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Rough Set Approximations and Freedom of Choice

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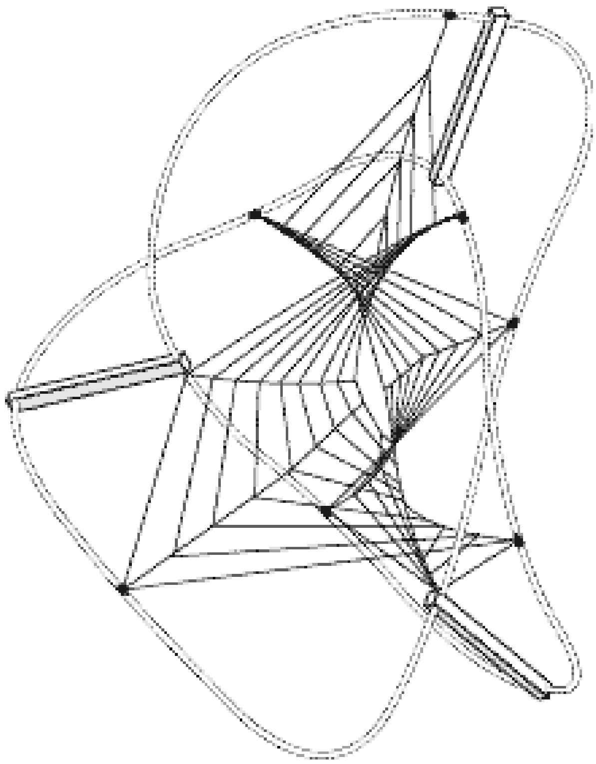

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Constraints and the Measurement of Freedom of Choice *

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Abstract

This paper introduces considerations about constraints in the construction of measures of an agent's freedom. It starts with motivating the exercise from both the philosophical and the informational point of view. Then it presents two rankings of opportunity sets based on information about the extent of options and the constraints that a decision maker faces. The first ranking measures freedom as variety of choice; the second as non-restrictedness in choice.

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1 Introduction

The literature on non-welfaristic foundations of normative economics addresses the question of whether there are things other than preferences which are important for an individual making a decision. This problem can be understood in two ways. We could imagine, to start with, things other than preference satisfaction, which make a difference to what an agent chooses (on this subject see, for instance, Sen (1993a), Baigent and Gaertner (1996) and Gaertner and Xu (1996a), Gaertner and Xu (1996b) for some formal characterizations). Alternatively, we could interpret the problem as one which asks whether other things may, while not affecting the individual's choice, guided by preference satisfaction, influence her 'well-being'; in this situation the problem posed is that the individual's preference satisfaction is not the sole determinant of her overall satisfaction.

The so-called 'Freedom of Choice Literature' (FCL) has worked within the boundaries of the second suggestion and we shall do so, too, in this paper. More precisely, the literature has suggested that assessments of states of affairs should take into consideration the extent of opportunities an individual has open for choice. In the literature, information about extent of options has often been interpreted as information about freedom of choice.

Basically, the idea is that, for an individual, to pick up x out of a state of affairs A which contains other possibilities is not the same as to be forced to get it from a state where x is the only available option. This is because, irrespective of which option the agent considers as the best, availability of opportunities reflects a certain degree of freedom for the decision maker which is impaired if the extent of options contracts, even if the most preferred alternative remains available.

In the measurement of availability of opportunities, FCL has mainly followed two approaches. The first (see Klemisch-Alhert (1993), Pattanaik and Xu (1990), Pattanaik and Xu (1995), Suppes (1987)), which reads freedom as 'availability of choices', claims that liberty coincides with having access to options. It can therefore be assessed by looking at how wide is the extent of options that are open for choice to individuals. The second (see Puppe (1996), Sen (1991), Sen (1993b)), which interprets liberty as 'effective freedom', suggests that, although having options is important to an agent's freedom, a proper assessment of the extent of liberty cannot avoid reference to the decision maker's preference structure. This is because an individual is free to the extent that he or she can access *de facto* relevant opportunities.

Within the boundaries of these approaches, discussions of freedom involve two kinds of considerations: they concern whose freedom is at stake (presumably an economic agent) and what the options available to the person are. No reference has been made to the role that information provided by constraints play in measuring an individual's liberty.

This paper's aim is bridging this gap. It introduces constraints into measures of freedom of choice and explores the limitations of the literature's framework in embodying philosophically meaningful notions of constraints. More specifically we show that constraints are relevant in the measurement of liberty and we provide two arguments why it should be so. The

first, philosophical in nature, is that discourses of freedom are usefully addressed within a framework (the syntax of MacCallum) in which the agent's impediments count in the definition and assessment of his or her liberty. The second is that constraints in fact possess an informational content beyond that embodied in both availability of choice and the decision maker's preference structure.

Once the reason why impediments should play a role is made explicit, we introduce an analytical framework based on rough set approximations which captures their informational content within the axiomatic structure of the freedom of choice literature. In so doing we are also able to show the limits of the FCL's framework in dealing with the notion of impediment.

The paper is ideally organized in three parts. The first includes section 2; it introduces constraints in the picture, discusses their nature and explains why they are useful in the assessment of the extent of freedom. The second part presents the analytical framework and is made up of two sections. The first, section 3, introduces the notation, the intuition and the formal content of rough set approximations. The second, section 4 presents some rankings of states of affairs based on rough approximations and responds to some objections that they have undergone. The third ideal part is composed by section 5 which displays final considerations where the role constraints play in the freedom of choice literature is assessed and some ways towards further research are delineated.

2 A role for constraints

The freedom of choice literature does not give autonomous role to constraints in the measurement of liberty in the sense that it regards impediments as the mere complement of opportunities. This is due to the fact that its philosophical underpinning is the Berlinian notion of positive freedom, as Sen (1988) explicitly states. According to Berlin (1969), individual liberty is either negative or positive. The former is concerned with delimiting an area of non-interference, a sphere, so to speak, in which no constraints are imposed to an individual in the exercise of his or her liberty. Positive freedom, on the contrary, has to do with the extent to which a person is "his own master", to use Berlin's words, with the limits within which his decisions depend on himself and not on external forces or circumstances.

