Strategy-proofness and single-crossing*

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Abstract

This paper characterizes the family of non dictatorial and unanimous social choice functions that can be implemented in dominant strategies over the domain of single-crossing preferences. The main result is that this family coincides with the class of *positional dictator* choice rules, which is obtained from the extended median rule by varying the distribution of $n-1$ parameters at the extremes of the real line. Interestingly, the paper shows that strategy-proofness cannot be guaranteed in the case of other median voter schemes. This contrast with the results on single-peakedness, where the extended median rule has been proved to be strategy-proof without any restriction on the distribution of the fixed ballots.

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1 Introduction

It is well known in economic theory and positive political science that voting, in general, can fail to produce well defined collective outcomes. The

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Condorcet paradox, for example, shows that the conflict of interests in a society may be such that none of the feasible alternatives has the support of a majority of voters against any other alternative. On the other hand, it is also known that no aggregation method via voting is free of individual and group manipulation (Gibbard [7] and Satterthwaite [16]).

To overcome these negative results, it is common in social choice theory to place restrictions on individual preferences. If alternatives can be placed over the real line, as for instance when different levels of a public good or different tax rates are the subject of collective choice, a *natural* preference restriction is *single-crossing*.[1] This restriction makes sense in many political settings. In few words, a society has single-crossing preferences if, given any two policies, one of them more to the right than the other, the more rightist is an individual (with respect to another individual) the more he will "tend to prefer" the right-wing policy over the left-wing one.[2] For instance, if alternatives are tax rates and individuals are *ordered* according to their income, this restriction simply means that, the richer is an individual, the lower will be the tax rate he will tend to prefer.

Thus, unlike single-peakedness, single-crossing is a restriction that imposes limitations *across* individual preferences, on the character of the voters' heterogeneity, rather than on the shape of individual preferences. The main idea behind it is that, in many circumstances, *ordering* people according to a single parameter (like income, productivity, intertemporal preferences, ideological position, etc.) is more natural than ordering alternatives. This condition projects the conflict of interests among individuals over a unidimensional space; and then the *types* of the agents are located over this left-right scale in such a way that, for any pair of alternatives, the set of types preferring one of the alternatives all lie to one side of those who prefer the other.

Technically, single-crossing not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule. In effect, the core is simply the ideal point of the median type agent, where the latter is defined over the ordering of individual types that makes the preference profile single-crossing. This result is well known since the seminal works of Roberts [12] and Grandmont [6]; and, more recently, due to the theoretical contributions of Rothstein [13] and [14], Gans and Smart [5] and Austen-Smith and Banks [1]. It is sometimes referred to as the *Representative Voter Theorem* (henceforth, RVT) or, alternatively, as "the second version" of the *Median Voter Theorem*.

The main problem with this result is that, unlike the original Median Voter Theorem over single-peaked preferences, whose noncooperative foundation was provided by Black [2], first, and then by Moulin [9], the RVT is

[1]The other one is, of course, single-peakedness.

[2]The exact meaning of "tend to prefer" is made clear in Section 3, Definition 8.

based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere* voting. Clearly, this assumption is difficult to maintain in applications that focus on policy choices made in strategic frameworks. Hence, a natural question arises with respect to its applicability in those models.

This issue has been recently considered in Saporiti and Tohmé [15]. We show there that single-crossing guarantees not only majority voting equilibria, but also the existence of non manipulable social choice rules. In particular, this is true for the median choice rule, which is found to be strategy-proof as well as group strategic-proof over the full set of alternatives and over every possible policy *agenda*. As a by-product, we also prove that the outcome predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism. This mechanism is a two-stage voting procedure where, first, individuals select a representative among themselves, and then the representative voter chooses the policy implemented by the planner.

Taken this work as the starting point, in this paper we characterize the family of non dictatorial and unanimous social choice functions that can be implemented in dominant strategies over the domain of single-crossing preferences. The main result is that this family coincides with the class of *positional dictator* choice rules, which is obtained from the extended median rule by varying the distribution of $n-1$ parameters at the extremes of the real line.

This class is first shown to be *conditional* strategy-proof. Conversely, every non dictatorial, unanimous and conditional strategy-proof social choice function is proved to be a member of this family. Since single-crossing is not a product set, we use the term "conditional" to distinguish our definition of strategy-proofness, which is similar to the one used by Campbell and Kelly [3] and [4], from the standard definition for Cartesian preference domains. In words, this property requires that each individual has incentives to report his true preferences in all those cases where, given the preferences reported by the others, he could do so without violating the domain condition. However, it puts not restrictions on declarations in other situations.

In addition, we also enlarge the domain of single-crossing preferences to make it equivalent to *order-restriction* (Rothstein [13] and [14]), by allowing the orders over the set of alternatives and the set of individual types to change from one profile to another. All previous results are extended to this larger domain in a simple way.

Finally, we show that conditional strategy-proofness is a necessary condition for a social choice function to be implemented in dominant strategies. And we provide a mechanism that carries out this for every positional dictator choice rule defined over single-crossing preferences.

The rest of the paper is organized as follows. In Section 2 we present the model, the notation and definitions. In Section 3, we restrict the

domain of admissible preferences, by introducing the formal definition of single-crossing and *broad* single-crossing. We also discuss their relation with single-peakedness, which is the other standard domain restriction in one-dimensional models of voting. Sections 4 and 5 characterize the family of non dictatorial and unanimous social choice functions that can be implemented in dominant strategies over the single-crossing domain. Final remarks are made in Section 6.

2 Preliminaries

We consider a society with a finite number of agents, represented by the elements of the set $I = \{1, \ldots, n\}$, where $|I| = n$ is odd and $n > 2$. These agents face a collective choice problem, which consists in choosing an alternative (for example, a tax rate) from a finite subset of the real line. They make this choice by voting.

