(U,V)-ordering and a duality theorem for risk aversion and Lorenz-type orderings

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Abstract

There is a duality theory connecting certain stochastic orderings between cumulative distribution functions F_1, F_2 and stochastic orderings between their inverses F_1^{-1}, F_2^{-1}. This underlies some theories of utility in the case of the cdf and deprivation indices in the case of the inverse. Under certain conditions there is an equivalence between the two theories. An example is the equivalence between second order stochastic dominance and the Lorenz ordering. This duality is generalised to include the case where there is "distortion" of the cdf of the form $v(F)$ and also of the inverse. A comprehensive duality theorem is presented in a form which includes the distortions and links the duality to the parallel theories of risk and deprivation indices. It is shown that some well-known examples are special cases of the results, including some from the Yaari social welfare theory and the theory of majorization.

Key Words Income inequality, Prospect Theory, Stochastic orderings, Utility theory, Yaari's Functionals
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1 Introduction

Many results in mathematical utility theory and in the parallel theories of poverty can be cast in terms of stochastic orderings; a standard reference is Shaked and Shanthikumar (2007). Moreover, it is becoming clear from work in these areas that there is a duality between orderings based on income distributions expressed via the cumulative income distribution (cdf) F and certain orderings on the quantile function F^{-1}; see, in particular, Yaari (1987). Such matters are also of growing interest in financial and insurance risk areas. A useful text is Müller and Stoyan (2002). An example of the duality is that between second order stochastic dominance (SSD) and the so-called Generalized Lorenz ordering $\leq^{GL}$:

$$\int_{-\infty}^{x} F_1 dt \geq \int_{-\infty}^{x} F_2 dt \text{ for all } x \in \mathbf{R} \Leftrightarrow \int_{0}^{\alpha} F_1^{-1} dt \leq \int_{0}^{\alpha} F_2^{-1} dt \text{ for all } 0 \leq \alpha \leq 1. \quad (1)$$

This result has been studied in important contributions by Muliere and Scarsini (1999) and Ogryczak and Ruszczynski (2002), who show the link to Fenchel duality. The book by Pallaschke and Rolewicz (1997) covers relevant generalisations of Fenchel duality. Sordo and Ramos (2007) cover similar ground with a discussion of the literature, going back to Lorenz (1905). A generalisation of (1) is at the heart of this paper.

There is also an equivalence between orderings and their characterizations in terms of a set of order preserving functions, expected utilities in the language of economics. Thus, first order stochastic dominance defined as $F_1(x) \geq F_2(x)$ for all x, holds if and only if

$$E(u(X)) \leq E(u(Y))$$

where $X \sim F_1$ and $Y \sim F_2$, for all non-decreasing $u(\cdot)$. For second order stochastic dominance (SSD) the equivalence is for all non-decreasing concave $u(\cdot)$ for which expectations exist. There is a utility theory associated with F^{-1} which is often referred to as the dual theory of *rank-dependent* utilities: see Atkinson (1970), Sen (1973), Cowell (1977). For the moment we simply note that these dual utilities measure the rank position of an individual (in terms of income) in the population, as opposed to the actual level of income. There are a variety of duality *theorems* relating the utility theories based on F and F^{-1}. Recent examples are Maccheroni, Muliere and Zoli (2005) and Chateauneuf and Moyes (2004).

A rich and related area is the study of "distortions" on a cumulative distribution function F. It is fundamental to our approach that there are two types of distortion and that they can be applied to F or to F^{-1}. One type takes the form of a direct transformation of the cdf: $v(F)$, for some function $v(\cdot)$. The other type is applied to the base measure so that, for an integrable function $g(\cdot)$, $\int g(t)dt$ becomes $\int g(t)du(t)$. For F^{-1} we have analogous distortions. Distortions have been studied in Prospect Theory using terminology such as "probability weighting", see Kahneman and Tversky (1992).

It is a main aim of the paper to present a general type of stochastic ordering which helps to unify the above theories. This is done via our main duality theorem, Theorem 1 in Section 3. The proof, in Appendix 1, uses the promised generalisation of (1), Lemma 1, which incorporates distortions of both types just mentioned, applied both to F and, swapped over, to F^{-1}. Furthermore, each side of the duality has an equivalent representation in terms of utility functions which, perhaps surprisingly, uses the same class of functions as are used for the distortion. This means that Theorem 1 has four equivalent parts. Our stochastic ordering is defined given two base (distortion) functions u_0 and v_0. The duality involving F and F^{-1} and the utility versions are then fixed. The proofs are somewhat technical because we assume general cdf's. Appropriate forms of some standard results are needed, such as integration by parts, and these are put into Appendix 2.

The test of a theory may be the range of its special cases. In Section 6 we cover such cases and also issues such as dominated risk aversion and inequality aversion.

2 Orderings and duality

We start with a general definition which involves a simultaneous distortion of the cdf's and the measure. Let U, V be classes of functions u, v, where $u : \boldsymbol{R} \rightarrow \boldsymbol{R}$ and $v : [0,1] \rightarrow [0,1]$.

All such functions will be of bounded variation, so that we can take the associated measures. The notation $\int f(x)du(x)$ and $\int g(\alpha)dv(\alpha)$ will be used for the Lebesgue-Stieltjes integrals, but when the integrating variable is clear, it will be shortened to $\int f(x)du$ and $\int g(\alpha)dv$. Throughout the paper it will be understood that when u or v is right continuous the associated Stieltjes integral will be extended to half-open intervals inclusive of the upper end-point, whereas if the function is left continuous the Stieltjes integral will be extended to intervals half-open to the right.

Definition 1 *Let U, V be classes of functions u, v and let F_1, F_2 be two cdf's. We say that F_1 **is less than** F_2 **in the** (U, V)**-ordering** if*

$$\int_{-\infty}^{\infty} v(F_1(x))du(x) \geq \int_{-\infty}^{\infty} v(F_2(x))du(x) \tag{2}$$

for all $u \in U$, $v \in V$ for which the integrals exist.

This paper is also concerned with dual orderings which take the form given by the following definition. For a cdf F we define F^{-1} in the usual way:

$$F^{-1}(\alpha) = \inf\{x : F(x) \geq \alpha\},\ \ \alpha \in (0,1)$$

and, following the standard, take F to be right continuous so that F^{-1} is left continuous.

Definition 2 *Let $\tilde{U}, \tilde{V}$ be classes of functions $\tilde{u}, \tilde{v}$ and let F_1, F_2 be two cdf's. We say that F_1 **is less than** F_2 **in the dual** $(\tilde{U}, \tilde{V})$**-ordering** if*

$$\int_0^1 \tilde{u}(F_1^{-1}(\alpha))d\tilde{v}(\alpha) \leq \int_0^1 \tilde{u}(F_2^{-1}(\alpha))d\tilde{v}(\alpha) \tag{3}$$

for all $\tilde{u}$ in $\tilde{U}$ and $\tilde{v}$ in $\tilde{V}$.

A duality theorem in the context of Definitions 1 and 2 is a collection $\{U, V, \tilde{U}, \tilde{V}\}$ such that the (U, V)-ordering and the dual $(\tilde{U}, \tilde{V})$-ordering are mathematically equivalent. A well-known example is for the ordinary (first order) stochastic dominance

$$F_1(x) \geq F_2(x) \text{ for all } x \in R \iff F_1^{-1}(\alpha) \leq F_2^{-1}(\alpha) \text{ for all } \alpha \in (0,1)$$

where $U = \{\text{all indicator functions } \mathbb{I}_{[c,\infty)}(x)\}$, $V = \{\text{identity on } [0,1]\}$, $\tilde{U} = \{\text{identity on } \boldsymbol{R}\}$, $\tilde{V} = \{\text{all indicator functions } \mathbb{I}_{[p,1)}(\alpha)\}$.