Limiting the domain to the construction of measures of positive freedom makes availability of opportunities the most important information to assess in a measure of liberty. This is because, within certain interpretations of positive freedom (Miller 1991), an individual's decisions depend on himself to the extent that he has alternatives to choose from. A measure of positive freedom requires therefore an assessment of the agent's available opportunities. Constraints become irrelevant in this picture since they count only in so far as they delimit the set of accessible options. Impediments can therefore be treated as complements of options: whenever there is an alternative available for choice there is not an impediment to access it and vice versa.

Yet, Berlin's analysis is neither the only possible framework where discourses of freedom can be addressed, nor necessarily the most appropriate. If we can show that other structures support discourses of freedom which require considerations of constraints along with assessments of the extent of options, it is likely that exclusive focus on opportunities becomes a liability. There are in fact two arguments for wider foundations of the freedom of choice literature.

2.1 The philosophical argument

The first argument is philosophical. Addressing discourses of freedom within Berlin's dichotomic framework deprives the debates of its nuances, flattens the internal diversity of liberty and forces the different parties into one of the two camps only. Discourses about freedom should instead be grounded on a more articulated framework in which agents, options and constraints retain an independent role and contribute to shaping conceptions of freedom.

According to the political theorist Gerald MacCallum (1967), philosophical discourses of liberty should be structured within a triadic syntax. In his own words: "[. . .] freedom is [. . .] always of something (an agent or agents), from something, to do, not do, become or not become something; it is a triadic relation. Taking the format "x is (is not) free from y to do (not do, become, not become) z", x ranges over agents, y ranges over "preventing conditions" as constraints, restrictions, interferences and barriers, and z ranges over actions or conditions of character or circumstance" (p. 314).

MacCallum makes two points. The first is that freedom should be conceived of as a relation, not a property: it is not something an individual possesses, but a relation between agents. The second is that the dicothomy between negative and positive freedom no longer holds. This is because, once MacCallum's syntax is adopted, the opposition between 'freedom to' (the positive interpretation) and 'freedom from' (the negative account) makes no further sense as liberty becomes at the same time freedom *from* impediments and *to* do or choose. Instead of dividing authors in two separate camps, his framework provides a "guiding syntax", in the form of a necessary condition for the discussion of freedom and shifts in all contexts, practical as well as philosophical, the debate on the range of the 'term variables' (Kristjánsson 1996).

Suppose we are observing a situation in which a person has been locked in a room and cannot go out. Writers adhering to the view that liberty is a negative concept would certainly define the circumstance as one of unfreedom if the door has been locked on purpose by another person in the pursuit of his own goals. But what if the door has been inadvertently locked by a careless porter who negligently does not check that all people have left the building? Would, in this latter case, adherents to negative liberty be prohibited from saying that this is a case of unfreedom? Similarly for positive freedom supporters. Suppose a person, having money and passport, is free to travel abroad. If, at some point, she finds herself without enough money for traveling, would scholars in the 'positive camp' be in a contradiction if they claim that

she remains free? Confining philosophers in these two camps appears exceedingly restrictive. After all, one could find herself on the 'positive side' without being forced to buy all the consequences that such an adherence bears with. Or, more precisely, adhering to one's own understanding of liberty is the same as qualifying or putting limits to the consequences one is prepared to subscribe, which in turn requires discussing what is an available option or a limiting impediment.

Instead of separating writers into two different camps, MacCallum suggests reinterpreting philosophical discourses about liberty in terms of the three elements of his syntax. Finer distinctions then emerge between the different positions which reflect the ways in which the 'term variables' have been delineated.

2.2 The informational argument

The second reason why constraints may be relevant in a measure of freedom is that they introduce new information in the assessment. To illustrate, consider an example. Suppose an agent initially has the choice of a blue car. We want to know whether her freedom is enhanced if she is given a blue car and a red car or if, instead, she is given the option of a blue car or a travel pass. Intuitions lead some to conjecture that, in the first case, the agent's freedom is not enhanced by the further option while, in the second, it is. In general, it looks as if the point of this example is to drive home the message that adding to an agent's choice set options which are, in some sense, similar to some element in the set does not increase her freedom. In this framework the absence of some elements should be considered as constraining freedom of choice, while that of others should not. Since the agent will not probably consider a blue car and a travel pass as similar, adding the latter to her opportunity set increases her freedom. By the same token, if an individual sees a red and a blue car as the same kind of object, adding the former to the initial set of options does not alter the extent of freedom she enjoys.

Information about the qualitative characteristics of the options, i.e., if they are similar or not, may be interpreted in terms of constraints. The red car is 'prevented' by its similarity to the (already available) blue one from being an element which increases the decision maker's freedom of choice. Similarity constraints then deny an individual the opportunity to act in a *de facto* different way, given her available possibilities.

The role of this further piece of information can be appreciated in a more general context by defining a mapping $S : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$ (the power set of a universe X minus the empty set) which maps an opportunity set A to the set of its similar alternatives $S(A)$. In general, such a mapping brings to the measurement of freedom information not recoverable from the analysis of the opportunity set or the preference structure. This is because letting the decision maker enjoy access to the options in $S(A)$ does not modify the extent of his effective freedom since they do not provide him with new, distinguishable alternatives to choose from. This is certainly not the case if similarity plays no role. In a ranking like the simple cardinality-based ordering (Pattanaik and Xu 1990) where distinctions among elements have no informational

role, it is not possible to separate options on the basis of their contribution to the agent's freedom.