The set of possible outcomes is $X = \{x_1, \ldots, x_m\}$, $|X| > 2$, where X is a finite subset of the extended non-negative real line $\mathbf{R}^*_+ = \Re_+ \cup \{+\infty\}$. The set X is such that $x_j \leq x_k$ for $j \leq k$, where the linear order $\leq$ is the usual order on $\mathbf{R}^*_+$. Following the standard notation, for a vector $(x_1, \ldots, x_n) \in (\mathbf{R}^*_+)^n$, we let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(\hat{x}_i, x_{-i}) = (x_1, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n)$, where $\hat{x}_i \in \mathbf{R}^*_+$. In addition, for any group of agents $S \subseteq I$, we denote $(x_S, x_{\bar{S}}) = ((x_i)_{i \in S}, (x_j)_{j \in \bar{S}})$, where $\bar{S} = I \backslash S$.

Let $P(X)$ be the set of all complete, transitive and antisymmetric binary orderings of X. We say $P(X)$ is the *universal domain* of individual preferences.[3] We assume agent i's preferences over X are completely characterized by a single parameter $\theta_i \in \Theta \subset \Re$. As usual, we interpret θ_i as being agent i's *type*. That is, we assume there exists a function $\phi : \Theta \to P(X)$ that assigns a unique binary relation $\phi(\theta) \in P(X)$ to each type $\theta \in \Theta$. Then, we say that $\succ_i$ represents the preferences of an agent i of type θ_i if,

$$\forall x, \, y \in X, \; x \succ_i y \; \Leftrightarrow \; x \, \phi(\theta_i) \, y.$$

The following example, taken from Persson and Tabellini [17], illustrates how these preferences can arise naturally in political-economic models:

Example 1 Consider the following simplified version of the redistributive distortionary taxation model of Roberts [12]. Suppose individual $i \in I$ has preferences $u(c_i, l_i) = c_i + v(l_i)$, where c_i denotes individual consumption, l_i leisure time and the function v is such that $v_l > 0$ and $v_{ll} \leq 0$.[4] The

[3] Indifference between alternatives is not allowed. This is a quite common assumption when the set of alternatives is finite. In this paper, it is adopted to simplify the proofs of our main results.

[4] As usual, v_l and v_{ll} represent, respectively, the first and the second derivate of $v(l_i)$.

individual's budget constraint is $c_i \leq (1-t)h_i + f$, where $t \in (0,1)$ is an income tax rate, $f \in \Re_+$ a lump-sum transfer and h_i the individual labor supply. The real wage is exogenous and normalized at unity. Individuals are heterogenous in a productivity parameter $\theta_i \in \Theta \subset \Re$, which is distributed in the population with mean $\bar{\theta}$. Given these different productivities, each individual i faces an "effective" time constraint $1 - \theta_i \geq l_i + h_i$. Finally, the government runs a balanced budget, so that $nf \leq t \sum_{i \in I} h_i$.

Solving the model and substituting the solution into the individual utility function, the *induced* preferences of i over different tax rates can be expressed as $w_i(t) = w(t, \theta_i) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta})$, where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply. Thus, for each individual $i \in I$, $w_i(t)$ is completely determined by θ_i. □

Let $\tau(\succ_i)$ be agent i's most preferred alternative in X according to his preference relation $\succ_i$. A preference profile on X, associated to a profile of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$, is an n tuple $(\succ_1, \ldots, \succ_n) = (\phi(\theta_1), \ldots, \phi(\theta_n))$ in $P(X)^n$. We assume each agent observes θ, so there is complete information among agents about their preferences over X.[5] Extending our earlier conventions to preference profiles, we have that $\succ_{-i} = (\succ_1, \ldots, \succ_{i-1}, \succ_{i+1}, \ldots, \succ_n)$ and, for any group of agents $S \subseteq I$, $(\succ_S, \succ_{\bar{S}}) = ((\succ_i)_{i \in S}, (\succ_j)_{j \in \bar{S}})$. Similarly, the profile obtained by changing agent i's preferences for $\hat{\succ}_i \in P(X)$ is $(\hat{\succ}_i, \succ_{-i}) = (\succ_1, \ldots, \succ_{i-1}, \hat{\succ}_i, \succ_{i+1}, \ldots, \succ_n)$. Finally, given a profile $\succ = (\succ_1, \ldots, \succ_n) \in P(X)^n$, we denote $\Theta(\succ) = \{\theta \in \Theta : \exists\, i \text{ such that } \succ_i = \phi(\theta)\}$ the set of *actual* types associated to $\succ$.

These preferences can be aggregated. The input for this aggregation process is the set of *declarations* of the individuals. These declarations are intended to provide information about their true types, although their sincerity may not be ensured.

The aggregation process is represented by a social choice function. Let $D(X) \subset P(X)^n$ be the set of *admissible* preference profiles. Notice that $D(X)$ is not required to be a product set. A *resolute* social choice function is a single-valued mapping $f : D(X) \to X$ that associates to each preference profile $(\succ_1, \ldots, \succ_n) \in D(X)$ a *unique* outcome $f(\succ_1, \ldots, \succ_n) \in X$. We denote r_f the range of f, $r_f = \{x \in X : \exists \succ \in D(X) \text{ such that } f(\succ) = x\}$.

We are interested in resolute social choice functions that satisfy the following properties on $D(X)$. The main one is that agents, acting individually or in groups, never have incentives to misrepresent their preferences. To define this idea, we introduce the following notation. Given a preference profile $(\succ_1, \ldots, \succ_n) \in D(X)$, for any coalition $S \subset I$, let $\psi_{\bar{S}}(\succ_S) = \{\hat{\succ}_{\bar{S}} \in P(X)^{n_{\bar{S}}} : (\succ_S, \hat{\succ}_{\bar{S}}) \in D(X)\}$ and $\psi_S(\succ_{\bar{S}}) = \{\hat{\succ}_S \in P(X)^{n_S} : (\hat{\succ}_S, \succ_{\bar{S}}) \in D(X)\}$, where $n_{\bar{S}} = |\bar{S}|$ and $n_S = n - n_{\bar{S}}$:[6]

[5] In Section 5, we discuss the importance of this assumption for the implementation and the implications of relaxing it.