Importantly, using integration by parts under suitable conditions, each of the inequalities in Definitions 1 and 2 may have an equivalent version in terms of expected utility. In that case the (U, V)-ordering is equivalent to the statement

$$\int_{-\infty}^{\infty} u(x)dv(F_1(x)) \leq \int_{-\infty}^{\infty} u(x)dv(F_2(x)) \tag{4}$$

for all u in U and v in V. The dual $(\tilde{U}, \tilde{V})$-ordering is equivalent to

$$\int_0^1 \tilde{v}(\alpha) d\tilde{u}(F_1^{-1}(\alpha)) \geq \int_0^1 \tilde{v}(\alpha) d\tilde{u}(F_2^{-1}(\alpha)) \tag{5}$$

for all u in $\tilde{U}$ and v in $\tilde{V}$. It would not be too presumptuous to say that the majority of stochastic orderings defined in the literature (see the list at the end of Shaked and Shanthikumar, 2007) are of the form (2), (3), (4) or (5). An equivalence theorem of the type mentioned above would state that under suitable conditions (2), (3), (4) and (5) are equivalent.

In (4) and (5) the functions $u(x)$ and $\tilde{v}(x)$, respectively, can be considered as utility functions, so that as we move to the dual versions, that is from (4) to (5) , the roles of the distortions are reversed. This discussion should explain roughly why our main result, Theorem 1, has four main parts.

3 The upper (u_0, v_0)-ordering and a duality theorem

We start with two functions, u_0, v_0 which define our basic stochastic ordering, following a few definitions. Throughout the paper "increasing" will mean non-decreasing (unless otherwise stated), and similarly for "decreasing". All functions will be of bounded variation on compact intervals and integration is Lebesgue-Stieltjes. Unless otherwise stated, when we integrate with respect to a measure defined by a right continuous function the integral will be extended to intervals of the form (a,b], and when with respect to a left continuous function to intervals [a,b).

Definition 3 *The pair of functions* (u_0, v_0) *is called* ***a standard pair*** *if*
(i) $u_0 : \boldsymbol{R} \to \boldsymbol{R}$ *is increasing and left continuous,*
(ii) $v_0 : [0,1] \to [0,1]$ *is increasing, right continuous, and* $v_0(0) = 0, v(1^-) = 1$.

Definition 4 *For a function* $u_0 : \boldsymbol{R} \to \boldsymbol{R}$ *the class of* u_0*-concave functions on* $\boldsymbol{R}$ *is defined as the class of functions* $\{u : \boldsymbol{R} \to \boldsymbol{R}\}$ *such that*

$$u(x) = \int_{-\infty}^{x} k(s) du_0(s).$$

for some bounded decreasing function $k(x)$ *on* $\boldsymbol{R}$. *Similarly define the class of* v_0*-concave functions,* $\{v : [0,1] \to [0,1]\}$ *as those for which*

$$v(\alpha) = \int_0^{\alpha} \tilde{k}(t) dv_0(t),$$

for some bounded decreasing $\tilde{k}$ *on* $[0,1]$.

We can interpret Definition 4 as saying that $k(x)$ is the Radon-Nikodym derivative of u with respect to u_0

$$k(x) = \frac{du}{du_0}$$

and is decreasing; similarly for v and v_0.

If we add the condition that u_0 and u are increasing (see Definition 3), we have that $k(x) \geq 0, x \in \boldsymbol{R}$. We have a similar interpretation for $\tilde{k}(\alpha) \geq 0,\ \alpha \in [0,1]$.

For differentiable $u(x)$ and $u_0(x)$, $u_0(x) > 0$, u being u_0-concave is equivalent to $\frac{u'}{u_0'}$ decreasing, which in turn, if the functions are twice differentiable and the second derivatives non-zero, is equivalent to

$$-\frac{u''}{u'} \geq -\frac{u_0''}{u_0'} \tag{6}$$

When u is a utility function, $-\frac{u''(x)}{u'(x)}$ is the measure of *absolute risk aversion.* By the Arrow-Pratt Theorem, (Arrow, 1974; Pratt, 1964), u and u_0 satisfy (6) if and only if

$$u(x) = \phi(u_0(x)) \text{ for some concave increasing function } \phi. \tag{7}$$

There are similarly versions of (6) and (7) for the v_0-concave functions of Definition 4. We can also define u_0- and v_0-convex functions which will be discussed in Section 3.

The stochastic ordering we introduce in this paper is a special example of the (U, V)-ordering discussed above.

Definition 5 *Given two cdf's F_1 and F_2 and a standard pair (u_0, v_0), according to Definition 3 we define the* ***upper*** (u_0, v_0)***-stochastic ordering*** $F_1 \prec^{(u_0,v_0)} F_2$ *as:*

$$\int_{-\infty}^{\infty} v_0(F_1(x))du(x) \geq \int_{-\infty}^{\infty} v_0(F_2(x))du(x),$$

for all u in U^0, the class of u_0-concave increasing functions.

The main result of the paper is the next theorem which is an example of a duality referred to above. As presaged it says first that $F_1 \prec^{(u_0,v_0)} F_2$ is equivalent to an ordering, involving F_1^{-1} and F_2^{-1}, in which the roles of u_0 and v_0 are reversed and secondly that there are equivalent utility versions, again using u_0 and v_0 in reverse "distortion" roles.

Theorem 1 *Let u_0, v_0 be a standard pair and U^0 and V^0 the u_0-concave v_0-concave increasing classes, respectively. Let F_1 and F_2 be cdf's which satisfy the following conditions*

(a) $\int_{-\infty}^{\infty} |u_0(x)| dv_0(F(x)) < \infty$

(b) $\int_{[0,p)} v_0(\alpha) du_0(F^{-1}(\alpha)) < \infty$ *for all $p < 1$.*

Then the following are equivalent:

(i) $F_1 \prec^{(u_0,v_0)} F_2$

(ii) $\int_0^1 u_0(F_1^{-1}(\alpha))dv(\alpha) \leq \int_0^1 u_0(F_2^{-1}(\alpha))dv(\alpha)$, *for all* v *in* V^0

(iii) $\int_{-\infty}^{\infty} u(x)dv_0(F_1(x)) \leq \int_{-\infty}^{\infty} u(x)dv_0(F_2(x))$ *for all* u *in* U^0

(iv) $\int_0^1 v(\alpha)du_0(F_1^{-1}(\alpha)) \geq \int_0^1 v(\alpha)du_0(F_2^{-1}(\alpha))$ *for all* v *in* V^0.

There are also four more equivalent statements which are obtained by transformation of variables in $(i)-(iv)$. For example $(i)^*$ is obtained form (i) by a transformation $\alpha = F(x)$ (see (42), Section 9.2 of Appendix 2) and we then obtain a formula with the same structure as (iv) but with the zero suffix moved from u_0 to v; similarly for $(ii)^*$ to $(iv)^*$.

$(i)^*$ $\int_0^1 v_0(\alpha)du(F_1^{-1}(\alpha)) \geq \int_0^1 v_0(\alpha)du(F_2^{-1}(\alpha))$ for all $u \in U^0$

$(ii)^*$ $\int_{-\infty}^{\infty} u_0(x)dv(F_1(x)) \leq \int_{-\infty}^{\infty} u_0(x)dv(F_2(x))$ for all $v \in V^0$

$(iii)^*$ $\int_0^1 u(F_1^{-1}(\alpha))dv_0(\alpha) \leq \int_0^1 u(F_2^{-1}(\alpha))dv_0(\alpha)$, for all $u \in U^0$

$(iv)^*$ $\int_{-\infty}^{\infty} v(F_1(x))du_0(x) \geq \int_{-\infty}^{\infty} v(F_2(x))du_0(x)$ for all $v \in V^0$.

The Proof of Theorem 1 is given in Appendix 1. At the centre of the proof is the following "double distortion" version of statement (1), also proved in Appendix 1.