The greater informational richness generated by the map S may also be captured by the definition of another mapping $S^* : \mathcal{X} \to \mathcal{X}$, where $S^*(A) := \{x : x \in X - A \text{ and } x \in S(A \cup \{x\})\}$. This is a mapping from A to the set of elements in $X - A$ which, if added to A, would not strictly increase the freedom offered by A. So S basically induces two partitions: A is partitioned into $S(A)$, the subset of elements of A which, if removed from A, would not decrease the freedom offered by A, and $A - S(A)$ the subset of elements of A which would; $X - A$ is partitioned into $S^*(A)$, the set of elements of $X - A$ which, if added to A would not increase the freedom offered by A, and $X - A - S^*(A)$, the set of elements of $X - A$ which would. Once again, this information is not available from the mere observation of the available options, if considerations of similarity are not embodied in the analysis.

We suggest that the definition of a non-trivial mapping S can be interpreted as the specification of a restriction of the relevant constraints on freedom, non-recoverable from the agent's opportunity set. The exception is given by the limit cases of S, specifying that all constraints or no constraints are relevant. This leads to a ranking of freedom collapsing respectively to Pattanaik and Xu's simple cardinality-based ordering or to the indirect utility ranking based on comparisons of preferred elements of opportunity sets. Given the many arguments made against both these rankings, it does not seem farfetched to argue that a widening of the informational basis is needed by way of, for instance, a mapping like S. The important part is the appreciation that, in moving away from the indirect utility ranking and the simple cardinality ordering, more information has to be introduced in the scheme, and that such information tries to discriminate which constraints are relevant and which are not in the measurement of freedom.

2.3 Similarity constraints and two measures of freedom

The introduction of this idea of a constraint is particularly helpful in the construction of freedom rankings. Return to the car example. If we have a blue car, to be constrained from having a red car is not as restrictive to one's freedom as to be constrained from being able to purchase an airline ticket or a travel card. We might argue that the inability to choose something which an agent thinks shares many characteristics with something she already has access to (i.e., blue car) is not a constraint upon her freedom. On the other hand, her inability to choose something which she sees as different with respect to anything she already has does constitute such a constraint. This kind of intuition points to qualitative variety of choice as an interpretation of freedom. In this case we might consider as the set of options, the absence of which constrains freedom, the subset of $X \setminus H = H^c$ containing elements towards which the agent is not indifferent with respect to some element in H. Call this set $H^{\#}$. In general we will expect that $H \cup H^{\#} \neq X$. Here we can use $H^{\#}$ to extract more information than we had by considering just H^c: it is impossible to identify $H^{\#}$ if our knowledge is restricted to

the agent's preference profile $\mathcal{P}$ and her opportunity set H. Further on we will characterize a ranking $\succeq_O$ on the basis of these ideas.

We could, of course, have a very different idea of when an agent is meaningfully constrained in her choices. Choice is not (just) about being able to choose between a bottle of wine and a banana. It is rather about one's choice of what bottle of wine to buy being totally unrestricted and autonomous. We may group together elements of a universe that share similar characteristics, that are perhaps substitutes in an economic sense, or indeed following any criterion that places together elements that an agent would want to choose from in specific situations in order to feel that her choice is in some sense complete, totally autonomous. Then, in a given choice set H, the relevant elements for an agent's freedom are those that make up fully such groupings. These elements will generally form a proper subset of the agent's choice set and could be used to yield a measure of the extent that an agent's choice is autonomous. The complement of such subset, call it $H^\flat$, will be such that, in general, $H^c \subset H^\flat$. We will characterize a ranking $\succeq_I$ along such lines, too.

3 Power Sets and Rough Approximations

Rough approximations are pairs of sets which 'encircle' any given subset of a universe from the 'inside' and from the 'outside'. They were introduced as such by Pawlak (1982) in order to describe approximate knowledge of sets and their application has been extremely diverse (for some recent work on the abstract structures which arise, see Cattaneo (1996)).

The intuitive idea is to start with a set and to split it up into mutually exclusive 'boxes'. Such boxes form a partition (and therefore induce an equivalence relation) on our set. Then, for any subset of our set, we can find two subsets which 'approximate' our set from the inside and from the outside. These approximating subsets are composed of boxes of our partition: the inner approximation is the union of all the boxes fully contained in our set, while the outer approximation is the union of all the boxes which have at least one element in common with our set.

The formal analysis presented in the latter part of this paper is independent of the criterion we choose to construct our boxes, but the interpretation and the application of such an analysis will not be. One general idea which can motivate this construction is that an agent will be able to express an opinion as to when two options from which the choice is being made are on a par from the point of view of freedom.

An obvious possibility is to follow the suggestion of Jones and Sugden, when they say that "one kind of choice that seems *not* to be very significant is one in which there is no reason for choosing one option rather than another (that is, the case of indistinguishable options)" (Jones and Sugden 1982, p. 59). For instance, a choice between two cars produced on a production line a few minutes apart is not a significant choice: we might assume that these two cars are on a par in terms of the contribution they make to the agent's freedom. We

might put them in the same box.

It is quite conceivable, and in general more interesting and useful, that an agent might feel two things to be on a par in terms of the contribution they make to the agent's freedom, while still wanting to express a preference between them. An option along these lines is to construct the boxes in terms of similar characteristics of the elements, one which we will favor in examples to be given in the next section. Many other options are of course available. In general it would appear that again the justification of a partition imposed on a set of options will be sensitive to the kind of evaluative judgement one is trying to make. Put in other words, the nature of the assessment will determine what kind of information is necessary in order to make the judgement.

Formally, we denote by X a finite universe of alternatives. Capital letters $A, B, \ldots$ denote subsets of X, i.e., elements of the power set of X, which is denoted by $\mathcal{P}(\mathcal{X})$, except that we use I, J to denote index sets like $\{1, 2, \ldots, n\}$. $\mathcal{X}$ denotes $\mathcal{P}(\mathcal{X}) \setminus \{\emptyset\}$, the set of non-empty subsets of X. Binary relations are represented either as sets of pairs $(x, y) \in X \times X$ or, in the case of preference orderings and extensions on the power set, as $\geq$ and $\succeq$ respectively. It should be clear from the context whether $\geq$ is the order relation on numbers or whether it represents preference orderings.