[6] If $S = \{i\}$, then $\psi_{-i}(\succ_i) = \{\hat{\succ}_{-i} \in P(X)^{n-1} : (\succ_i, \hat{\succ}_{-i}) \in D(X)\}$ and $\psi_i(\succ_{-i}) =$

Definition 1 *f is manipulable on $D(X)$ if there exists $i \in I$, $\succ_{-i} \in \psi_{-i}(\succ_i)$ and $\hat{\succ}_i \in \psi_i(\succ_{-i})$ such that $f(\hat{\succ}_i, \succ_{-i}) \succ_i f(\succ_i, \succ_{-i})$.*

If f is not manipulable on $D(X)$, then we say f is *conditional strategy-proof* on $D(X)$, noted f is CSP. The term "conditional" stands for the fact that $\succ_{-i}$ is required to belong to $\psi_{-i}(\succ_i)$ and $\hat{\succ}_i$ to $\psi_i(\succ_{-i})$.

This definition of strategy-proofness is similar to the definition used by Campbell and Kelly [3] and [4] over the preference domain where a Condorcet winner always exists. It says that a resolute social choice function f is (conditional) strategy-proof on $D(X)$ if for *any* preference declaration $\succ_{-i}$ that the rest of the agents could make, each individual $i \in I$ considers the outcome generated by declaring his true preference relation, $f(\succ_i, \succ_{-i})$, at least as good as $f(\hat{\succ}_i, \succ_{-i})$, where the latter results from i's deviation to any other ordering $\hat{\succ}_i$. However, since $D(X)$ is not necessarily a product set, Definition 1 explicitly requires that the profile that contains i's true preferences, $(\succ_i, \succ_{-i})$, and the deviation, $(\hat{\succ}_i, \succ_{-i})$, both belong to the set of admissible preferences.[7]

That is, a social choice function is CSP on $D(X)$ if in those profiles where each individual can report his true preferences, this constitutes a dominant strategy for him, compared with any other deviation that is admissible according to the domain restriction. However, CSP puts no restrictions on individuals' declarations in other profiles.

Proceeding in a similar way, we can also define conditional group strategy-proofness, to capture the possibility of group deviations in the admissible domain.

Definition 2 (CGSP) *f is conditional group strategy-proof on $D(X)$ if, for every $S \subseteq I$ and every $\succ_{\bar{S}} \in \psi_{\bar{S}}(\succ_S)$, there does not exist $\hat{\succ}_S \in \psi_S(\succ_{\bar{S}})$ such that $f(\hat{\succ}_S, \succ_{\bar{S}}) \succ_i f(\succ_S, \succ_{\bar{S}})$ for all $i \in S$.*

Other properties that we may seek in a social choice function are *non dictatorship*, *unanimity* and *Pareto efficiency*. These conditions are well known and require no further comments.

Definition 3 (ND) *f is non dictatorial on $D(X)$ if for every $i \in I$ there exists $\succ \in D(X)$ such that $f(\succ) \neq \tau(\succ_i)$.*

Definition 4 (U) *f is unanimous on $D(X)$ if for all $x \in X$, and all $\succ \in D(X)$, such that $\tau(\succ_i) = x$ $\forall i \in I$, $f(\succ) = x$.*

Definition 5 (P) *f is Pareto efficient on $D(X)$ if, for all $\succ \in D(X)$, $f(\succ) \in \{x \in X : \nexists\, y \in X \backslash \{x\}$ such that $y \succ_i x$ $\forall i \in I\}$.*

$\{\hat{\succ}_i \in P(X) : (\hat{\succ}_i, \succ_{-i}) \in D(X)\}$.

[7] If $D(X)$ is a product set, i.e., if $D(X) = \prod_{i=1}^n D_i(X)$, this condition simply means that $\succ_j \in D_j(X)$ for all $j = 1, \ldots, n$, and $\hat{\succ}_i \in D_i(X)$.

One last property a social choice function may satisfy is *tops-onliness.* We say that f is *tops-only* on $D(X)$ if, for any profile $(\succ_1, \ldots, \succ_n) \in D(X)$, the outcome $f(\succ_1, \ldots, \succ_n)$ is determined exclusively by the individuals' most preferred alternatives $\tau(\succ_1), \ldots, \tau(\succ_n)$.

Definition 6 (TO) *f is tops-only on $D(X)$ if, for all $\succ, \hat{\succ} \in D(X)$, such that $\tau(\succ_i) = \tau(\hat{\succ}_i)$ for all $i \in I$, $f(\succ) = f(\hat{\succ})$.*

The tops-only property dramatically constraints the scope for manipulation. No agent can expect to be able to affect the social outcome without modifying the peak of his reported preference ordering. However, as we show latter, this condition is related to CSP. More precisely, Corollary 3 below proves that, on the domain of single-crossing preferences (yet to be defined), every ND, U and CSP social choice rule is also TO.

We now define the *extended median rule.* This social choice function plays a crucial role in our characterizations of Sections 4 and 5. For any odd positive integer k, let $m^k : (\mathbf{R}_+^*)^k \to \mathbf{R}_+^*$ be the *k-median function* on $(\mathbf{R}_+^*)^k$. That is, for all $x \in (\mathbf{R}_+^*)^k$, let $m^k(x)$ be the *k-median* of $x = (x_1, \ldots, x_k)$ if and only if $|\{x_i \in \mathbf{R}_+^* : x_i \leq m^k(x)\}| \geq \frac{(k+1)}{2}$ and $|\{x_j \in \mathbf{R}_+^* : m^k(x) \leq x_j\}| \geq \frac{(k+1)}{2}$. Since k is odd, this function is always well defined.

Definition 7 *f^e on $D(X)$ is the extended median rule if there exist $\alpha_1, \ldots, \alpha_{n+1} \in \mathbf{R}_+^*$ such that, for all $\succ \in D(X)$,*

$$f^e(\succ) = m^{2n+1}(\tau(\succ_1), \ldots, \tau(\succ_n), \alpha_1, \ldots, \alpha_{n+1}).$$

We denote $\mathcal{E} = \{f^e : (\alpha_1, \ldots, \alpha_{n+1}) \in (\mathbf{R}_+^*)^{n+1}\}$ the family of instances of f^e, obtained by redistributing the parameters (also called *fixed ballots* or *phantom voters*) $\alpha_1, \ldots, \alpha_{n+1}$ in $(\mathbf{R}_+^*)^{n+1}$. A particular case of interest within this family is the following. Let $\alpha_1 = \ldots = \alpha_{\frac{n+1}{2}} = 0$ and $\alpha_{\frac{n+1}{2}+1} = \ldots = \alpha_{n+1} = +\infty$. Then, for all $\succ \in D(X)$,

$$f^e(\succ) = m^{2n+1}(\tau(\succ_1), \ldots, \tau(\succ_n), \underbrace{0, \ldots, 0}_{\frac{(n+1)}{2} \text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{\frac{(n+1)}{2} \text{ times}}).$$