Lemma 1 *Let* (u_0, v_0) *be a standard pair and let* F_1, F_2 *be two cdf's on* $\boldsymbol{R}$ *satisfying the condition (a), above, then the following are equivalent:*

(i) $\int_{-\infty}^{c} v_0(F_1(x))du_0 \geq \int_{-\infty}^{c} v_0(F_2(x))du_0$ *for all* c *in* $\boldsymbol{R}$

(ii) $\int_0^p u_0(F_1^{-1}(\alpha))dv_0 \leq \int_0^p u_0(F_2^{-1}(\alpha))dv_0$ *for all* p *in* $[0,1)$.

We show that statements (i) and (ii) of Lemma 1 are equivalent to statements (i) and (ii), respectively, of Theorem 1.

Note that Theorem 1 shows that the upper (u_0, v_0)-ordering is equivalent to two inequalities for the "expected utilities" under the distortion, namely:

$$\mathrm{E}(u(X)) \leq \mathrm{E}(u(Y)),$$

for all $u \in U^0$ where $X \sim v_0(F_1)$, $Y \sim v_0(F_2)$, and also:

$$\mathrm{E}(u_0(X)) \leq \mathrm{E}(u_0(Y)),$$

where $X \sim v(F_1)$, $Y \sim v(F_2)$ for all $v \in V^0$.

4 A convex version

A review of the stochastic orderings literature points to several results in which convex increasing functions are used combined with the survivor function $G(x) = 1 - F(x)$. The authors pondered as to whether there are two rather separate theories or whether the duality theory of Section 2 can be applied without too much additional labour to obtain a convex version. We believe that indeed the latter is the case and this section develops such a result. First, we need to define u_0-convexity.

Definition 6 *A function $u : \boldsymbol{R} \to \boldsymbol{R}$ is said to be u_0-convex if*

$$u(x) = \int_{-\infty}^{x} m(t) du_0(t)$$

where $m(\cdot)$ is increasing on $\boldsymbol{R}$. Similarly for v_0-convex functions using an increasing $\tilde{m}$ on $[0, 1]$.

We start with a preamble giving transforms which yield a convex version of Theorem 1. If $X \sim F(x)$ is a random variable, then the cdf of $-X$ is

$$F_{-X}(x) = 1 - F((-x)^-) = 1 - F(-x^+), \tag{8}$$

and its inverse cdf is

$$F_{-X}^{-1}(\alpha) = -F_X^{-1}((1-\alpha)^+) = -F_X^{-1}(1-\alpha^-).$$

Also, for any standard pair (u_0, v_0) define

$$\tilde{u}_0(x) = -u_0(-x^-)), \quad \tilde{v}_0(\alpha) = 1 - v_0(1 - \alpha^+) \tag{9}$$

and note that $(\tilde{u}_0, \tilde{v}_0)$ is still a standard pair.

Next, select one of the expressions used in Lemma 1, e.g.

$$\int_{-\infty}^{c} v_0(F(x)) du_0(x),$$

and replace $u_0(x)$ by $\tilde{u}_0(x)$, $v_0(x)$ by $\tilde{v}_0(x)$ and $F(x)$ by $F_{-X}(x)$. Making the transformation $z = -x$, we have

$$\begin{aligned}
\textstyle\int_{-\infty}^{c} \tilde{v}_0(F_{-X}(x)) d\tilde{u}_0(x) &= -\textstyle\int_{-\infty}^{c} \tilde{v}_0(1 - F((-x)^-)) du_0(-x^-) \\
&= -\textstyle\int_{-c}^{\infty} \tilde{v}_0(1 - F(z^-)) du_0(z) \\
&= -\textstyle\int_{-c}^{\infty} (1 - v_0(1 - (1 - F(z^-))^+) du_0(z) \\
&= -\textstyle\int_{-c}^{\infty} (1 - v_0(F(z))) du_0(z)
\end{aligned} \tag{10}$$

These calculations lead naturally to the following definition.

Definition 7 *For cdf's F_1 and F_2 and standard pair (u_0, v_0) define the* ***lower*** *$(u_0, v_0)-$* ***ordering*** *$F_1 \prec_{(u_0,v_0)} F_2$ by*

$$-\int_{-\infty}^{\infty}(1 - v_0(F_1(x))du(x) \geq -\int_{-\infty}^{\infty}(1 - v_0(F_2(x))du(x)$$

for all increasing u_0-convex functions u.

The convex version of Theorem 1 is obtained by applying $\tilde{u}_0$, $\tilde{v}_0$, F_{-X_1} and F_{-X_2} throughout and then converting back to statements about u_0, v_0, F_1 and F_2 and using (8) and (9). It should be added that after this conversion condition (a) from Theorem 1 remains the same, but (b) changes to $(b)'$ below. A compact way of summarizing the analysis is to say that it is a development of the statement

$$F_{X_1} \prec_{(u_0,v_0)} F_{X_2} \Leftrightarrow F_{-X_1} \prec^{(\tilde{u}_0,\tilde{v}_0)} F_{-X_2}. \tag{11}$$

This could be taken as an equivalent definition.

Theorem 2 *Let (u_0, v_0) be a standard pair and U_0 and V_0 the u_0-, v_0-convex increasing classes, respectively. Let F_1 and F_2 be cdf's which satisfy condition (a) of Theorem 1 together with*

(b)′ $\int_{-\infty}^{\infty}(1 - v_0(F(x)))du_0(x) < \infty$

Then the following are equivalent:

(i)′ $F_1 \prec_{(u_0,v_0)} F_2$

(ii)′ $\int_0^1 u_0(F_1^{-1}(\alpha))dv(\alpha) \leq \int_0^1 u_0(F_2^{-1}(\alpha))dv(\alpha)$, *for all v in V_0*

(iii)′ $\int_{-\infty}^{\infty} u(x)dv_0(F_1(x)) \leq \int_{-\infty}^{\infty} u(x)dv_0(F_2(x))$ *for all u in U_0*

(iv)′ $-\int_0^1(1 - v(\alpha))du_0(F_1^{-1}(\alpha)) \geq -\int_0^1(1 - v(\alpha))du_0(F_2^{-1}(\alpha))$ *for all v in V_0.*

Important aspects of Theorem 2 are that u_0-concave and v_0-concave are replaced respectively by u_0-convex and v_0-convex and that the utility version for $v(\alpha)$, namely (iv), has as similar structure to $v_0(\alpha)$ in Definition 7. The new condition, $(b)'$, controls the existence of the integrals as $x \to \infty$ and uses a distortion generalisation of the survivor function: $1 - v_0(F(x))$. This requires that $u(x)$ does not increase too fast as $x \to \infty$. This is to be compared to (b) of Theorem 1 which says that $u(x)$ should not decrease too fast as $x \to -\infty$.

5 A combined (u_0, v_0)-ordering

Now we combine Theorems 1 and 2 and the upper and lower (u_0, v_0)-orderings and require that $(a), (b)$ and $(b)'$, in those theorems, all hold. First, let us impose no condition on u_0 except bounded variation. Then any such u_0-concave function $u(x)$ can be represented as

$$u(x) = u_1(x) + u_2(x)$$

where $u_1(x)$ is u_0-concave increasing and $-u_2(x) = u_3(x)$ is u_0-convex increasing. This is established by breaking $k(x)$ (see Definition 4) into non-negative and non-positive parts: $k(x) = k^+(x) + k^-(x)$. Then if we have inequalities involving integrals du_1 and reverse inequalities involving du_3, *with the same integrand,* we can achieve bounds with no extra conditions on u_0.

Definition 8 *For cdf's F_1 and F_2 and a pair of function (u_0, v_0) on $\boldsymbol{R}$ and $[0, 1]$ respectively of bounded variation define the* ***double*** (u_0, v_0)-***ordering*** *as*

$$F_1 \preceq_{(u_0,v_0)} F_2 \;\Leftrightarrow F_1 \;\prec^{(u_0,v_0)} F_2 \;\; \text{and} \;\; F_2 \;\prec_{(u_0,v_0)} F_1$$

Motivated by the above discussion we have

Lemma 2 *For cdf's F_1 and F_2 and a pair of functions (u_0, v_0) as in Definition 8, and satisfying (b) and (b)′ from Theorems 1 and 2, $F_1 \preceq_{(u_0,v_0)} F_2$ is equivalent to the statement*

$$\int_{-\infty}^{\infty} v_0(F_1(x))du(x) \le \int_{-\infty}^{\infty} v_0(F_2(x))du(x), \tag{12}$$

for all u_0-concave functions u (not necessarily increasing).