Now, suppose we have a set X. It is possible to define a partition $\mathcal{E}(X) := \{E_i : i \in I\}$ (where $E_i \in \Pi(X)$) on X, with I an index set, for instance $I := \{1, 2, \ldots, n\}$. The $E_i \in \Pi(X)$ are called 'elementary sets'. The partition $\mathcal{E}(X)$ can also be characterized by the equivalence relation $\mathcal{R} \subseteq X \times X$ defined as

$$(x, y) \in \mathcal{R} \iff \exists E \in \mathcal{E}(X) : x, y \in E.$$

In other words, let us start with a partition of the universe X into elementary sets based on whatever we feel is a convenient criterion to split our space. Then it is possible to construct the equivalence relation on the basis of the following proposition: '$\langle$ equivalent $\rangle$ if and only if $\langle$ member of the same elementary set $\rangle$' and proceed from here.

For any subset $H \in \mathcal{P}(X)$ we can construct the rough approximation of H

$$r(H) := (\mathbb{I}\,(H), \mathbb{C}(H)),$$

composed of two *definable sets*. These are the subsets of X obtained as the set-theoretic union of elementary subsets, i.e., $A = \cup\{E_j : j \in J\}$ where $J \subseteq I$. The collection of all such definable sets including the empty set is denoted by $\mathbb{C}\ \mathbb{O}\,(T)$.

$\mathbb{I}\,(H)$ and $\mathbb{C}\,(H)$ are two particular definable sets with respect to H: $\mathbb{I}\,(H)$ is the *inner* (or *lower*) *approximation* for H

$$\mathbb{I}\,(H) := \cup\{A \in \mathbb{C}\ \mathbb{O}\,(T) : A \subseteq H\};$$

$\mathbb{C}\,(H)$ is the *outer* (or *upper*) *approximation* for H

$$\mathbb{C}\,(H) := \cap\{A \in \mathbb{C}\ \mathbb{O}\,(T) : H \subseteq A\} = \cup\{E_h \in \ \mathcal{E}(X) : H \cap E_h \neq \emptyset\}.$$

Example 3.1 Consider a universe $X = \{a, b, c, d, e\}$ and a partition with $E_1 = \{a, b, c\}$ and $E_2 = \{d, e\}$. Now the set $A = \{a, d\}$ has as its inner approximation the union of all the definable sets contained in A. This clearly is $\emptyset$, the empty set. Its outer approximation is the union of all the definable sets which have at least an element in common with A, in this case X. If we consider the set $B = \{d, e\}$, this is equal to E_2, so in this case $\mathbb{I}(B) = \mathbb{C}(B) = E_2$.

Example 3.2 Suppose $X = \{a, b, c, d, e, f, k, l\}$. We can define three partitions: with $E_1 = \{a, b, c\}$ which includes necessary goods like food, clothes and water; $E_2 = \{d, e, f\}$ which encompasses luxury goods like yachts, air conditioning and jewels; $E_3 = \{k, l\}$ which includes, say, motorbikes and cars. If $A = \{a, b, c, e\}$, then $\mathbb{I}(A) = \{a, b, c\}$ and $\mathbb{C}(A) = \{a, b, c, d, e, f\}$. In this case, the inner approximation yields something like a 'subsistence' set: it tells us that the only element of our partition over which our agent has complete choice is such set; on the other hand the outer approximation also contains our 'luxury goods' partition: it tells us that our agent has access to some, although not all, luxury goods, too. Considering, instead, $B = \{a, b, c, e, l\}$, the outer approximation is just X: our agent has access to some goods out of all our partitions.

The structure based on the power set of the approximation space T is naturally equipped with two set-theoretical complementations,

$$^c : \Pi(X) \to \Pi(X), H \mapsto H^c = X - H$$

and

$$^\# : \Pi(X) \to \Pi(X), H \mapsto H^\# = X - \mathbb{C}(H),$$

linked by the rule for all $H \in X$,

$$H^\# \subseteq H^c.$$

From these it is possible to define a third complementation

$$^\flat : \Pi(X) \to \Pi(X), H \mapsto H^\flat := H^{c\#c} = X - \mathbb{I}(H).$$

Then, from the ordering point of view, we have the following chain of inclusions:

$$H^\# \subseteq H^c \subseteq H^\flat.$$

In our basic scheme, $H^\#$ is just the complement of the *outer* approximation of a given set H, while $H^\flat$ is the complement of the *inner* approximation of H. The similarity in notation with the discussion of the previous section is clearly motivated. If, for instance, we introduce a variety based notion of constraints, the complement of the outer approximation might represent elements which constrain the freedom of an individual both by their absence from H and by their failure to be similar to some element in H, as they do not belong to any equivalence class that intersects H.

4 Some examples of rankings

This section characterizes two examples of rankings that, while conserving something of the framework that Pattanaik and Xu (1990) introduce, circumvent the problems that are raised about some of the axioms they propose. The starting point here is a set of states X which need not be necessarily finite. We will assume that the number of elements of the partition is finite: this allows for some simplification of the axioms necessary to characterize the rankings.

4.1 The outer approximation ranking

The first ranking is defined in such a way that, for all A and $B \in \mathcal{X}$,

$$A \succeq B \iff |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| \geq |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}|, \tag{1}$$

where sets such as $\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}$ are subsets of $\mathcal{P}(\mathcal{X})$. This just counts the number of boxes in the outer approximation of a set A. We adopt the convention that the measure which counts the boxes in the outer approximation of a set A is written as $||A||$.