This rule is the well known *median choice rule*, noted f^m, and it can be rewritten as

$$f^m(\succ) = m^n(\tau(\succ_1), \ldots, \tau(\succ_n)).$$

Proceeding in a similar way, we can derive other rules from f^e, by restricting the parameters $\alpha_1, \ldots, \alpha_{n+1}$ to take some particular values in $\mathbf{R}_+^*$. Notice that, if for example $\alpha_1 =, \ldots, = \alpha_{n+1} = \alpha$, then f^e is completely insensitive to the preferences reported by the individuals. That is, for all $\succ \in D(X)$

$$f^e(\succ) = m^{2n+1}(\tau(\succ_1), \ldots, \tau(\succ_n), \underbrace{\alpha, \ldots, \alpha}_{(n+1) \text{ times}}) = \alpha.$$

We might want to exclude such undesirable voting rules and, in particular, require Pareto efficiency. In order to allow the extended median rule f^e to satisfy Pareto efficiency, we eliminate the possibility of inefficiency by setting $\alpha_n = 0$ and $\alpha_{n+1} = +\infty$. Therefore, we obtain the following restriction of f^e: for all $\succ \in D(X)$,

$$f^{e^*}(\succ) = m^{2n-1}(\tau(\succ_1), \ldots, \tau(\succ_n), \alpha_1, \ldots, \alpha_{n-1}),$$

which is the *efficient extended median rule* with $n-1$ parameters. The set of all such efficient rules is denoted $\mathcal{E}^* = \{f^{e^*} : (\alpha_1, \ldots, \alpha_{n-1}) \in (\mathbf{R}^*_+)^{n-1}\}$.

Finally, suppose each parameter α_i is restricted to take its value at either zero or infinity. That is, assume $\alpha_i \in \{0, +\infty\}$ for all $i = 1, \ldots, n-1$. For this particular case, where each phantom voter is either a *leftist* or a *rightist*, the social choice rules obtained from f^{e^*} are the class of *positional dictator* choice rules (Moulin [10]).

These rules select the jth peak between the tops of the reported preference orderings, for some $j \in \{1, \ldots, n\}$. For example, if $j = 1$, we have the *leftist rule*, which always chooses the smallest reported peak. Of course, the median choice rule f^m is also a particular case. We denote by d_j the positional dictator that always selects the alternative placed at the jth position in the sequence $\tau(\succ_1), \ldots, \tau(\succ_n)$. This rule is obtained from f^{e^*} by distributing $n-j$ fixed ballots at zero and the remaining $j-1$ at infinity. The family of all such rules is denoted $\mathcal{D} = \{d_j : j = 1, \ldots, n\} \subset \mathcal{E}^*$.

In the following sections, we study how well these rules perform, according to the manipulation criteria given above, on the domain of single-crossing preferences. But before that, we introduce in the next section the formal definition of this preference domain, and we discuss its relation with single-peakedness, which is the other important preference restriction in unidimensional models of voting.

3 Single-crossing

We consider in this section an admissible domain of preferences over the real line that puts restrictions *across* individual preferences, i.e., over the entire profile, rather than on the shape of each individual preference relation. This domain is the set of single-crossing preference profiles.[8]

[8]Other restrictions related with single-crossing are *hierarchical adherence*, *intermediateness*, *order-restriction* and *unidimensional alignment*. For more on them, see Roberts [12], Grandmont [6], Rothstein [13] and [14], Gans and Smart [5], Myerson [11], Austen-Smith and Banks [1], List [8] and Saporiti and Tohmé [15].

Definition 8 *A profile* $(\phi(\theta_1), \ldots, \phi(\theta_n))$ *is single-crossing on* X *if for all* $x, y \in X$*, and all* $i, j \in I$*, if* $y > x$*,* $\theta_j > \theta_i$*, and* $y\,\phi(\theta_i)\,x$*, then* $y\,\phi(\theta_j)\,x$*.*

We denote $SC(X)$ the set of all single-crossing preference profiles on X (with respect to the linear order $\geq$). In the political arena, this domain makes sense if, for example, individual types are interpreted as being different ideological characters, arranged in a left-right scale, and alternatives are policies or candidates located on a unidimensional space. Put in this way, Definition 8 says that, given any two policies, one of them more to the right than the other, the more rightist a type the more will he *tend to prefer* the right-wing policy over the left-wing one.

The recent interest on this restricted domain of preferences is due to the fact that, like single-peakedness, single-crossing has been shown to be sufficient to guarantee the existence of majority voting equilibria.

However, apart from this fact, it should be clear that both conditions are independent, in the sense that neither property is logically implied by the other. For instance, in Example 1 the profile of induced preferences $(w_1, \ldots, w_n)$ satisfies single-crossing on the interval $(0, 1)$, since for any two policies $t^{'}, t^{''} \in (0, 1)$, $t^{'} > t^{''}$, $w(t^{'}, \theta) - w(t^{''}, \theta)$ is strictly increasing in θ. However, for $h(t)$ sufficiently convex, it violates single-peakedness. Examples 2 and 3 below also illustrate this point.

Example 2 Suppose three agents, with types $\theta_1 < \theta_2 < \theta_3$, and three alternatives x, y, $z \in \Re_{++}$, where $x < y < z$. Assume preferences are as in Table 1. Then, the profile is single-crossing on $\{x, y, z\}$. However, for any ordering of the alternatives, it violates single-peakedness.[9] $\square$

Example 3 Consider three agents, 1, 2 and 3, and four alternatives, w, x, y, $z \in \Re_{+}$. The profile of Table 2 is single-peaked with respect to the linear order $w < x < y < z$. However, if each preference ordering is associated with a different type and each agent is identified with its corresponding type, then for every ordering of the types, $(\succ_1, \succ_2, \succ_3)$ violates single-crossing.[10] $\square$

Table 1: Single-crossing.