Proof. To establish the inequality in the Lemma we add the inequalities in definitions of upper and lower orderings which, because of the reversals, are in the right direction. It is important too that assuming both (b) and $(b)'$ gives the existence of the relevant integrals. To establish the converse we can make $k^+(x)$ and $k^-(x)$ in the construction alternatively zero.

Drawing on similar arguments to those for Theorems 1 and 2 we can establish the following.

Theorem 3 *Let F_1 and F_2 be two cdf's and let (u_0, v_0) be a standard pair. Assume $(a), (b)$ and $(b)'$ from Theorems 1 and 2 hold; then the following are equivalent*

(i)″ $F_1 \;\prec_{(u_0,v_0)} F_2$

(ii)″ $\int_0^1 u_0(F_1^{-1}(\alpha))dv(\alpha) \le \int_0^1 u_0(F_2^{-1}(\alpha))dv(\alpha)$, *for all v_0-concave v*

(iii)″ $\int_{-\infty}^{\infty} u(x)dv_0(F_1(x)) \le \int_{-\infty}^{\infty} u(x)dv_0(F_2(x))$ *for all u_0-concave u*

(iv)″ $\int_0^1 v(\alpha)du_0(F_1^{-1}(\alpha)) \le \int_0^1 v(\alpha)du_0(F_2^{-1}(\alpha))$ *for all v_0-concave v*

6 Some Examples

6.1 Both u_0 and v_0 are the identity

Let $u_0(x) = x$ for all $x \in \boldsymbol{R}$ and $v_0(\alpha) = \alpha$ for all $\alpha \in [0, 1]$. Then u_0-concave means concave on $\boldsymbol{R}$ and v_0-concave means concave on $[0, 1]$. Similarly for u_0-convex and v_0-convex. The upper (u_0, v_0)-ordering $F_1 \prec^{(u_0,v_0)} F_2$ is

$$\int_{-\infty}^{\infty} F_1(x)du(x) \geq \int_{-\infty}^{\infty} F_2(x)du(x) \tag{13}$$

for all increasing concave $u(\cdot)$ for which the integrals exist. Condition (a) of Theorem 1 means the existence of the expected values $E(X), E(Y)$, where $X \sim F_1$ and $Y \sim F_2$. In this case some of the equivalent statements of Theorems 1, 2 and 3 are well-known, but others are not easily found in the literature.

Theorem 1 states that (13) is equivalent to

$$\int_{-\infty}^{\infty} u(x)dF_1(x) \leq \int_{-\infty}^{\infty} u(x)dF_2(x) \tag{14}$$

for all increasing concave $u(\cdot)$. This is known as the increasing concave ordering: $F_1 \leq_{icv} F_2$ (Shaked and Shanthikumar, 2007). For continuous cdf's the equivalence of (13) and (14) is well-known.

Taking $u_x(z) = z$ if $z \in (-\infty, x]$, and $u_x(z) = x$ otherwise, for all $x \in R$, (13) is equivalent to

$$\int_{-\infty}^{x} F_1(z)dz \geq \int_{-\infty}^{x} F_2(z)dz \qquad \text{for all } x \in R. \tag{15}$$

This is the Second Order Stochastic Dominance ($F_1 \leq_{SSD} F_2$) ordering. The equivalence of $\leq_{SSD}$ and $\leq_{icv}$ is also well-known, see Muller and Stoyan (2002).

Further, by Theorem 1, (13) is also equivalent to

$$\int_{0}^{1} F_1^{-1}(\alpha)dv(\alpha) \leq \int_{0}^{1} F_2^{-1}(\alpha)dv(\alpha) \tag{16}$$

for all increasing concave $v(\cdot)$, and also to

$$\int_{-\infty}^{\infty} v(F_1(x))dx \geq \int_{-\infty}^{\infty} v(F_2(x))dx \tag{17}$$

for all increasing concave $v(\cdot)$. This equivalence seems to be new.

By taking $v_p(\alpha) = \alpha$ if $\alpha \in [0, p]$, and $v_p(\alpha) = p$ otherwise, for all $\alpha \in [0, 1]$, (16) becomes

$$\int_{0}^{p} F_1^{-1}(\alpha)d\alpha \leq \int_{0}^{p} F_2^{-1}(\alpha)d\alpha \qquad \text{for all } p \in [0, 1] \tag{18}$$

which is the Generalized Lorenz ordering, also called *Inverse Stochastic Dominance* (Muliere and Scarsini, 1989). The equivalence of (18) and $\leq_{icv}$ can be found in several papers, at

least for special cases. Recently Sordo and Ramos (2007) have proved it under general conditions.

With u_0 and v_0 the identity, the lower (u_0, v_0)-ordering $F_1 \leq_{(u_0,v_0)} F_2$ can be expressed as:

$$\int_{-\infty}^{\infty} G_1(x)du(x) \leq \int_{-\infty}^{\infty} G_2(x)du(x) \tag{19}$$

for all increasing convex $u(\cdot)$, where $G_j(x) = 1 - F_j(x)$, $(j = 1, 2)$. By Theorem 2, (19) is equivalent to

$$\int_{-\infty}^{\infty} u(x)dF_1(x) \leq \int_{-\infty}^{\infty} u(x)dF_2(x) \tag{20}$$

for all increasing convex $u(\cdot)$. This is known as the increasing convex ($\leq_{icx}$) ordering. It is also equivalent to

$$\int_0^1 F_1^{-1}(\alpha)dv(\alpha) \leq \int_0^1 F_2^{-1}(\alpha)dv(\alpha) \tag{21}$$

for all increasing convex $v(\cdot)$. The equivalence of $\leq_{icx}$(20) and (21) is a well-known result, see Shaked and Shanthikumar (2007), Theorem 4.A.4.

By taking $v_p(\alpha) = \alpha$ if $\alpha \in (p, 1]$, $v_p(\alpha) = p$ otherwise, for all $\alpha \in [0, 1]$, (21) becomes also equivalent to

$$\int_p^1 F_1^{-1}(\alpha)d\alpha \leq \int_p^1 F_2^{-1}(\alpha)d\alpha \qquad \text{for all } p \in [0, 1] \tag{22}$$

and the equivalence of $\leq_{icx}$ and (22) is Theorem 4.A.3 of Shaked and Shanthikumar (2007).

6.2 Majorization

Majorization is an ordering $\prec$ of real vectors which indicates that the components of a vector are less spread out than another. This notion arises in a variety of contexts and a really thorough discussion of it is the recent fundamental book by Marshall, Olkin and Arnold (2011). Given the vectors $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ and $\boldsymbol{y} = (y_1, y_2, ..., y_n)$ let $x_{(1)} \geq x_{(2)} \geq ... \geq x_{(n)}$ and $y_{(1)} \geq y_{(2)} \geq ... \geq y_{(n)}$ be the rearranged coordinates; one of several equivalent definitions of $\boldsymbol{y} \prec \boldsymbol{x}$ is

$$\begin{aligned} \sum_{i=1}^{k} x_{(i)} &\geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, ..., n-1 \\ \sum_{i=1}^{n} x_{(i)} &= \sum_{i=1}^{n} y_{(i)} \end{aligned}$$

Removing the "equal means" condition (bottom line) defines two extensions of this ordering, that are perhaps less well known: *lower weak majorization* $\boldsymbol{y} \prec_w \boldsymbol{x}$ **:**

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, ..., n$$

and *upper weak majorization* $\boldsymbol{y} \prec^w \boldsymbol{x}$

$$\sum_{i=n-k}^{n} x_{(i)} \leq \sum_{i=n-k}^{n} y_{(i)} \qquad k = 0, 1, ..., n-1 \tag{23}$$

