_P^{A-A}(f) = \int_S \left(\sum_{x \in supp\, f(s)} u(x) f(s) \right) dP(s).$$

Moreover, U *is cardinally unique and* $\mathcal{C}$ *is a singleton iff* $\succsim$ *satisfies Ind.*

Idea of the proof

The G-S axiomatisation extends properly the A-A setup, so the proof proceeds by extending the A-A result to ensure the existence of a functional $J : \mathbb{R} \to \mathbb{R}$ which represents E_P^{G-S}

Key Lemmas

1. CIn and UAv are necessary and sufficient to ensure that J is a *superlinear capacity* (concavity of the outer measure)
2. If J is a superlinear capacity that there exists a compact and closed subset $\mathcal{C}$ of $\Delta(\Sigma)$ such that for all acts f

$$E_P^{G-S}(f) = \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(f) \right\}$$

Understanding $\mathcal{C}$

A twofold interpretation

The smaller the cardinality of $\mathcal{C}$:

1. the agent possesses "better" or "more precise" information
2. "less ambiguity" or "more confidence"

Fact (Ghirardato et al., 2004)

$\mid \mathcal{C} \mid = 1$ iff $\forall f, g \in F$

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Criticisms

- The asymmetric averaging modelled by E^{G-S} appears unnecessarily asymmetric (or pessimistic)
- The problem of determining uniquely the probability measure is lifted to determining uniquely $\mathcal{C}$

More on uncertainty aversion

1) Diversification

- originates with the Babylonian Talmud (1/3, or $1/N$ rule)
- central in Portfolio Optimisation (Mean-Variance Analysis) and in Risk Measures (coherent and convex risk measures (Yor, 2008))
- hedging against uncertainty
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2) Preference for randomisation

- experimental evidence (Hsu et al., 2005)
- Games against nature / informed opponent

Criticism

The justification is not intuitive for asymmetric weights (i.e. $\alpha \neq \frac{1}{2}$)

Outline

CAPACITIES

A CAPACITY is a normalised and monotone set function $\nu : \Sigma \to [0, 1]$, that is it satisfies

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(C.2) $A \subseteq B \to \nu(A) \leq \nu(B)$

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(C.3')

$$\nu(\cup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{\#I+1} \nu(\cap_i \in IA)$$

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A probability function is an additive 2-monotone capacity

(Schmeidler, 1989)

Ellsberg 2 urns problem is rephrased as follows. You must toss a coin and you have a choice between

X bet on your coin (known to be "fair")

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Schmeidler 1989 insists that rational agents should strictly prefer X which is inconsistent with probability.

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Capacities as uncertainty measures

Since capacities are superadditive, i.e.

$$\nu(H \cup T) > \nu(H) + \nu(T)$$

the asymmetric assignment $\nu(H) = \nu(T) = .4$, $\nu(H \cup T) = 1$ and would not be *irrational*.

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- However, it is mathematically far from trivial...

THE CHOQUET INTEGRAL

Choquet, G. (1953). Theory of capacities. *Ann. Inst. Fourier*, Grenoble 5, 131–295.

Let $\phi : S \to \mathbb{R}$ be a nonnegative function.
Then the
CHOQUET INTEGRAL OF ϕ WITH RESPECT TO THE CAPACITY ν is

$$\int_S \phi(s) d\nu(s) = \int_0^\infty \nu\left(\{s \in S \mid \phi(s) \geq t\right) dt,$$

where the rhs integrals are taken in the sense of Riemann.

A difficult concept

- Technically complex owing to the non additivity of the measure
- No clear "behavioural" interpretation of the resulting notion of expectation
- Schmeidler's key contribution consists in his axiomatic justification of the Choquet expected utility functional obtained by restricting A-A Ind to

Comonotonic Acts

For $f, g \in F$ it is never the case that $f(s) \succ f(s')$ and $g(s) \prec g(s')$

Choquet rationality

Theorem (Schmeidler's representation theorem)

The following are equivalent:

1. $\succsim \subseteq F^2$ *satisfies WeO, Con, ComInd, Mon*
2. $\exists$ *nonconstant* $U : C \to \mathbb{R}$ *and a unique capacity* $\nu : \Sigma \to [0, 1]$ *s.t.*

$$f \succsim g \Leftrightarrow E_P^{Ch}(f) \geq E_P^{Ch}(g),$$

where

$$E_P^{Ch}(f) = \int_S \left(\sum_{x \in supp\, f(s)} U(x) f(s) \right) d\nu(s).$$

Moreover, U is cardinally unique.

Relation with D-S Belief functions and G-S

Fact (Schmeidler, 1989)

1. E^{Ch} and E^{G-S} are not nested
2. yet they coincide when the beliefs of the agent are represented by a 2-monotone (a.k.a *supermodular*) belief function

Fact (Ghirardato, 2010)

- E^{Ch} can be "rationalised" by a supermodular belief function, which coincides with E^{G-S}.
- Hence E^{Ch} never violates the norm of E^{G-S}
- But the convers doesn't hold: G-S rationality can be seen to violate Ch rationality, which is therefore more restrictive.

