Decision Making Under Uncertainty

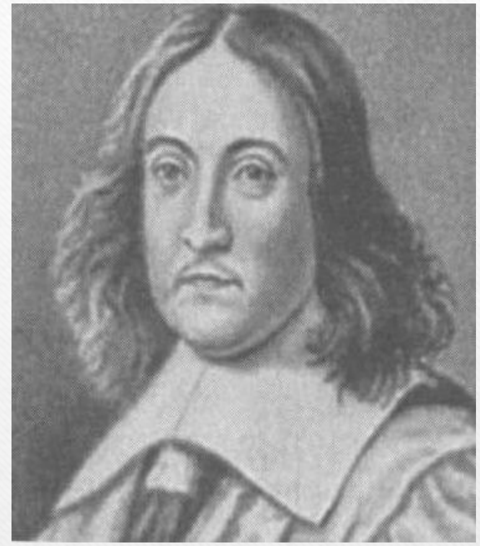
First Masterclass

Outline

- A short history
- Decision problems
- Uncertainty
- The Ellsberg paradox
- Probability as a measure of uncertainty
- Ignorance

Probability

Blaise Pascal



Pierre Fermat

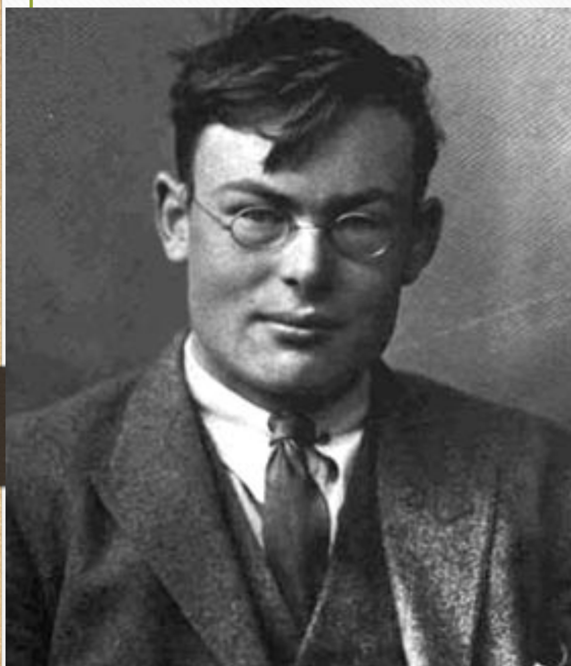
- Probability measures the opposite of uncertainty
- Laws of probability and chance developed through discussions of gambling problems
- Gamble to maximise expected monetary gain

Bernouilli's Hypothesis

- Principle of maximising expected monetary value leads to the St. Petersburg paradox.
- Bernouilli's solution was the concept of expected benefit or **utility**.

Daniel Bernouilli (1700-1782)