Our theory gives results well known or easily derived directly. Let X and Y be random variables on finite supports on the line, such that $\Pr(X = x_i) = \Pr(Y = y_i) = 1/n$ for all i. The upper (u_0, v_0)-ordering $F_1 \prec^{(u_0,v_0)} F_2$ with u_0 the identity and v_0 the identity is upper weak majorization $\boldsymbol{y} \prec^w \boldsymbol{x}$ (in the reverse ordering of the vectors). By Lemma 1, $\prec^{(u_0,v_0)}$ is defined by $\int_0^p F_1^{-1}(\alpha))d\alpha \leq \int_0^p F_2^{-1}(\alpha)d\alpha$ for all $0 < p \leq 1$, which is precisely (23). An increasing concave function $u(x)$ yields positive decreasing increments $u(x_{(n-i+1)}) - u(x_{(n-i)})$ and $u(y_{(n-i+1)}) - u(y_{(n-i)})$ for $i = 1, ..., n-1$. Similarly v increasing concave means that the increments $b_{(i)} = v(\frac{i}{n}) - v(\frac{i-1}{n})$ yield a positive decreasing sequence, and the equivalent statements (i) to (iv) in Theorem 1 can be translated into equivalent statements (24) to (27) below. Equivalence with (24) to (26) can be found in Marshall *et al* (2011), whereas (27) can be easily obtained from the definition

(i)

$$\sum_{i=1}^{n-1} \frac{i}{n}(u(x_{(n-i)}) - u(x_{(n-i+1)}) \geq \sum_{i=1}^{n-1} \frac{i}{n}(u(y_{(n-i)}) - u(y_{(n-i+1)})) \tag{24}$$

for any increasing concave real function $u(\cdot)$;

(ii)

$$\sum_{i=1}^{n} b_{(i)} x_{(i)} \leq \sum_{i=1}^{n} b_{(i)} y_{(i)} \tag{25}$$

for any decreasing sequence $1 \geq b_{(1)} \geq b_{(2)} \geq ... \geq b_{(n)} \geq 0$;

(iii)

$$\sum_{i=1}^{n} u(x_i) \leq \sum_{i=1}^{n} u(y_i) \tag{26}$$

for all increasing concave real functions $u(\cdot)$;

(iv)

$$\sum_{i=1}^{n} v(\frac{i}{n})(x_{(n-i)} - x_{(n-i+1)}) \geq \sum_{i=1}^{n} v(\frac{i}{n})(y_{(n-i)} - y_{(n-i+1)}) \tag{27}$$

for all increasing concave real functions $v(\cdot)$, where $x_{(0)} = y_{(0)} = K$ is any real number.

Similarly, for uniform distributions with finite supports on the real line, the lower (u_0, v_0)-ordering $F_1 \prec_{(u_0,v_0)} F_2$ when u_0 and v_0 are the identity becomes $\boldsymbol{x} \prec_w \boldsymbol{y}$ and the equivalence of $(i), (ii), (iii)$ and (iv) of Theorem 2 gives equivalent statements in the theory of lower weak majorization.

It is well known that standard majorization $\boldsymbol{x} \prec \boldsymbol{y}$ holds iff $\boldsymbol{x} \prec^w \boldsymbol{y}$ and $\boldsymbol{x} \prec_w \boldsymbol{y}$. Thus $\boldsymbol{x} \prec \boldsymbol{y}$ is a special case of the double ordering of Theorem 3 with u_0, v_0 both the identity.

It may be interesting to consider extensions when u_0 is a different increasing function. For instance, $u_0(x) = \log x$ yields the ordering known as log-majorization (including the weak versions). Theorems 1, 2 and 3 in this case provide non-trivial new results.

6.3 Social welfare functionals

The usual *Lorenz ordering*

Definition 9 *$F_1 \leq^L F_2$ (F_2 is Lorenz-better than F_1) iff*

$$\frac{1}{\mu_1}\int_0^p F_1^{-1}(\alpha)d\alpha \leq \frac{1}{\mu_2}\int_0^p F_2^{-1}(\alpha)d\alpha \qquad \forall\, p \in [0,1] \tag{28}$$

where $\mu_1 = E(X_1)$ and $\mu_2 = E(X_2)$, where $X_1 \sim F_1$ and $X_2 \sim F_2$

is often used for comparing inequality in the distribution of some measure of well-being, such as wealth, income, consumption, health, education, quality of life or a combination thereof. For finite populations it means that F_2 can be obtained from F_1 by means of a sequence of Pigou-Dalton transfers. It is not the same as the Generalized Lorenz order $\leq^{GL}$ of (18); $\leq^L$ implies $\leq^{GL}$ when $\mu_1 \leq \mu_2$, but not conversely. Roughly speaking, $F_1 \leq^{GL} F_2$ means that there exists a variable V with $V \sim G$ such that G is greater than F_1 with respect to first order stochastic dominance, and F_2 has the same mean as G but is more equal than G, namely $G \leq^L F_2$, and $E(V) = E(X_2)$. In the case of finite populations of the same size n, $\leq^{GL}$ is the reverse of *upper weak majorization* of vectors. Thus $\leq^{GL}$ appears to be an appropriate tool for ranking distributions from a social welfare viewpoint.

A large family of social welfare functions are the so-called *utilitarian* ones, which, (up to an increasing transform), are of the form $E\left(u(X)\right)$ for a given concave increasing utility function $u(\cdot)$; the concavity of u reflects *inequality aversion.* This is possible because under general conditions welfare indicators of the expected utility form with concave u are consistent with second order stochastic dominance and thus with $\leq^{GL}$ because of (1). In other words, equivalence (1) acts as a "bridge" between the theory of decision making under risk and the theory of social choice, which makes it possible to transfer results from one theory to the other, with inequality aversion as the natural equivalent of risk aversion. This "bridge" can be widened by means of Theorem 1, which, extending (1), allows one to import social indicators from utilities (and viceversa) in a wider context. We shall now give one example.

It has been recently argued that it would be more appropriate to compare distributions by new "weighted" orderings of Generalized Lorenz-type that take into account individual deprivation, satisfaction and so on. Chateauneuf and Moyes (2004) suggest orderings of the form

$$\int_0^p F_1^{-1}(\alpha)df_0(\alpha) \leq \int_0^p F_2^{-1}(\alpha)df_0(\alpha) \qquad \forall\, p \in [0,1) \tag{29}$$

for given $f_0 : [0,1] \to [0,1]$ increasing and such that $f_0(0) = 0$, $f_0(1) = 1$; the intuitive meaning is that f_0 is a *weighting function:* when $f_0(p)$ is differentiable, a positive weight $w_0(p) = \frac{\partial f}{\partial p}$ is attached to all income ranks p. Different choices of f reflect different properties: for instance for measures of absolute deprivation f_0 should be star-shaped from above

at 0 and for absolute satisfaction star-shaped from above at 1. This is closely related to the dual theory of rank-dependent social welfare, due to Yaari (1987), whose indicators are of the form

$$Y(F) = \int_{-\infty}^{\infty} x df(F(x)) = \int_0^1 F^{-1}(\alpha) df(\alpha) \tag{30}$$

The most popular social welfare function of the form (30) is the *S-Gini function*, where $f_0(p) = p^\rho$ with $\rho > 1$ (Donaldson and Weymark, 1980; Yitzhaki, 1983). Note that the classical Gini inequality index is associated to the S-Gini social welfare function with $\rho = 2$. The parameter ρ can be seen as a measure of inequality aversion. The ordering (29) means that F_2 is preferred to F_1 by all Yaari's functionals with f belonging to a suitable class.

We now state two simple consequences of Theorem 1 in Section 3 that link the Atkinson-type "utilitarian" welfare indicators to the present context.