Revance to MSU

Despite the criticisms, several interesting aspects:

- Common formalism in the theory of Fuzzy Sets (along with Sugeno integrals)
- Close links with cooperative game theory (Core of a Game, Shapley value)
- Relevance in the (weighted) aggregation of expert opinion
- Applied in automated multi-criteria decision making (R package KAPPALAB)

Grabisch, M., & Labreuche, C. (2010). A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *Annals of Operations Research*, 44, 247–286.

Further reading

- Schmeidler, D. (1989). Subjective Probability and Expected Utility without Additivity. *Econometrica*, 57(3), 571-587.
  - The paper containing the original proof of the representation theorem
- Wakker, P. P. (2010). Prospect theory: for risk and ambiguity. Management. *Cambridge University Press*.
  - relates Choquet expected utility (via Rank-Dependent Utility)to Prospect Theory
- Augustin, T., Coolen, F. P. A., de Cooman, G., & Troffaes, M. C. M. (Eds.). (2014). Introduction to Imprecise Probabilities. *Wiley*.
  - Puts E^{Ch} in the general framework of imprecise probabilities

Outline

Variational expectation

Maccheroni, F., Marinacci, M., & Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6), 1447–1498.

Further weakening of the A-A Ind axiom to
Wak Certainty Independence (WCIn)
For $f, g \in F, x, y \in X$ and for $\alpha \in (0, 1)$

$$\alpha f + (1 - \alpha)x \succ \alpha g + (1 - \alpha)x \Leftrightarrow \alpha f + (1 - \alpha)y \succ \alpha g + (1 - \alpha)y.$$

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Idea

The G-S CIn is required to hold only for fixed mixing coefficients. MMR prove that G-S CIn is equivalent $\forall f, g \in F, x, y \in X, \alpha, \beta \in (0,1)$ to

$$\alpha f + (1-\alpha)x \succ \alpha g + (1-\alpha)x \Leftrightarrow \alpha f + (1-\beta)y \succ \alpha g + (1-\beta)y.$$

MMR representation

Theorem ((Maccheroni et al., 2006))

The following are equivalent

1. $\succsim \subseteq \mathcal{A}$ *satisfies WeO, Mon, Con, WCInd, UAv*
2. $\exists$ *convex (etc.)* $c : \Delta(\Sigma) \to [0, \infty]$ *such that for all* $f, g \in F$

$$f \succsim g \Leftrightarrow \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(f) + c(P) \right\} \geq \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(g) + c(P) \right\},$$

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Moreover, for each $u : X \to \mathbb{R}$ $\exists! \, c$ *which satisfies the above, namely*

$$c(P) = \sup_{f \in F} \left\{ u(x_f) - \int_S u(f(s)) dP(s) \right\} \; \forall P \in \Delta(\Sigma)$$

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Ambiguity index

$c(P)$ has a natural interpretation as ambiguity index, which is maximal when $c(P)$ vanishes, giving us back the G-S expectation

MMR then prove that WCInd is enough to characterise the Hansen and Sargent functional, which brings us back to MaxEnt!

Model Uncertainty

Suppose a "true model of uncertainty" exists, say $Q \in \Delta(\Sigma)$ and the decision maker wants to take into account the confidence of their assessment by measuring the "distance" of P from Q

$$V(f) = \min_{P \in \mathcal{C}} \left\{ E_P^{A-A}(f) + \theta KL(P,Q) \right\}$$

where $KL(P,Q) = \sum_{s \in S} P(s) log \frac{P(s)}{Q(s)}$ and $\theta > 0$.

Hansen, L. P., & Sargent, T. J. (2001). Robust Control and Model Uncertainty. *AEA Papers and Proceedings*, 91(2), 60–66.

Outline

Recap

Objective uncertainty

Our relevant ignorance is confined to not knowing which state s will obtain

Subjective uncertainty

Objective uncertainty extends to not knowing the probability of each state

Severe uncertainty

Subjective uncertainty extends to *having to choose* one or more parameters in our model, including

- One (set of) probability measures to quantify the agent's uncertainty
- One state-space for the construction of the probability space
- One (complete) consequence space, etc.

Our question

How to do the *Benthamite calculation* under ambiguity

How to compare models

EMPIRICAL comparison of the outputs of numerical simulations

- quite promising in mathematical finance (asset pricing and portfolio management)
- dubious in policy-making under severe uncertainty (e.g. IPCC recommendations etc.)

FOUNDATIONAL assess the norms of decision making and uncertainty quantification by

- designing relevant decision problems (a là Ellsberg)
- investigating the desirability of the principles/properties/axioms leading to the norm in question

The symmetric solution

MaxEnt

Only the available information should be used: not having information is treated symmetrically as having information. minimise distance model from the "least biased" model

+ principle easy to understand
+ good philosophical back up (Objective Bayesian Epistemology)
* computationally amenable (if generally hard)
\- prone to modelling ambiguity (choice of the fundamental partition)

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E_P^{G-S} Extreme uncertainty aversion

- principle appears rather arbitrary

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E_P^{Var} Measure of Model uncertainty by minimising distance from the "true model"

- \+ probably the most expressive model
- \- (inevitably) complex

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P. Ghirardato, F. Maccheroni, and M. Marinacci. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118:133–173, 2004.

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