The Modern Theory

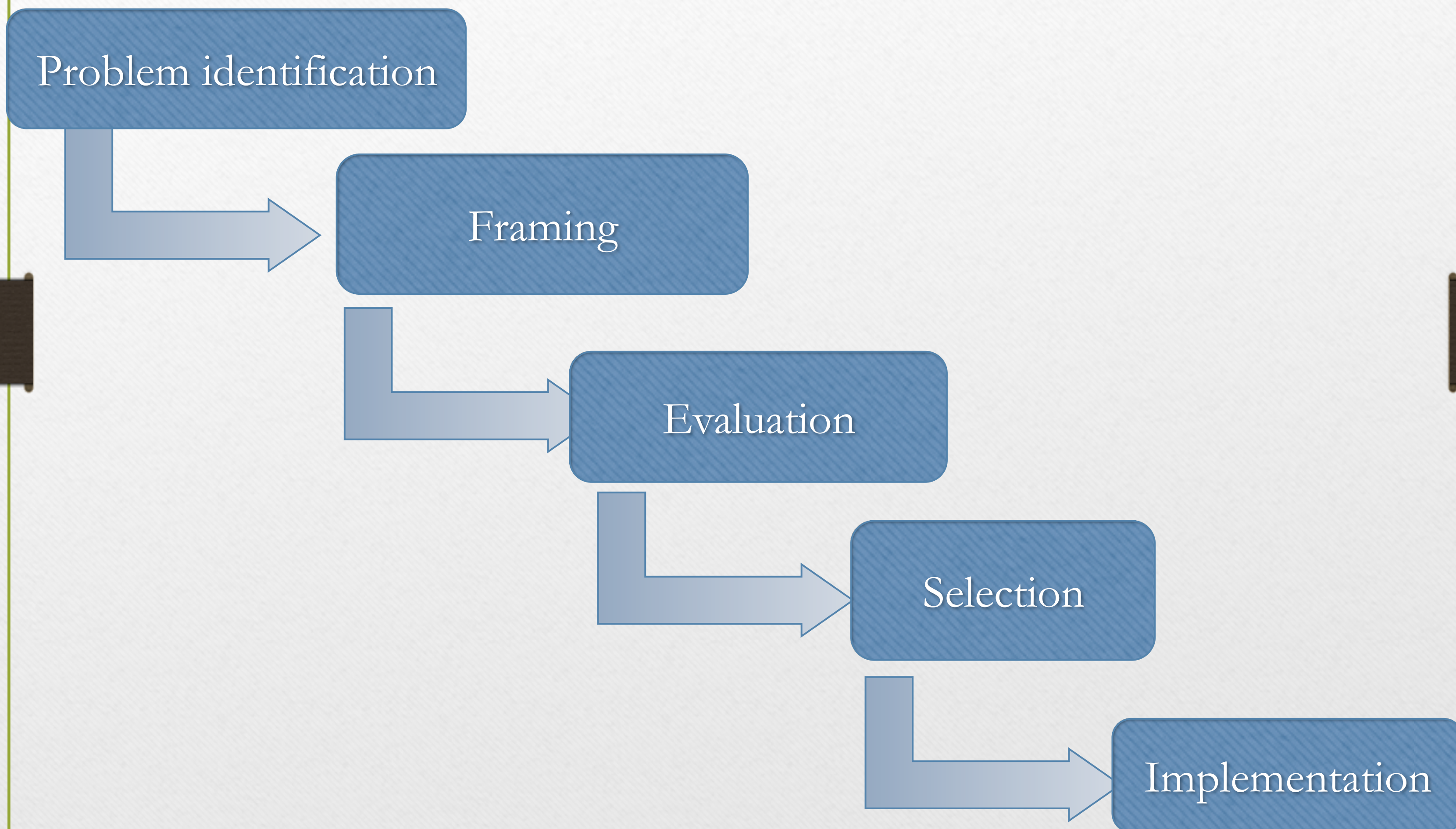


Frank Ramsey

John Von Neumann

Leonard Savage

Decision Processes



Choice and Value

Choice problem

- Given a set of alternatives and what we know about them, which ones should and/or can we select?
- Formally, the problem of specifying a choice function.

Value-based Choice

A choice function is based on a value relation iff the value relation explains/justifies the choices it encodes.

- Formally, a mapping from alternatives and value relation to a subset of alternatives
- Optimising / maximising / satisficing choice
- Choice-value relations

Decision Problems Under Uncertainty

- A decision problem under uncertainty is characterised by:
    1. A set of alternatives or options
    2. The possible states of the world
    3. The possible consequences (in each state) of each action
- A decision frame consists of a decision problem plus the 'givens': canonically the agent's information/beliefs and values/preferences.
- A decision rule is a mapping from a decision frame to a (possibly partial) ordering of the alternatives.

Standard Decision Problems

In the absence of any option or state-space uncertainty, a decision problem can represented by a state-consequence matrix (or variants for people, times, places, etc.).

	States			
Actions	**S_1**	**S_2**	**…**	**S_n**
A	a_1	a_2	…	a_n
B	b_1	b_2	…	b_n

Relations and Orders

- Informational givens canonically specified by a pair of relations:
    - An 'at least as credible/probable' relation $\trianglerighteq$ on sets of states (events or propositions)
    - An 'at least as preferred/good' relation $\succsim$ on consequences (subset of propositions)
    - Open to different interpretations
- When these relations are partial orders – i.e. complete and transitive - they are numerically representable by credibility or utility functions (see Debreu, Krantz et al)
    - Continuity required for uncountable sets
    - When not complete, they are representable by sets of such numerical measures (Evren & OK, 2011)

Some Important Facts

Suppose that $\precsim$ and $\trianglerighteq$ are continuous partial orders defined on an atomless Boolean algebra of propositions Ω. Then:

1. If $\trianglerighteq$ is quasi-additive then it is representable by a probability.
2. If $\succsim$ is an averaging relation that coheres with $\trianglerighteq$ then $\succsim$ is representable by a desirability.
3. If these relations are not complete then they can be represented by sets of probability/desirability functions

Let $\alpha \wedge \gamma = \perp = \beta \wedge \gamma$.