Statement 1 *F_2 is preferred to F_1 by all the Yaari welfare measures (30) for which the function f indicates a degree of inequality aversion at least as great as a given f_0 if and only if F_2 is preferred to F_1 by all indicators of the form $\int_{-\infty}^{\infty} u(x) df_0(F(x))$, with concave increasing $u(x)$.*

Functionals of the form

$$\int_{-\infty}^{\infty} u(x) df_0(F(x)) \tag{31}$$

where $f_0 : [0,1] \to [0,1]$ is a strictly increasing and continuous distortion function, called a "perception", for which $f_0(0) = 0$ and $f_0(1) = 1$ and u_0 is a strictly increasing utility, are the building blocks of the *rank-dependent expected utility* (RDEU) theory developed by Quiggin (1983) and later incorporated into Cumulative Prospect Theory by Tversky and Kahneman (1992). Statement 1 is a simple corollary of Theorem 1 when u_0 is the identity and f_0 any given increasing function on $[0,1]$, by letting $v_0(\alpha) = 1 - f_0(1-\alpha)$, so that $0 \le v_0(\alpha) \le 1$, and v_0 is increasing. Increasing v_0-concave functions are all the increasing f_0-convex ones.

Another consequence of Theorem 1 refers to preferences expressed by expected utilities in the sense of (31) and is obtained when u_0 and f_0 are any two increasing functions:

Statement 2 *The preference of F_2 over F_1, when the perception is f_0 and u a utility with risk aversion at least as great as u_0, is equivalent to F_2 being preferred to F_1 by all decision-makers with utility u_0 and a weight function f expressing more inequality aversion than f_0.*

To the best of these authors' knowledge, this equivalence does not appear in the literature.

7 Discussion

Stochastic orderings are an attractive way of summarizing preferences between distributions such as in comparing portfolios, assessing risk in insurance, in individual decision-making

and in the study of income distribution and welfare. Our starting point in Definition 1 is to stress the use of partial orderings, that is group preferences, via the (U, V)-formulation.

Our version of this integral stochastic ordering, in Definition 5, captures the preference not simply of a single subject but those of a group of subjects each with a private utility and each at least as risk averse as a base subject represented by u_0. The duality theory says that this group defines a dual group described by utilities attached to the quantile function, with its own base utility v_0. Members of this dual group are at least as risk averse as the base subject in the dual realm. Moreover, the utility function for the base subject in the first group provides a (probability) distortion in the dual theory and vice versa.

8 Appendix 1

8.1 Proof of Theorem 1

The proof is in two stages. The first is to show that Lemma 1 holds at "crossing points" of F_1 and F_2, the second is to extrapolate the result between crossing points.

Definition 10 *For two cdf's F_1 and F_2 a* ***crossing interval*** *is a set $[a, b] \subset \boldsymbol{R}$ such that there exists $\epsilon > 0$ such that for all $0 < \epsilon_1, \epsilon_2 < \epsilon$*

(i) $F_1(a - \epsilon_1) < F_2(a - \epsilon_1)$

(ii) $F_1(b + \epsilon_2) > F_2(b + \epsilon_2)$

(iii) when $a < b$, $F_1(x) = F_2(x)$, for all $x \in (a, b)$

In this case we say that "F_1 up-crosses F_2". If the roles of F_1 and F_2 are reversed we say that "F_1 down-crosses F_2".

Thus an up-crossing (down-crossing) *point* x_0 is such that $F_1(x_0^-) \leq F_2(x_0^-) \leq F_2(x_0) \leq F_1(x_0)$ $(F_2(x_0^-) \leq F_1(x_0^-) \leq F_1(x_0) \leq F_2(x_0))$. We can similarly define crossing intervals $[\alpha, \beta] \subset [0, 1]$ for F_1^{-1} and F_2^{-1}. With care, we can make the crossing intervals match up: if x_0 is an up-crossing point of (F_1, F_2), then $[F_2(x_0^-), F_2(x_0)]$ is a down-crossing interval for (F_1^{-1}, F_2^{-1}) and similarly for the converse.

Lemma 3 *If x_0 is a crossing point for the pair (F_1, F_2), then*

$$\int_{-\infty}^{x_0} v_0(F_1(x))du_0(x) \geq \int_{-\infty}^{x_0} v_0(F_2(x))du_0(x)$$

implies

$$\int_0^p u_0(F_1^{-1}(\alpha))dv_0(\alpha) \leq \int_0^p u_0(F_2^{-1}(\alpha))dv_0(\alpha)$$

for all $p \in [F_2(x_0^-), F_2(x_0)]$ if F_1 up-crosses F_2 and for all $p \in [F_1(x_0^-), F_1(x_0)]$ if F_1 down-crosses F_2. Similarly, given an up-crossing value α_0 of (F_1^{-1}, F_2^{-1})

$$\int_0^{\alpha_0} u_0(F_1^{-1}(\alpha))dv_0(\alpha) \leq \int_0^{\alpha_0} u_0(F_2^{-1}(\alpha))dv_0(\alpha)$$

implies

$$\int_{-\infty}^{c} v_0(F_1(x))du_0(x) \geq \int_{-\infty}^{c} v_0(F_2(x))du_0(x)$$

for all c in $[F_2^{-1}(\alpha_0), F_2^{-1}(\alpha_0^+)]$. The same for a down-crossing value, changing the statements appropriately.

Proof. Change of variables is the key to the proof. Thus, by Lemma 4 in Appendix 2, and the discussion there, applied first to $F_1(x)$ and then to $F_2(x)$, we have

$$\int_{-\infty}^{x_0} v_0(F_1(x))du_0(x) + \int_0^p u_0(F_1^{-1}(\alpha))dv_0(\alpha) = \tag{32}$$

$$= v_0(F_1(x_0))u_0(x_0) + v_0(p)u_0(F_1^{-1}(p)) - v_0(F_1(x_0))u_0(F_1^{-1}(p))$$

and

$$\int_{-\infty}^{x_0} v_0(F_2(x))du_0(x) + \int_0^p u_0(F_2^{-1}(\alpha))dv_0(\alpha) = \tag{33}$$

$$= v_0(F_2(x_0))u_0(x_0) + v_0(p)u_0(F_2^{-1}(p)) - v_0(F_2(x_0))u_0(F_2^{-1}(p))$$

Let x_0 be an up-crossing point for the pair (F_1, F_2) and let $p \in [F_2(x_0^-), F_2(x_0)]$. For the purpose of proving the first implication in Lemma 3, we can assume, without loss of generality, that any crossing interval reduces to just a point, since open intervals on which $F_1(x) = F_2(x)$ (part (iii) of Definition 9) only contribute by adding a constant to the left hand sides of the last identities. By the same argument, for all $F_1(x_0^-) \leq F_2(x_0^-) \leq \alpha \leq F_2(x_0) \leq F_1(x_0)$ we have $F_1^{-1}(\alpha) = F_2^{-1}(\alpha) = x_0$, so without loss of generality we can assume, similarly, that $x_0 = F_1^{-1}(p) = F_2^{-1}(p)$. Then the right hand sides of (32) and (33) become equal and the first implication in Lemma 3 is true for up-crossing. The proof for down-crossing points is similar. Furthermore, the proof of the converse is now straightforward.

The value of Lemma 3 is to highlight "good" points where it is straightforward to prove the equivalence in Lemma 1. It is a little more straightforward to prove the reverse implication (ii) $\Rightarrow$ (i) in Lemma 1 first. Thus, assume that (ii) in Lemma 1 holds for all p. Then for any x_0 which belongs to a crossing interval the inequality (i) in Lemma 1 holds, as just shown. We now need to extend the proof essentially to the regions between crossing intervals.