Pattanaik and Xu (1990) discussion of the red car/blue car example suggests that not being able to access an option similar to some option already available to us is not a restriction on our freedom. At this point we can make one further interpretive step and claim that, once we have gained access to a dissimilar element, this will increase our freedom, but after this no further element similar to the new addition will do so. This implies that the constraints that restrict freedom are those that prevent us from accessing elements in $A^{\#}$, the complement of the outer approximation of a given choice set A. This leads to the following axiom:

$$\text{(I) } \forall A \in \mathcal{X}, \forall x \in \mathbb{C}(A) \iff A \cup \{x\} \sim A.$$

This, together with other axioms similar to those in Pattanaik and Xu (1995), will characterize the ranking rule. These axioms are

$$\begin{aligned}
&\text{(IM) } \forall A, B \in \mathcal{X} \text{ such that } A \succeq B \text{ and } \forall x \in X \text{ such that } x \in A^{\#}, \\
&\qquad A \cup \{x\} \succ B; \\
&\text{(ICOM) } \forall A, B \in \mathcal{X} \text{ and } \forall x \in A^{\#}, y \in B^{\#}, \\
&\qquad A \succeq B \iff A \cup \{x\} \succeq B \cup \{y\} \\
&\qquad \text{and } A \succ B \iff A \cup \{x\} \succ B \cup \{y\}.
\end{aligned}$$

These axioms capture in a natural way the ranking given by (1). Before proving this we give the following

Definition 4.1 $\hat{A}$ denotes any one of the maximal subsets of a set $A \in \mathcal{X}$ with the property that, $\forall x, y \in \hat{A}$, x, $y \notin E$ for any elementary set E.

The set $\hat{A}$ will be called the *set of representatives from* A. In general such subsets will not be unique, but for our purpose all we need is to be able to select one of them, and such a set we denote by $\hat{A}, \hat{B}$ and so on. We can now prove

Proposition 4.1 *Let* $\succeq$ *be a complete, reflexive and transitive relation.* $\succeq$ *satisfies* (I), (IM), (ICOM) *if and only if*

$$A \succeq B \iff |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| \geq |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}|.$$

Proof. The proof is very similar in structure to that of Proposition 5.1 of Pattanaik and Xu (1995).

Necessity is obvious; we prove only sufficiency. Suppose $\succeq$ satisfies the axioms. First we show that, $\forall A, B \in \mathcal{X}$,

$$\text{if } |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| = |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}| \text{ then } A \sim B. \tag{2}$$

By repeated applications of OI it is clear that $\hat{A} \sim A$ and $\hat{B} \sim B$, so we need to prove that $\hat{A} \sim \hat{B}$. Suppose that $|\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| = |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}| = n$. $||\hat{A}||$ and $||\hat{B}||$ will also be equal to n. If $||\hat{A}||$ were greater than n, then there would be more equivalence classes in $\hat{A}$ than in A. Similarly for $||\hat{B}||$ and B. But this is not possible as $||\hat{A}||$ and $||\hat{B}||$ are subsets of A and B, respectively. If their cardinality were less then $\hat{A}$ and $\hat{B}$ would not be maximal, i.e., we would be able to choose an element in an equivalence class in A or B which would satisfy the condition in the definition of $\hat{A}$ without being in $\hat{A}$.

Now $\forall x,\ y \in X$, $\{x\} \sim \{y\}$; suppose this does not hold, i.e., there are x, y such that $\{x\} \succ \{y\}$. Then it is easy to see that, given $||X|| < \infty$, repeated applications of OCOM lead to $\hat{X} \succ \hat{X}'$ (where $\hat{X}'$ is some other set of representatives from X), hence by OI $X \succ X$, which is a contradiction.

So consider $\hat{A} = \{a_1, a_2, \ldots, a_n\}$ and $\hat{B} = \{b_1, b_2, \ldots, b_n\}$; $\{a_1\} \sim \{b_1\}$; by OCOM, $\{a_1, a_2\} \sim \{b_1, b_2\}$. Repeated applications of OCOM clearly give us $\hat{A} \sim \hat{B}$, which proves (2).

We now show that $\forall A, B \in \mathcal{X}$,

$$\text{if } |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| > |\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}| \text{ then } A \succ B \tag{3}$$

Suppose that $|\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(B)\}| = n$ and $|\{E \in \mathcal{E}(X) : E \subseteq \mathbb{C}(A)\}| = n + m$, then it follows, for similar reasons to the previous instance, that $||\hat{A}|| = n$, $||\hat{B}|| = n + m$ and that $\hat{A} \sim A$, $\hat{B} \sim B$. We now have to show that $\hat{A} \succ \hat{B}$. Suppose $\hat{B} = \{b_1, b_2, \ldots, b_n\}$ and $\hat{A} = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{n+m}\}$. By (2) $\hat{B} \sim \{a_1, a_2, \ldots, a_n\}$.

Now by OM $\{a_1, a_2, \ldots, a_n, a_{n+1}\} \succ \hat{B}$. Repeated application of OM will give that $\hat{A} \succ \hat{B}$, which proves (3) and completes the proof.

Q.E.D.

The above result characterizes a ranking based on a notion of freedom as variety of choice by measuring the number of elementary sets an individual has some sort of access to, so that greater variety implies wider freedom. In practice, the extent to which different degrees of freedom are recorded will be extremely sensitive to the way we split the universe up. Suppose we put most of the elements that an individual has access to into one class, then the freedom enjoyed will be almost always the same. The simple cardinality ordering would arise in the opposite case, where each single element counts as different and therefore belongs to separate singleton elementary sets. In between lie the cases where this framework will more effectively yield information.

An example will help to make the point. There are discount supermarkets which offer products at an extremely cheap price. They achieve this by tying themselves to a single producer and stocking and selling only one brand of every product (bread, milk, cheese and so on). In deciding whether to shop in such stores or in more traditional ones, many people claim that they prefer the traditional ones as they offer more freedom in what one can buy. This claim is not concerned with whether there are 100 or 500 loaves of bread on a shelf, but whether there are one or more makes of bread on the shelf. This usage of the term 'freedom' is modeled by constructing boxes which group particular products of particular makes in each separate box and then checking how many boxes are represented in different stores. This provides a good example of how the outer approximation ranking just characterized may be used. It is worth stressing that the outer approximation ranking also makes sense of the freedom judgement implied in Pattanaik and Xu's examples of red cars, blue cars and trains: defining elementary sets by elements' shared characteristics, cars belong to the same set and thus more cars do not offer more freedom, while cars and trains belong to different sets, so adding the choice of taking the train to having a car does offer more freedom.