Quasi-additivity: $\alpha \succsim \beta \Leftrightarrow \alpha \vee \gamma \succsim \beta \vee \gamma$

Averaging: $\alpha \succsim \gamma \Leftrightarrow \alpha \succsim \alpha \vee \gamma \succsim \gamma$

Decision Making Under Uncertainty

The orthodoxy:

- All uncertainty in a standard decision problem is captured by a probability measure on events.
- Rationality requires maximising the expectation of utility relative to one's uncertainty and one's preferences.

Expected Utility Theory

- Probabilities for states and utilities for consequences are given (either objectively or subjectively).
- EU says pick the action which maximises expected benefit:

$$EU(A) = \sum_i U(a_i).\Pr(S_i)$$

States

Actions	**Pr(S_1)**	**Pr(S_2)**	**…**	**Pr(S_n)**
A	U(a_1)	U(a_2)	…	U(a_n)
B	U(b_1)	U(b_2)	…	U(b_n)

Justifications

- Arguments for Probabilism
  - Dutch book (De Finetti)
  - Axiomatic (Cox, van Fraassen)
  - Accuracy (Joyce)
- Arguments for EU
  - Long run optimisation
  - Axiomatic (Ramsey, Savage, etc.)

The Sure-Thing Principle

P1 (Ordering): $\succsim$ is a partial order

P2 (Separability): f $\succsim$ g iff f' $\succsim$ g'

	E	¬E
f	X	Y
g	X*	Y
f'	X	Z
g'	X*	Z

P3 (State-independence): f $\succsim$ g iff X $\succsim$ X*

Subjective Probability

- **Qualitative Probability** : Let E and F be any events and f and g the actions displayed below. Then:

  $E \trianglerighteq F$ iff $X \gtrsim Y$

	E	¬E
f	X	Y

	F	¬F
g	X	Y

P4: The relation $\trianglerighteq$ is complete

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Ellsberg's Paradox

	Red	Black	Yellow
Lottery A	100	0	0
Lottery B	0	100	0

	Red	Black	Yellow
Lottery C	100	0	100
Lottery D	0	100	100

Ellsberg's Prefs: $A \gtrsim B$ and $D \gtrsim C$

By P2: $A \gtrsim B \iff C \gtrsim D$

By P4: Red $\trianglerighteq$ Black iff $A \gtrsim B$

Black or Yellow $\trianglerighteq$ Red or Yellow iff $D \gtrsim C$

Representing Uncertainty

Is uncertainty adequately represented by numerical probability?

- Paradox of ideal evidence (Peirce, Popper).
- Risk versus uncertainty (Knight, Keynes) Keynes on Uncertainty
- Ambiguity and Ellsberg's paradox
- Bounded agents and incompleteness