Thus, suppose that for a given x_0 there is no p such that (x_0, p) is a crossing pair. Let

$$x_1 = \sup\{x' : \text{ such that } x' < x_0 \text{ and } x' \text{ is a crossing point}\}$$

then x_1 belongs to a crossing interval and assume

$$\textstyle\int_{-\infty}^{x_1} v_0(F_1(x))du_0 \quad \geq \quad \int_{-\infty}^{x_1} v_0(F_2(x))du_0$$

from the assumption. If at this crossing interval F_1 up-crosses F_2 then

$$v_0(F_1(x)) \geq v_0(F_1(x))$$

for all $x_1 \leq x \leq x_0$ and

$$\begin{aligned}
\int_{-\infty}^{x_0} v_0(F_1(x))du_0 &= \int_{-\infty}^{x_1} v_0(F_1(x))du_0 + \int_{x_1}^{x_0} v_0(F_1(x))du_0 \\
&\geq \int_{-\infty}^{x_1} v_0(F_2(x))du_0(x) + \int_{x_1}^{x_0} v_0(F_2(x))du_0 \\
&= \int_{-\infty}^{x_0} v_0(F_2(x))du_0
\end{aligned}$$

Note that possibly $x_1 = -\infty$, in which case $v_0(F_1(x)) \geq v_0(F_2(x))$ for all $x \leq x_0$ and the assertion remains true.

If F_1 down-crosses F_2 at x_1, let

$$x_2 = \inf\{x' : \text{such that } x' > x_0 \text{ and } x' \text{ is a crossing point}\}.$$

so that x_2 is an up-crossing point for (F_1, F_2). Note that in this case possibly $x_2 = +\infty$.

Now assume

$$\int_{-\infty}^{x_2} v_0(F_1(x))du_0 \;\; \geq \;\; \int_{-\infty}^{x_2} v_0(F_2(x))du_0.$$

Also, since $v_0(F_1(x)) \leq v_0(F_2(x))$ for all $x_0 \leq x \leq x_2$, we obtain

$$\begin{aligned}
\int_{-\infty}^{x_0} v_0(F_1(x))du_0 &= \int_{-\infty}^{x_2} v_0(F_1(x))du_0 - \int_{x_0}^{x_2} v_0(F_1(x))du_0 \\
&\geq \int_{-\infty}^{x_2} v_0(F_2(x))du_0 - \int_{x_0}^{x_2} v_0(F_2(x))du_0 \\
&= \int_{-\infty}^{x_0} v_0(F_2(x))du_0
\end{aligned}$$

The forward implication, (i) $\Rightarrow$ (ii) in Lemma 1 follows on similar lines, starting with an arbitrary $p \in [0, 1]$. This completes the proof of Lemma 1.

The next step in the proof of Theorem 1 involves a mixing argument. We start with (i) in Lemma 1 and claim that for any given $c \in \boldsymbol{R}$

$$\begin{array}{rcl}
\int_{-\infty}^{c} v_0(F_1(x))du_0(x) & \geq & \int_{-\infty}^{c} v_0(F_2(x))du_0(x) \\
& \Leftrightarrow & \\
\int_{-\infty}^{\infty} \int_{-\infty}^{c} v_0(F_1(x))du_0(x)d\mu(c) & \geq & \int_{-\infty}^{\infty} \int_{-\infty}^{c} v_0(F_2(x))du_0(x)d\mu(c),
\end{array}$$

for all non-negative bounded σ-finite measures $d\mu(c)$ on $\boldsymbol{R}$. Next, introducing the indicator function $\mathbb{I}_{(-\infty,c)}(x)$, reversing the integrals and using Fubini's Theorem, which holds because

of the boundedness of $d\mu(c)$ and condition (a), we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} v_0(F_1(x))\mathbb{I}_{(-\infty,c)}(x)du_0(x)\right]d\mu(c) &\geq \\
&\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} v_0(F_2(x))\mathbb{I}_{(-\infty,c)}(x)du_0(x)\right]d\mu(c) \\
&\Leftrightarrow \\
\int_{-\infty}^{\infty} v_0(F_1(x))\left[\int_{-\infty}^{\infty}\mathbb{I}_{(-\infty,c)}(x)d\mu(c)\right]du_0(x) &\geq \\
&\int_{-\infty}^{\infty} v_0(F_2(x))\left[\int_{-\infty}^{\infty}\mathbb{I}_{(-\infty,c)}(x)d\mu(c)\right]du_0(x) \\
&\Leftrightarrow \\
\int_{-\infty}^{\infty} v_0(F_1(x))k(x)du_0(x) \geq &\int_{-\infty}^{\infty} v_0(F_2(x))k(x)du_0(x) \\
&\Leftrightarrow \\
\int_{-\infty}^{\infty} v_0(F_1(x))du(x) \geq &\int_{-\infty}^{\infty} v_0(F_2(x))du(x),
\end{aligned}$$

where $k(x)=\int_{-\infty}^{\infty}\mathbb{I}_{(-\infty,c)}(x)d\mu(c)$ is a non-negative decreasing bounded function and we define $u(x)$ so that $k(x)du_0 = du$. But such a u is precisely a u_0-concave increasing function satisfying Definition 4. A similar argument applies to statement (ii) in Lemma 1 and we obtain the equivalent version. Thus, we have shown that $(i) \Leftrightarrow (ii)$ in Theorem 1. Note that we use condition (a) to obtain Fubini in this case and a bounded decreasing function $\tilde{k}(\alpha),\ \alpha\in[0,1]$, as in Definition 4.

Finally, that condition (iii) is equivalent to (i), and (ii) is equivalent to (iv) follows from the version of integration by parts in Section 6.1 of Appendix 2, the discussion therein and conditions (a) and (b) in Theorem 1. This is to obtain bounded integrals. This ends the proof of Theorem 1.

9 Appendix 2

9.1 Integration by parts

Because we use a nonstandard version of the "integration by parts" theorem, we include a full proof here.

Theorem 4 *Let U and V denote two real functions of finite variation on each compact interval of the real line, with U left continuous and V right continuous. Then for each pair of real numbers $a < b$*

$$\int_{(a,b]} U(x)dV + \int_{[a,b)} V(x)dU = U(b)V(b) - V(a)U(a)$$

Proof. Define the measures

$$\begin{aligned}
\mu\{[a,b)\} &= U(b)-U(a) \\
\nu\{(a,b]\} &= V(b)-V(a)
\end{aligned}$$

for each pair of real numbers $a < b$. The statement of the theorem is equivalent to the following

$$\textstyle\int_{(a,b]} (U(x) - U(a))\, d\nu + \int_{[a,b)} (V(x) - V(a))\, d\mu =$$

$$= U(b)V(b) - V(a)U(a) - V(a)\,(U(b) - U(a)) - U(a)\,(V(b) - V(a))\,,$$

which is also equivalent to

$$\int_{(a,b]} (U(x) - U(a))\, d\nu + \int_{[a,b)} (V(x) - V(a))\, d\mu = (U(b) - U(a))\,(V(b) - V(a))\,.$$

We observe that

$$\begin{aligned}\int_{[a,b)} (V(x) - V(a))\, d\mu &= \int_{[a,b)} \nu\{y \in (a,b] : y \le x\}\mu(dx) \\ &= \mu \otimes \nu\{(x,y) \in [a,b) \times (a,b] : y \le x\}\end{aligned}$$

by Fubini's theorem. Similarly

$$\begin{aligned}\int_{(a,b]} (U(x) - U(a))\, d\nu &= \int_{(a,b]} \mu\{x \in [a,b) : x < y\}\nu(dy) \\ &= \mu \otimes \nu\{(x,y) \in [a,b) \times (a,b] : x < y\}.\end{aligned}$$

Adding up these two identities, we obtain

$$\begin{aligned}\int_{(a,b]} (U(x) - U(a))\, d\nu + \int_{[a,b)} (V(x) - V(a))\, d\mu &= \mu \otimes \nu\{[a,b) \times (a,b]\} \\ &= (U(b) - U(a))\,(V(b) - V(a))\,,\end{aligned} \tag{34}$$

an equivalent statement to the assert of this theorem.