$\phi(\theta_1)$	$\phi(\theta_2)$	$\phi(\theta_3)$
x	x	z
y	z	y
z	y	x

Table 2: Single-peakedness.

$\succ_1$	$\succ_2$	$\succ_3$
x	z	y
y	y	x
z	x	w
w	w	z

[9] Notice that every alternative appears in the bottom row of the table.

[10] Notice that it violates single-crossing not only for the ordering of alternatives $w < x < y < z$, but also for every ordering of them.

From the perspective of the analysis of strategy-proofness, there is also a *substantial* difference between these two preference domains. While the domain of single-peaked preferences is a product set, single-crossing is not. This is because each individual ordering in a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$ may depend on other orderings, in the way specified in Definition 8.

This has two important implications. First of all, strategy-proofness becomes a *conditional* property of the social choice functions (see Definitions 1 and 2 above and the discussion that follows Definition 1). Second, the Revelation Principle does not apply. That is, even if a social choice function is found to be (conditional) strategy-proof on $SC(X)$, the mechanism implementing it cannot be a *direct* one.[11] We will return to this point in Section 5.

Now, we introduce a property of single-crossing that is used in the next section to prove several results.

Lemma 1 *For any profile $(\phi(\theta_1), \ldots, \phi(\theta_n)) \in SC(X)$ and each agent i, there are at least two alternative orderings, $\phi(\hat{\theta}_i)$ and $\phi(\bar{\theta}_i)$, such that $\phi(\hat{\theta}_i)$, $\phi(\bar{\theta}_i) \in \psi_i(\phi(\theta)_{-i})$.*

Proof Consider a profile $\phi(\theta) \in SC(X)$. First, we prove our claim for a particular agent. Suppose, by contradiction, for all $i \in I$, and all $\phi(\hat{\theta}_i)$ and $\phi(\bar{\theta}_i)$, either $(\phi(\hat{\theta}_i), \phi(\theta)_{-i}) \notin SC(X)$ or $(\phi(\bar{\theta}_i), \phi(\theta)_{-i}) \notin SC(X)$. For a type $\theta_j \in \Theta(\theta)$, let $L(\theta_j) = \{\theta_i \in \Theta(\theta) : \theta_i < \theta_j\}$ and $H(\theta_j) = \{\theta_i \in \Theta(\theta) : \theta_i > \theta_j\}$.[12]

If $|\Theta(\theta)| > 2$, it is straightforward to see that there is a type $\theta_i \in \Theta(\theta)$ such that $H(\theta_i) \neq \emptyset$ and $L(\theta_i) \neq \emptyset$. Then, we can define $\theta^{\max} = \min H(\theta_i)$ and $\theta^{\min} = \max L(\theta_i)$. By definition, $\phi(\theta_i)$ and $\phi(\theta^{\min})$ are different, as well as $\phi(\theta_i)$ and $\phi(\theta^{\max})$. Moreover, $(\phi(\theta^{\max}), \phi(\theta)_{-i})$ and $(\phi(\theta^{\min}), \phi(\theta)_{-i})$ are in $SC(X)$. Contradiction.

On the other hand, if $|\Theta(\theta)| = 1$, it is trivial to find an individual and a pair of alternative orderings for this agent, such that the new profiles are in $SC(X)$. Therefore, let us consider the case where $|\Theta(\theta)| = 2$. Without loss of generality, suppose that $\Theta(\theta) = \{\theta^1, \theta^2\}$, where $\theta^1 < \theta^2$. Assume that $\phi(\theta^1)$ and $\phi(\theta^2)$ differ in the pair (w, z), $w > z$. Then, $z\,\phi(\theta^1)\,w$ and $w\,\phi(\theta^2)\,z$. Define a type $\bar{\theta} > \theta^1$, such that $\phi(\bar{\theta})$ coincides with $\phi(\theta^1)$ for every pair of alternatives, except for (z, w), and set $w\,\phi(\bar{\theta})\,z$. If $\phi(\bar{\theta}) \neq \phi(\theta^2)$, we are done.

Otherwise, if $\phi(\bar{\theta}) = \phi(\theta^2)$, just consider any pair $(x, y) \subset X$, $x > y$, such that $x \neq w$ or $y \neq z$. This pair exists because $|X| > 2$. Assume $x\,\phi(\theta^2)\,y$. Define a type $\theta^{'} < \theta^2$ such that $y\,\phi(\theta^{'})\,x$. Then, for an agent of type θ^2 we have two alternative orderings $\phi(\theta^{'})$ and $\phi(\theta^1)$ compatible

[11] Recall that a direct mechanism is a game form in which the strategy set of each agent is the set of all possible individual characteristics.

[12] Along this proof, we assume that for every $\theta', \theta'' \in \Theta(\theta)$, $\phi(\theta') \neq \phi(\theta'')$.

with the domain restriction. Contradiction. That means that $\phi(\theta^1)$ is such that $x_1\,\phi(\theta^1)\,x_2\,\phi(\theta^1)\,x_3\,\ldots\,x_{m-1}\,\phi(\theta^1)\,x_m$, where $m = |X|$.[13] But then there must exist a pair $(a, b) \subset X$, say $a > b$, and a type $\theta'' > \theta^2$ such that $\phi(\theta'')$ coincides with $\phi(\theta^2)$, but $b\,\phi(\theta^2)\,a$, while $a\,\phi(\theta'')\,b$. Furthermore, $(\phi(\theta''_i), \phi(\theta)_{-i}) \in SC(X)$, where i is an agent of type θ^2. Again, agent i and the pair $\phi(\theta'')$ and $\phi(\theta^1)$ gives the desired contradiction.