It is also worth repeating how this ranking incorporates a particular notion of constraints on freedom: given an agent, and given an opportunity set, depriving an agent of the choice of an element similar to some other element in the opportunity set does not constitute a restriction on the agent's freedom, while the converse is not true.

4.2 The inner approximation ranking

We now prove a similar result for a ranking based on the inner approximation of sets $\mathbb{I}$. We will need the following axioms:

$$\begin{aligned}
\text{(I')} \ & \forall A \in \mathcal{X}, \forall x \in A \text{ and } x \in A^{\flat} \implies A \setminus \{x\} \sim A \\
\text{(NM)} \ & \forall A, B \in \mathcal{X} \text{ such that } A \succeq B \text{ and } \forall x \in X \text{ such that } x \in \mathbb{I}(B), \\
& A \succ B \setminus \{x\} \\
\text{(DECOM)} \ & \forall A, B \in \mathcal{X} \text{ and } \forall x \in \mathbb{I}(A), y \in \mathbb{I}(B) \\
& A \succeq B \iff A \setminus \{x\} \succeq B \setminus \{y\} \\
\text{and } & A \succ B \iff A \setminus \{x\} \succ B \setminus \{y\}
\end{aligned}$$

Proposition 4.2 *Let $\succeq$ be a complete, reflexive and transitive relation. $\succeq$ satisfies* (I'), (NM), (DECOM) *if and only if*

$$A \succeq B \iff |\{E : E \subseteq \mathbb{I}(A)\}| \geq |\{E : E \subseteq \mathbb{I}(B)\}|.$$

Proof. Necessity again can be checked straightforwardly. We prove sufficiency only. Suppose $\succeq$ satisfies the three axioms. Again we begin by showing that, $\forall A, B \in \mathcal{X}$,

$$|\{E : E \subseteq \mathbb{I}(A)\}| = |\{E : E \subseteq \mathbb{I}(B)\}| \implies A \sim B \tag{4}$$

Repeated applications of II show that $\mathbb{I}(A) \sim A$ and $\mathbb{I}(B) \sim B$, so it remains to prove that $\mathbb{I}(A) \sim \mathbb{I}(B)$. Suppose that $|\{E : E \subseteq \mathbb{I}(A)\}| = |\{E : E \subseteq \mathbb{I}(B)\}| = n$. By definition, $||\mathbb{I}(A)|| = ||\mathbb{I}(B)|| = n$, too. Under the assumption that the number of equivalence classes in $\mathcal{E}(\mathcal{X})$ is finite, suppose that $|\{E : E \subset X\}| = n + m$. Then we can construct a chain $A_0 = \mathbb{I}(A), A_1, \ldots, A_m = X$ of definable sets obtained by adding an equivalence class at each element of the chain. Similarly we can construct a chain $B_0 = \mathbb{I}(B), B_1, \ldots, B_m = X$.

Now consider two elements $x \in A_m - A_{m-1}$, $y \in B_m - B_{m-1}$. Application of DECOM to $X \sim X$ yields $X - \{x\} \sim X - \{y\}$. Now repeated application of II to $X - \{x\}$ yields that $X - \{x\} \sim A_{m-1}$ and similarly $X - \{y\} \sim B_{m-1}$, hence $A_{m-1} \sim B_{m-1}$. Repeating this procedure we prove that $\mathbb{I}(A) = A_0 \sim B_0 = \mathbb{I}(B)$, which proves (4).

Next we show that, $\forall A, B \in \mathcal{X}$,

$$|\{E : E \subseteq \mathbb{I}(A)\}| > |\{E : E \subseteq \mathbb{I}(B)\}| \implies A \succ B \tag{5}$$

Suppose that $|\{E : E \subseteq \mathbb{I}(B)\}| = n$ and $|\{E : E \subseteq \mathbb{I}(A)\}| = n + l$, $l \leq m$. As before, we need to prove that $\mathbb{I}(A) \succ \mathbb{I}(B)$ and we begin by constructing two chains $A_0 = \mathbb{I}(A), A_1, \ldots, A_m = X$ and $B_0 = \mathbb{I}(B), B_1, \ldots, B_l, \ldots, B_m = X$. Repeating the previous procedure of applying II and DECOM proves that $A_0 \sim B_l$. Now choose an $x \in B_l - B_{l-1}$. By NM we have that $A_0 \succ B_l - \{x\}$. Repeated applications of II imply that $A_0 \succ B_{l-1}$. Repeating this procedure down the chain shows that $A_0 \succ B_0$, hence $\mathbb{I}(A) \succ \mathbb{I}(B)$, establishing (5) and therefore completing the proof.

Q.E.D.

This characterization identifies a ranking of opportunity sets in terms of a notion of freedom understood as completeness of choice: an agent here is more free when her choice is made over a larger number of 'complete' sets of elements. In a sentence, the inner approximation ranking rests on the idea that 'take it or leave it' is not a condition of freedom and it tries to measure the extent to which an individual is not compelled by such a condition.

The relevant constraints on freedom here are those that deny us choice from a complete set. To be deprived of individual elements in cases like this is not usually as important as to be deprived of access to the appropriate sets as the following example makes clear.

An example may provide a better grip on the underlying idea of the inner approximation ranking. Suppose we have two kinds of medical insurance policies. Both offer access to medical care for all known possible diseases and accidents; one, however, restricts access only to certain doctors, while the other covers expenses for whoever specialist the insured might want to consult. If asked to evaluate which policy offers the most freedom to the agent who has to choose between them, we would choose the second one because it is unrestricted in the access it gives to medical expertise in each of the fields.