We first apply Theorem 4 taking (u_0, v_0) to be a standard pair, and letting $U(x) = u_0(x)$ and $V(x) = v_0(F(x))$. Then under condition (a) of Theorem 1 we obtain

$$\begin{aligned}\int_{(-\infty,c]} u_0(x)dv_0(F(x)) + \int_{(-\infty,c)} v_0(F(x))du_0(x) &= u_0(c)v_0(F(c)) - \lim_{a\to-\infty} u_0(a)v_0(F(a)) \\ &= u_0(c)v_0(F(c))\end{aligned} \tag{35}$$

Then we apply Theorem 4 again taking $U(\alpha) = u_0(F^{-1}(\alpha))$ and $V(\alpha) = v_0(\alpha)$ and, again under condition (a) of Theorem 1, obtain

$$\begin{aligned}\int_{(0,p]} u_0(F^{-1}(\alpha))dv_0(\alpha) + \int_{[0,p)} v_0(\alpha)du_0(F^{-1}(\alpha)) &= u_0(F^{-1}(p))v_0(p) - \lim_{\alpha\to 0} u_0(F^{-1}(\alpha))v_0(\alpha) \\ &= u_0(F^{-1}(p))v_0(p).\end{aligned} \tag{36}$$

9.2 Change of variables

A classical mathematical result states.

Theorem 5 *(Change of variables). Let (Ω, Ξ, μ) be a measure space and $\varphi : \Omega \to \boldsymbol{R}$ a measurable function. For any Borel set A consider the measure $\mu\varphi^{-1}(A) = \mu(\varphi^{-1}(A))$. Let f be measurable real function on the real line $\mathbf{R}$. Then for all A*

$$\int_{\varphi^{-1}(A)} f(\varphi(\omega))\mu(d\omega) = \int_A f(x)\mu\varphi^{-1}(dx). \tag{37}$$

We apply this theorem to the sets $A = (-\infty, a]$ and the function $\varphi(\alpha) = F^{-1}(\alpha)$ where F is a cdf so that $\Omega = (0, 1)$. Recall

$$F(x) \geq \alpha \iff F^{-1}(\alpha) \leq x \qquad \forall x \in \boldsymbol{R}, \forall \alpha \in [0, 1] \tag{38}$$

hence

$$F(x) < \alpha \iff F^{-1}(\alpha) > x \qquad \forall x \in \boldsymbol{R}, \forall \alpha \in [0, 1]. \tag{39}$$

It is easy to see that $\varphi^{-1}(a, b] = (F(a), F(b)]$ for all $a, b \in \boldsymbol{R}$. Take $f = u_0$ and μ the measure on $[0, 1]$ defined by the function v_0 (which can be thought of as a cdf). Then the measure λ on $\boldsymbol{R}$ defined by the distribution function $v_0(F)$, namely $\lambda_{(v_0F)}\{(a, b]\} = v_0(F(b)) - v_0(F(a))$, is the same as $\mu\varphi^{-1}$ and the RHS of (37) becomes $\int_{(-\infty,a]} u_0(x)dv_0(F(x))$. Furthermore the LHS becomes $\int_{(0,F(a)]} u_0(F^{-1}(\alpha))dv_0(\alpha)$. Hence

$$\int_{(-\infty,a]} u_0(x)dv_0(F(x)) = \int_{(0,F(a)]} u_0(F^{-1}(\alpha))dv_0(\alpha) \qquad \text{for any given } a \in \boldsymbol{R}.$$

A very similar proof yields:

$$\int_{(0,p)} v_0(\alpha)du_0(F^{-1}(\alpha)) = \int_{(-\infty,F^{-1}(p))} v_0(F(x))du_0(x) \qquad \text{for any given } p \in (0, 1]\ .$$

Clearly we can replace $u_0(\cdot)$ by $u(\cdot)$ and $v_0(\cdot)$ by $v(\cdot)$, and also let $a \to \infty$ and $p \to 1$, thus we have

$$\int_{-\infty}^{\infty} u_0(x)dv(F(x)) = \int_0^1 u_0(F^{-1}(\alpha))dv(\alpha) \tag{40}$$

$$\int_{-\infty}^{\infty} u(x)dv_0(F(x)) = \int_0^1 u(F^{-1}(\alpha))dv_0(\alpha) \tag{41}$$

$$\int_{-\infty}^{\infty} v_0(F(x))du(x) = \int_0^1 v_0(\alpha)du(F^{-1}(\alpha)) \tag{42}$$

$$\int_{-\infty}^{\infty} v(F(x))du_0(x) = \int_0^1 v(\alpha)du_0(F^{-1}(\alpha)) \tag{43}$$

Lemma 4 *For any cdf F satisfying conditions (a) of Theorem 1 we have*

$$\int_{-\infty}^{x_0} v_0(F(x))du_0(x) + \int_0^{\alpha_0} u_0(F^{-1}(\alpha))dv_0(\alpha) = v_0(\alpha_0)u_0(x_0)$$

whenever either $\alpha_0 = F(x_0)$ or $x_0 = F^{-1}(\alpha_0)$ (or both).
Furthermore, for all $F(x_1^-) \leq \alpha_1 \leq F(x_1)$ and/or $F^{-1}(\alpha_1) \leq x_1 \leq F^{-1}(\alpha_1^+)$

$$\begin{aligned}\int_{-\infty}^{x_1} v_0(F(x))du_0(x) + \int_0^{\alpha_1} u_0(F^{-1}(\alpha))dv_0(\alpha) = \\ = v_0(F(x_1))u_0(x_1) + v_0(\alpha_1)u_0(F^{-1}(\alpha_1)) - v_0(F(x_1))u_0(F^{-1}(\alpha_1)).\end{aligned}$$

Proof. All integrals are bounded because of (a). We prove the first statement. Fix x_0 and consider the case $\alpha_0 = F(x_0)$. By change of variables in the second term and integration by parts

$$\begin{aligned}\int_{-\infty}^{x_0} v_0(F(x))du_0 + \int_{-\infty}^{x_0} u_0(x)d(v_0F) &= v_0(F(x_0))u_0(x_0) \\ &= v_0(\alpha_0)u_0(x_0).\end{aligned}$$

Now fix α_0 and assume $x_0 = F^{-1}(\alpha_0)$. Then again applying change of variables and integration by parts but to the inverse F^{-1} we have

$$\begin{aligned}\int_0^{\alpha_0} v_0(a)d(u_0F^{-1}) + \int_0^{\alpha_0} u_0(F^{-1}(\alpha))dv_0 &= v_0(\alpha_0)u_0(F^{-1}(\alpha_0)) \\ &= v_0(\alpha_0)u_0(x_0).\end{aligned}$$

To prove the second statement of the Lemma assume $F(x_1^-) \leq \alpha_1 \leq F(x_1)$. Then $F^{-1}(\alpha_1) = F^{-1}(F(x_1))$ and the function $u_0(F^{-1}(\alpha))$ is constant for all $\alpha_1 \leq \alpha \leq F(x_1)$. By the first part of the Lemma

$$\begin{aligned}&\textstyle\int_{-\infty}^{x_1} v_0(F(x))du_0(x) + \int_0^{\alpha_1} u_0(F^{-1}(\alpha))dv_0(\alpha) = \\ &\textstyle= v_0(F(x_1))u_0(x_1) - \int_{\alpha_1}^{F(x_1)} u_0(F^{-1}(\alpha))dv_0 \\ &= v_0(F(x_1))u_0(x_1) - u_0(F^{-1}(\alpha_1))[v_0(F(x_1)) - v_0(\alpha_1)] \\ &= v_0(F(x_1))u_0(x_1) + v_0(\alpha_1)u_0(F^{-1}(\alpha_1)) - v_0(F(x_1))u_0(F^{-1}(\alpha_1)).\end{aligned}$$

Similarly when $F^{-1}(\alpha_1) \leq x_1 \leq F^{-1}(\alpha_1^+)$, which ends the proof of Lemma 4.

Remark The above Lemma continues to hold replacing $v_0(\alpha)$ by any $v(\alpha) \in V^0$ and/or $u_0(x)$ by any $u(x) \in U^0$.

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