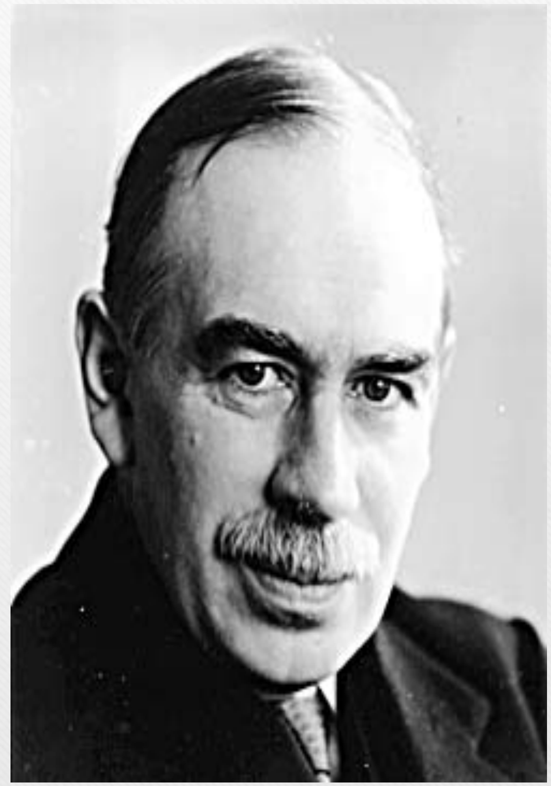
Alternatives to Probability

- Dempster-Shafer belief functions (convex capacities, non-additive probabilities)
- Sets of probabilities (intervals, lower-upper probabilities)
- Second-order measures (confidences, reliabilities)
- What does this mean for how we make decisions?

- Decision Making under Ignorance

Keynes on Uncertainty

By "uncertain" knowledge, let me explain, I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty; … . Even the weather is only moderately uncertain. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper … . **About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know.** Representing Uncertainty



Decision Making under Ignorance

In cases of ignorance your information consists of a utility on consequences and nothing more.

Actions	**S_1**	**S_2**	**…**	**S_n**
A	$U(a_1)$	$U(a_2)$	…	$U(a_n)$
B	$U(b_1)$	$U(b_2)$	…	$U(b_n)$

(Column group heading: **States**)

Ordinal Rules

- Dominance: Don't choose dominated alternatives.

$$\forall i, U(a_i) \geq U(b_i) \Rightarrow A \gtrsim B$$

- Maximin: Choose the action with best worst outcome.

$$A \gtrsim B \Leftrightarrow min[U(a_i)] \geq min[U(b_i)]$$

- Maximax: Choose the action with best best outcome.

$$A \gtrsim B \Leftrightarrow max[U(a_i)] \geq max[U(b_i)]$$

- These rules require only ordinal utility information

- Maximin (maximax) is unreasonably pessimistic (optimistic)
- Dominance can conflict with both Maximin and Maximax e.g. Car dominates Take Bus
  - Leximin and Leximax avoid this problem
- Dominance assumes independence of states from acts, but can this be assumed under ignorance?

	Traffic States		
Actions	**Heavy**	**Medium**	**Light**
Take Bus	-2	0	10
Walk	-1	-1	-1
Car	-2	2	10

Hurwicz Criterion

- Let h be a measure of your pessimism taking values in [0,1]
- **Hurwicz Index**: $\mathrm{H(A)} = h.Max[U(a_i)] + (1 - h)Min[U(a_i)]$
- Choose the action which maximises the Hurwicz index e.g. Take Bus is strictly preferred to Walk iff h < 2/3

Traffic States

Actions	**Heavy**	**Medium**	**Light**
Take Bus	-2	-2	1
Walk	-1	-1	-1

Hurwicz Criterion (cont.)

Issues:

- Is pessimism a stable psychological attitude?
- Gives invariant advice (only) if utilities are cardinally measurable
- Also conflicts with dominance

Traffic States

Actions	Heavy	Medium	Light
Take Bus	-2	-2	1
Walk	-1	-1	-1
Car	-2	1	1

Maximean

- Choose the action with greatest average benefit:

$$V_{mean}(\mathrm{A}) = \frac{\sum_{i=1}^{n} U(a_i)}{n}$$

- Main problem: not partition-independent e.g. {heavy, medium, light} versus {heavy, not-heavy}

	Traffic States		
Actions	**Heavy**	**Medium**	**Light**
Take Bus	-4	1	1
Walk	-1	-1	-1

Minimising Regret

Regret Index: $r(a_i) = \max\big(u(x_i)\big) - u(a_i)$

Minimax Regret: Choose the action which minimises the maximum regret.

	Traffic States		
Actions	**Heavy**	**Medium**	**Light**
Take Bus	-2 [1]	-2 [1]	1 [0]
Walk	-1 [0]	-1 [0]	-1 [2]

Minimising Regret

Conflicts with independence of irrelevant alternatives

Actions	Traffic States: Heavy	Medium	Light
Take Bus	-2 [1]	-2 [3]	1 [0]
Walk	-1 [0]	-1 [2]	-1 [2]
Car	-2 [1]	1 [0]	1 [0]