In the example at hand the equivalence classes are made up of doctors specialized in different fields. Dentists, brain surgeons or orthopaedists, each belong to different equivalence classes. In practice, what may well happen is that agents who subscribe to the first insurance policy and are in need of a dentist will be constrained to see Dr. Teeth, the only dentist whose service is covered by the policy. On the contrary, people who have subscribed to the second policy will be able to choose between Dr. Teeth and Dr. Tooth. Their freedom of choice will terefore be less restricted. The latter insurance policy then freedom-wise dominates the former. The example portrays faithfully the kind of idea that is captured by the inner approximation ranking: an agent has more freedom to the extent that he or she has access to the largest possible number of equivalence classes fully contained in his or her opportunity set.

4.3 Counterarguments and counterexamples

The measures constructed in this paper have undergone a number of observations. We start with the following remark. The rankings are characterized without assuming the axiom (INS) of Pattanaik and Xu (which can be deduced from the others) and, in the case of the second ranking without explicitly using it. (INS) states that $\forall x,\ y \in X,\ \{x\} \sim \{y\}$. The fact that we can do without it is due to the assumption that the number of equivalence classes is finite. One could generalize the result to ranking subsets containing finitely many equivalence classes where the overall number is denumerable, but then (INS) would probably be needed explicitly. Note that Sen strongly criticizes (INS), with intuitive examples suggesting that if an agent strongly prefers x to y then she will feel more free choosing from $\{x\}$ than from $\{y\}$.

This is then generalized to an argument for the necessity of taking preferences into account in ranking opportunity sets in terms of the freedom they offer. In general, however, one need not reject (INS) simply on the basis that preferences are important and must be taken into account in measuring freedom. The rankings discussed in this paper can be constructed on the basis of preferences defining the equivalence classes, and (INS) is still satisfied. The point is therefore that, whatever the plausibility of such a ranking, an assumption such as (INS) is not in principle incompatible with incorporating information about preferences, a remark which has also been well argued and developed by Pattanaik and Xu (1995).

An objection which has been moved to this paper's framework is that all that is needed to characterize freedom in the outer approximation ranking is an appropriate redefinition of the unit of measure of an opportunity. If we consider the basic opportunity for choice to be not a single loaf of bread, but a particular make, then one can characterize the above example by means of a simple cardinality ordering and our analysis becomes superfluous. But this is to mistake the nature of the choice that the agent is facing in these situations. The agent is choosing a single loaf of bread and it is in the context of this choice that the above example is set. In the example this conclusion might be suggested by the fact that the agents's choice basically involves just the choice of make: she is arguably indifferent to which particular loaf she will choose. But it is not hard to imagine situations where the choice is not reducible in this way. For instance, do we buy bread in a bakery that sells just bread or one that sells bread and lots of other products like cakes, biscuits and so on? In a situation like this, judgement of the freedom of choice offered is amongst kinds of goods sold, while there still remains a genuine choice to be made between different types and sizes of goods. In this case such a redefinition of the unity of opportunity would give the wrong characterization of the agent's situation.

A more interesting technical counterexample to the outer approximation ranking is the following.

Example 4.1 Consider the comparison between the sets

$$A := \{\text{blue Ford limousine, blue Ford station wagon}\}$$

and

$$B := \{\text{blue Ford limousine, green GM station wagon}\}.$$

Intuition suggests that if we want to measure freedom as variety of choice, B should dominate A since it offers a choice of colour, a choice of make and a choice of type (i.e., limousine vs. station wagon), whereas A does not offer a choice either of make or colour. The only partition which would yield the ranking intuition suggests is

$$\mathcal{E}_1(X) := \{\{\text{blue Ford lim, blue Ford sw}\}, \{\text{green GM sw}\}\}.$$

Note that the outer approximation measure based on the partition $\mathcal{E}_1$ would also yield

$$C := \{\text{green GM sw}\} \sim D := \{\text{blue Ford lim, blue Ford sw}\},$$

clearly a counterintuitive result since the set on the right hand side offers a choice of type which is not available on the left hand side.

The oddity of the result is certainly dependent on the chosen partition. As a matter of fact, if we had partitioned the universe according to

$$\mathcal{E}_2(X) := \{\{\text{blue Ford lim}\}, \{\text{green GM sw, blue Ford sw}\}\},$$

the outer approximation measure would have respected intuition leading to the conlcusion that $D \succ C$. Yet, it would have also yielded $A \sim B$, in contradiction with what intuition has suggested initially.

The example is trying to convey the idea that similarity is a complicated concept and not necessarily captured by one and only one partition.

The example is 'technically' correct, but its point is misleading, for the judgement involved is not between one similarity characteristic, but three. In other words, there are three different dimensions of the options which may serve in a judgement about their similarity: colour (green, blue), make (Ford, GM) and type (limousine, station wagon). The example is playing with the fact that, in the partition, options are gathered together according to either make or colour while intuition suggests that $D \succ C$ because of a similarity judgement which involves types.

Yet, the example also suggests that in a measure of freedom as variety of choice we may not want to separate the dimensions over which variety is assessed. In other words, we may wish that variety is evaluated looking at all dimensions in which it appears, simultaneously, rather than one at a time. As a consequence, while we believe that the oddity highlighted in the example is due to its playing with non-homogeneous dimensions of variety at the same time, we are also aware of the fact that a way for aggregating them should be found. Consider the following. When $\mathcal{E}_1$ is the selected partition, C and D are freedom-wise indifferent. Instead, if $\mathcal{E}_2$ is the prevalent partition, $D \succ C$. Hence, in general, it is reasonable to define $A \succ B$ if $A \succeq B$ for all partitions and there exists at least one partition for which $A \succ B$. This clearly gives the right result for C and D in the example at hand.

4.4 Which is the 'correct' ranking?

It goes without saying that in many cases one would not want to make an evaluative judgement on the basis of the ranking rule based on inner approximations, just as in many cases one would not want to make such judgements on the basis of the ranking which relies on outer approximations. Pattanaik and Xu (1995) want to say, when comparing two sets {blue car, red car} and {blue car, train}, that the latter offers more freedom than the former. But, taking a leaf out of Sen's book, suppose an agent is a car lover and hates trains; furthermore the agent likes red cars, which are to the agent but one step from a Ferrari. If, with Sen, we want to say that in some sense impediments on the agent satisfying their preferences are impediments

on their freedom, then we would not judge the agent to be more free in choosing from the second set.