We have seen that there exists $i \in I$ for whom the claim is true. Now we see that this can be extended to every other agent. Consider agent $j \neq i$, with type $\theta_j \neq \theta_i$. Define two preference relations for j: $\phi(\hat{\theta}_j) = \phi(\hat{\theta}_i)$ and $\phi(\bar{\theta}_j) = \phi(\bar{\theta}_i)$, where $\phi(\hat{\theta}_i)$ and $\phi(\bar{\theta}_i)$ are two alternative orderings of i. Then, for $\hat{\theta}_j$ and $\bar{\theta}_j$ appropriately located on $\Re$, it follows that $(\phi(\hat{\theta}_j), \phi(\theta_i), \phi(\theta)_{-\{j,i\}}) \in SC(X)$, as well as $(\phi(\bar{\theta}_j), \phi(\theta_i), \phi(\theta)_{-\{j,i\}})$. Thus, if $\phi(\bar{\theta}_j) \neq \phi(\theta_j)$ and $\phi(\hat{\theta}_j) \neq \phi(\theta_j)$, we are done. Otherwise, if for example $\phi(\bar{\theta}_j) = \phi(\theta_j)$, then we can simply consider $\phi(\theta'_j) = \phi(\theta_i)$. Since the argument can be repeated for each j, this establishes the truth of the claim. □

Finally, we enlarge the domain of single-crossing preferences, by allowing the orders over X and Θ to change from one profile to another.[14]

Definition 9 *A profile $\phi(\theta) = (\phi(\theta_1), \ldots, \phi(\theta_n))$ is broad single-crossing over X if there exists a permutation $\gamma_\theta : \Theta \to \Theta$ and a linear order $>_\theta$ over X such that, for all $x, y \in X$, and all $i, j \in I$, if $y >_\theta x$, $\gamma_\theta(\theta_j) > \gamma_\theta(\theta_i)$, and $y\,\phi(\theta_i)\,x$, then $y\,\phi(\theta_j)\,x$.*

We denote $BSC(X)$ the set of all preference profiles on X that satisfy the previous definition. The next example shows that $SC(X) \subset BSC(X)$.

Example 4 Consider the preferences of Tables 3 and 4, over $X = \{x, y, z\} \subset \Re_+$. Then, $(\phi(\theta_1), \phi(\theta_2), \phi(\theta_3)) \in BSC(X)$ for $x < z < y$ and $\theta_1 < \theta_3 < \theta_2$. Similarly, $(\phi(\hat{\theta}_1), \phi(\hat{\theta}_2), \phi(\hat{\theta}_3)) \in BSC(X)$ for $x < y < z$ and $\hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3$. However, they are not simultaneously in $SC(X)$, since the profile of Table 4 requires z not to be between x and y, while the profile of Table 3 that either $x < z < y$ or $y < z < x$. □

Table 3: Left.

$\phi(\theta_1)$	$\phi(\theta_2)$	$\phi(\theta_3)$
x	y	z
z	z	y
y	x	x

Table 4: Right.

$\phi(\hat{\theta}_1)$	$\phi(\hat{\theta}_2)$	$\phi(\hat{\theta}_3)$
x	x	z
y	z	y
z	y	x

[13]Remember that X is such that $x_j \leq x_k$ for $j \leq k$.

[14]Saporiti and Tohmé [15] shows that this larger domain, called *broad single-crossing*, is equivalent to order-restriction (Rothstein [13] and [14]).

4 Conditional strategy-proofness

In this section, we characterize the family of social choice functions that satisfy ND, U and CSP on broad single-crossing preferences. We begin by showing that every positional dictator choice rule $d_j \in \mathcal{D}$ is CGSP over $SC(X)$.

Proposition 1 *Each $d_j \in \mathcal{D}$ is CGSP over $SC(X)$.*

Proof Consider a rule $d_j \in \mathcal{D}$ and a profile $(\succ_1, \dots, \succ_n) \in SC(X)$, with associated vector of types $(\theta_1, \dots, \theta_n)$. Suppose by contradiction there exists a coalition $S \subseteq I$, a declaration $\succ_{\bar{S}} \in \psi_{\bar{S}}(\succ_S)$ and a joint deviation $\hat{\succ}_S \in \psi_S(\succ_{\bar{S}})$ for S such that $d_j(\hat{\succ}_S, \succ_{\bar{S}}) \succ_i d_j(\succ_S, \succ_{\bar{S}})$ for all $i \in S$.

For simplicity, denote $d_j(\succ_S, \succ_{\bar{S}}) = \tau$ and $d_j(\hat{\succ}_S, \succ_{\bar{S}}) = \hat{\tau}$. By the assumed distribution of the parameters, τ and $\hat{\tau}$ coincide with the tops reported by some "real" voters. Denote these agents k and k' and their types θ_k and $\theta_{k'}$, respectively. Since $\tau \neq \hat{\tau}$, assume $\tau < \hat{\tau}$. Then, for all $i \in S$, $\tau(\succ_i) > \tau$. Suppose not. That is, assume $\tau(\succ_i) \leq \tau$ for some agent $i \in S$. If $\tau(\succ_i) = \tau$, then $\tau \succ_i \hat{\tau}$, which contradicts our hypothesis. Instead, suppose $\tau(\succ_i) < \tau$. Since $\hat{\tau} \succ_i \tau$ and $(\succ_1, \dots, \succ_n) \in SC(X)$, we have $\hat{\tau}\, \phi(\theta)\, \tau$ for all $\theta \geq \theta_i$. Then, $\theta_k < \theta_i$. And, again by the fact that $(\succ_1, \dots, \succ_n) \in SC(X)$, $\tau\, \phi(\theta_k)\, \tau(\succ_i)$ implies $\tau\, \phi(\theta_i)\, \tau(\succ_i)$. Contradiction. Hence, $\tau(\succ_i) > \tau$ for all $i \in S$. The rest of the proof is as follows.

By definition, $d_j(\succ_S, \succ_{\bar{S}}) = m^{2n-1}(\tau(\succ_1), \dots, \tau(\succ_n), \alpha_1, \dots, \alpha_{n-1}) = \tau$, while $d_j(\hat{\succ}_S, \succ_{\bar{S}}) = m^{2n-1}(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\succ_j)\}_{j \in \bar{S}}, \alpha_1, \dots, \alpha_{n-1}) = \hat{\tau}$. Two cases are possible. (1) For each $i \in S$, $\tau(\hat{\succ}_i) \geq \tau$. Then, $\hat{\tau} = \tau$. Contradiction. (2) For some $i \in S$, $\tau(\hat{\succ}_i) < \tau$. Then, if we rename $(\{\tau(\hat{\succ}_i)\}_{i \in S}, \{\tau(\succ_j)\}_{j \in \bar{S}}, \alpha_1, \dots, \alpha_{n-1})$ as $(y_1, \dots, y_{2n-1})$, we have that

$$|\, \{j \in \{1, \dots, (2n-1)\} : y_j \leq \tau\} \,| \geq n.$$

Therefore, $m^{2n-1}(y_1, \dots y_{2n-1}) \leq \tau$. That is, $d_j(\hat{\succ}_S, \succ_{\bar{S}}) \leq d_j(\succ_S, \succ_{\bar{S}})$, which contradicts our initial hypothesis. □

Thus, falling short of Moulin's [9] results, Proposition 1 shows that any extended median rule is CGSP (and, consequently, CSP) over single-crossing preference profiles, provided that each fixed ballot is placed at the extremes of $\mathbf{R}^*_+$, (i.e., at 0 or $+\infty$).[15] In the following corollary, we prove that this is also true over broad single-crossing.