The general claim is that there are many different kinds of ranking rules one can justify, depending on what the relevant information is for making an evaluative judgement. In particular, the answer to the question 'Which is the correct ranking?' must be that it depends. It follows that to point to the fact that certain ranking rules do not capture what we regard as intuitively correct in a certain situation is irrelevant to rejecting these rules: we also need an argument about why in such a case it should be inappropriate to rank opportunity sets, in terms of the freedom they offer, according to such a ranking. We have provided in this section examples of cases where the rankings we characterize seem appropriate: both capture plausible judgments that we would make under certain circumstances when asked to produce an evaluation of the freedom of an agent. For both rankings it is also possible to generate instances where the rankings they yield are counterintuitive, or just plainly wrong. This does not restrict the value of considering the two examples together, which is that if you feed in different notions of relevant constraints for freedom, you get out different ranking rules, which may be reasonably applied in different circumstances.

In a search for the correct ranking, the 'flexibility' of the rankings $\succeq_O$ and $\succeq_I$ might provide some further insight. For example, it could be of interest to have a 'weak' and a 'strong' ranking of freedom. Given a partition of the universe X, suppose we have that $A \succeq_O B$. This would give us an idea that the individual has more 'variety' in her choices in A than in B. But then we can consider whether $A \succeq_I B$ or not. If it does then the information that the two rankings give is in the same direction and we can say that A gives more freedom than B in a strong sense. If $B \succeq_I A$ then the information is conflicting. A offers more variety than B, but B offers in some sense a more complete choice. If we assume, say, that variety is more important than how 'complete' our choice is, then we may say that A gives more freedom than B in a weaker sense. Moreover, if the inner approximation of A includes (or, more properly, has more elementary sets than) the outer approximation of B, then adding elements to B without changing its outer approximation (no more variety) will never weaken the freedom advantage that A offers. In order for B to offer more freedom to the agent, genuine variety has to be added to it.

5 Constraints and FCL's framework

In this paper we have suggested that if an individual's opportunity set is enlarged by the introduction of a further option similar to another she already has, this does not increase her freedom of choice. The idea is that a similarity constraint operates in these cases which prevents the new option from counting in a measure of liberty.

One may object that stretching the notion of impediments to encompass similarity constraints is like pouring water in a glass of Chianti: it increases the liquid in the glass but

dilutes its nature and spoils its taste. The reason is simple: constraints are impediments to an individual's acting or choosing; similarity constraints cannot be considered as impediments since they do not prevent the agent from choosing or acting in a specific way since she can always access the already available similar option.

Although this objection is well grounded, it does not apply when we discuss about 'effective freedom', i.e., the freedom to access those options that the decision maker considers as *de facto* relevant. If we take this conception of freedom into consideration, the obstacles that an individual faces to access options have a two-faceted nature. They can be constraints imposed to her by other persons or by natural impediments; but they may also be informational constraints. A person may be unable to enlarge her own effective liberty if she is given options which are similar to alternatives already available. In this sense, both natural and man-made obstacles, on the one side, and similarity constraints, on the other, though different in nature, retain an important feature in common: they prevent options from counting in the measurement of liberty. In the first case they do that because they keep the option outside the agent's opportunity set. If some obstacle (natural or man-made) is preventing a person from doing something, 'doing something' is an alternative which does not belong to her opportunity set. In the second case, despite the fact that the option may now be selected by the decision maker, information about similarity makes it irrelevant to the assessment of her freedom.

While, on the one hand, this kind of considerations gives a role to informational constraints in the construction of freedom-based rankings of states of affairs, on the other it marks the limit within which an FCL-like environment for the measurement of freedom is able to encompass meaningful notions of impediments. To the extent that informational constraints do not share in full the nature of physical and man-made obstacles, the fact that FCL's framework hardly embodies fuller interpretation of impediments than similarity constraints shows with greater detail the limits of the literature and the way towards improvements. We believe that similar intuitions and motivations lay behind the recent interest shown by social choice theory towards characterizations of freedom in terms of a game-theoretic analysis, a position defended in detail, among the others, by Gaertner, Pattanaik and Suzumura (1992), and by Pattanaik and Suzumura (1996).

6 Conclusions

This paper defends two ideas

1. there are good reasons why constraints should play a role in the measurement of the extent of freedom. These reasons are due to the fact that liberty is better characterized by MacCallum's triadic syntax and that constraints provide information for the assessment which cannot be retrieved by the analysis of the available options only.

2. similarity constraints may be identified and used in the measurement of liberty, yielding rankings that provide further views and measures of an agent's freedom and shed light on the limits of FCL as a framework in which to address considerations of liberty.

The paper also applies these ideas to some examples from the freedom of choice literature in welfare economics; in so doing it tries to propose a solution to some debates in this literature.

The introduction of partitions in the formal analysis of the example from the welfare economics literature was motivated by a desire to enrich the notions of freedom that underlie earlier rankings by considering constraints, and we have done so by using the partitions to specify novel notions of relevant elements in the opportunity set. The two topics have quite a separate history and both merit further discussion and development in their own right. However, the interplay of the two has been useful in addressing some of the issues that have been raised in the freedom of choice literature, particularly the question of how to weaken in a satisfactory way independence conditions.

We have characterized two rankings which we believe can be usefully applied to analyze two notions of freedom in different choice situations. We make no claims for the superiority of one over the other, or indeed of either over all other characterizations of freedom present in the literature. It seems to us that both rankings capture ways in which agents employ considerations of freedom in choice situations.

We do believe that many ideas of what constitutes free choice are present in the economic domain. There might be grounds for claiming that there ought to be such different ideas of freedom, none taking precedence over the other, but each applicable in different domains of economic analysis. Whether such a normative attitude is tenable in the context of freedom remains a complex, open question.

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