Corollary 1 *Each $d_j \in \mathcal{D}$ is CGSP over $BSC(X)$.*

[15] Notice that placing some parameters of f^{e^*} at peaks of actual types, in addition to at zero or infinity, yields the same result. However, this is ruled out to ensure that the social choice function is independent of the particular profile of preferences considered.

Proof Assume, by contradiction, there exists a rule $f \in \mathcal{D}$ and a coalition $S \subseteq I$ such that S manipulates f at $(\phi(\theta)_S, \phi(\theta)_{\bar{S}}) \in BSC(X)$, via $\phi(\hat{\theta})_S \in \psi_S(\phi(\theta)_{\bar{S}})$. By definition, we have that for all $x, y \in X$, and all $i, j \in I$, $y >_\theta x$, $\gamma_\theta(\theta_j) > \gamma_\theta(\theta_i)$, and $y\, \phi(\gamma_\theta(\theta_i))\, x$ implies $y\, \phi(\gamma_\theta(\theta_j))\, x$. That is, the profile $(\phi(\gamma_\theta(\theta_1)), \ldots, \phi(\gamma_\theta(\theta_n)))$ is single-crossing over X under $>_\theta$, for the family of types $\gamma_\theta(\Theta)$ with the corresponding natural order.

On the other hand, notice that $(X, >_\theta)$ is isomorphic to X under the natural order $\geq$. That is, there exists $h : X \to X$ such that for any $x, y \in X$, if $x \geq_\theta y$, then $h(x) \geq h(y)$. Therefore,

$$\begin{aligned} f(\phi(\theta)) &= m^{2n-1}_{>_\theta}(\tau(\gamma_\theta(\theta_1)), \ldots, \tau(\gamma_\theta(\theta_n)), \alpha_1, \ldots, \alpha_{n-1}), \qquad (1) \\ &= m^{2n-1}(h \circ \tau(\gamma_\theta(\theta_1)), \ldots, h \circ \tau(\gamma_\theta(\theta_n)), \alpha_1, \ldots, \alpha_{n-1}), \end{aligned}$$

where $m^{2n-1}_{>_\theta}$ is the $(2n-1)$-median function on $(\mathbf{R}^*_+)^{2n-1}$, defined with respect to the linear order $\geq_\theta$, and $(\alpha_1, \ldots, \alpha_{n-1}) \in \{0, +\infty\}^{n-1}$. By the properties of the median function and the permutation γ_θ, we have that $m^{2n-1}(h \circ \tau(\gamma_\theta(\theta_1)), \ldots, h \circ \tau(\gamma_\theta(\theta_n)), \alpha_1, \ldots, \alpha_{n-1}) = m^{2n-1}(h \circ \tau(\theta_1), \ldots, h \circ \tau(\theta_n), \alpha_1, \ldots, \alpha_{n-1})$. By Proposition 1, no coalition can manipulate the outcome $m^{2n-1}(h \circ \tau(\theta_1), \ldots, h \circ \tau(\theta_n), \alpha_1, \ldots, \alpha_{n-1})$. But then, by (1), no coalition can manipulate f at $\phi(\theta)$ as well. Contradiction. □

Interestingly, strategy-proofness cannot be guaranteed in the case of other extended median rules, which do not restrict the fixed ballots to take their values at the extremes of the real line; i.e., that allow the collective outcome to be the top of a "fictitious" voter.

Consider for example the preference profile of Table 1, and any efficient extended median rule $f^{e^*} \notin \mathcal{D}$. Let $\alpha_1 = y$ and $\alpha_2 = +\infty$. Notice that α_1 does not coincide with neither the most preferred alternative of a real voter nor the extremes of $\mathbf{R}^*_+$. Then, $f^{e^*}(\phi(\theta_1), \phi(\theta_2), \phi(\theta_3)) = m^5(x, x, z, \alpha_1, \alpha_2) = y$. However, since y is the worst outcome for agent 2, he could deviate to $\phi(\hat{\theta}_2) = \phi(\theta_3)$, $\hat{\theta}_2 > \theta_2$, generating the profile $(\phi(\theta_1), \phi(\hat{\theta}_2), \phi(\theta_3)) \in SC(X)$, and the outcome $f^{e^*}(\phi(\theta_1), \phi(\hat{\theta}_2), \phi(\theta_3)) = m^5(x, z, z, \alpha_1, \alpha_2) = z$. Since $z\, \phi(\theta_2)\, y$, manipulation is successful.[16]

The reason why CSP is not preserved for any possible distribution of the fixed ballots is because, for such arbitrary distributions, the socially selected alternative need not be the most preferred alternative of a real type. However, without this information, single-crossing cannot rule out manipulation.

[Insert Figure 1 about here]

[16] Notice that, in the example, f^{e^*} is manipulable not only under dominant strategies, but also under Nash equilibrium. In effect, player 2's deviation occurs in a profile of declarations where agents 1 and 3 are reporting their true preferences.

To illustrate this, consider Figure 1. Assume f^{e^*} is *any* efficient extended median rule. Single-crossing does not restrict the shape of individual preferences. Therefore, the condition is compatible with preferences that do not decrease monotonically to both sides of the individual peak. That means that, in general, single-crossing cannot ensure that no agent has incentives to misrepresent his preferences. In the picture, for instance, $f^{e^*}(\hat{\succ}_i, \succ_{-i}) \succ_i f^{e^*}(\succ_i, \succ_{-i})$, so i would like to manipulate f^{e^*} at $(\succ_i, \succ_{-i})$ via $\hat{\succ}_i \in \psi_i(\succ_{-i})$.

However, this is not longer true for positional dictator choice rules. In this case, the fact that preferences are single-crossing is sufficient to rule out individual and group manipulation. Suppose for example that $f^{e^*}(\succ_i, \succ_{-i})$ is j's most preferred alternative on X. If $f^{e^*}(\hat{\succ}_i, \succ_{-i}) \succ_i f^{e^*}(\succ_i, \succ_{-i})$, like in the figure, single-crossing implies $f^{e^*}(\hat{\succ}_i, \succ_{-i}) \succ_k f^{e^*}(\succ_i, \succ_{-i})$ for all $\theta_k \geq \theta_i$. Thus, $f^{e^*}(\succ_i, \succ_{-i}) = \tau(\succ_j)$ requires $\theta_j < \theta_i$. But then agent i cannot improve by declaring $\hat{\succ}_i$. That is, i's preferences cannot have the shape of Figure 1. If this happens, $(\succ_i, \succ_{-i}) \in SC(X)$, $\tau(\succ_i) < f^{e^*}(\succ_i, \succ_{-i})$ and $f^{e^*}(\succ_i, \succ_{-i}) \succ_j \tau(\succ_i)$ implies $f^{e^*}(\succ_i, \succ_{-i}) \succ_i \tau(\succ_i)$.

Therefore, if the social choice function selects an individual peak for each single-crossing profile, the way in which this agent orders the alternatives is informative enough to reject any incentive for manipulation. Contrary, if some of the outcomes are not individual tops, we might still think that we can infer i's preferences from the rankings of the rest of the agents. However, the fact is that there exist single-crossing profiles where the way in which $j \in I \backslash \{i\}$ orders the alternatives bears no relation with the ordering of i. Therefore, in these cases it is impossible to guarantee that all individuals have the right incentives, in the sense that they prefer to report their actual preferences.

This contrasts with the results on single-peakedness, where the extended median rule has been proved to be strategy-proof without any restriction on the distribution of the phantom voters. Moreover, it suggests that the family of ND and U social choice functions that can be implemented in dominant strategies over the single-crossing domain is strictly smaller than the corresponding class over single-peakedness. The rest of this section and Section 5 deal with this conjecture.

Theorem 1 *f is ND, P, TO and CSP on $SC(X)$ if and only if $f \in \mathcal{D}$.*

Proof ($\Leftarrow$): Immediate from the definition of positional dictator choice rules and Proposition 1.

($\Rightarrow$): Suppose, by contradiction, f satisfies the hypothesis of Theorem 1, but $f \notin \mathcal{D}$. That is, assume for every combination $(\alpha_1, \ldots, \alpha_{n-1}) \in \{0, +\infty\}^{n-1}$, there exists a profile $\succ \in SC(X)$ such that $f(\succ) \neq m^{2n-1}(\tau(\succ_1), \ldots, \tau(\succ_n), \alpha_1, \ldots, \alpha_{n-1})$. If we denote by i^* the i-th position in the order of the reported peaks, this is equivalent to claim that, for every position

$i = 1, \ldots, n$, there exists a profile $\succ \in SC(X)$ such that

$$f(\succ) \neq \tau(\succ_{i^*}), \tag{$*$}$$

where agent i^* is the individual whose peak takes up the i-th place (according to the linear order $\leq$) in the distribution of tops $\tau(\succ_{1^*}), \ldots, \tau(\succ_{i-1^*}), \tau(\succ_{i^*}), \tau(\succ_{i+1^*}), \ldots, \tau(\succ_{n^*})$, generated by the profile $\succ \in SC(X)$.[17]

On the contrary, if there were a position, say the i-th, such that $f(\succ) = \tau(\succ_{i^*})$ for every $\succ \in SC(X)$, then we would get a contradiction, since

$$\tau(\succ_{i^*}) = m^{2n-1}(\tau(\succ_{1^*}), \ldots, \tau(\succ_{i^*}), \ldots, \tau(\succ_{n^*}), \underbrace{0, \ldots, 0}_{n-i \text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{i-1 \text{ times}}).$$

Consider a profile $\succ \in SC(X)$ where $(*)$ is satisfied. Let $f(\succ) = x \neq \tau(\succ_{i^*})$. We derive a contradiction in the following way. Consider an agent k and two alternative preferences, $\hat{\succ}_k$ and $\bar{\succ}_k$, such that they verify *simultaneously* the following properties: (i) $\hat{\succ}_k, \bar{\succ}_k \in \psi_k(\succ_{-k})$; (ii) $\tau(\bar{\succ}_k) = \tau(\succ_k)$; (iii) $f(\hat{\succ}_k, \succ_{-k}) = y \neq x$; and (iv) $f(\hat{\succ}_k, \succ_{-k}) \,\bar{\succ}_k\, f(\succ_k, \succ_{-k})$.

If k, $\hat{\succ}_k$ and $\bar{\succ}_k$ exist, then we get the desired contradiction. That is, we have a pair of preferences $\hat{\succ}_k$ and $\bar{\succ}_k$ such that $(\hat{\succ}_k, \succ_{-k}) \in SC(X)$ and $(\bar{\succ}_k, \succ_{-k}) \in SC(X)$, while $f(\hat{\succ}_k, \succ_{-k}) \neq f(\succ_k, \succ_{-k})$ and $\tau(\bar{\succ}_k) = \tau(\succ_k)$. Since f is tops-only, $f(\succ_k, \succ_{-k}) = f(\bar{\succ}_k, \succ_{-k})$. But then $f(\hat{\succ}_k, \succ_{-k}) \,\bar{\succ}_k\, f(\succ_k, \succ_{-k})$ implies that $f(\hat{\succ}_k, \succ_{-k}) \,\bar{\succ}_k\, f(\bar{\succ}_k, \succ_{-k})$, which contradicts the assumption that f is CSP on $SC(X)$. Summarizing, we obtain a contradiction by assuming $(*)$. Thus, there must exist a combination $(\alpha_1, \ldots, \alpha_{n-1}) \in \{0, +\infty\}^{n-1}$ such that $f \in \mathcal{D}$.

In the rest of the proof, we focus the analysis on the existence of k, $\hat{\succ}_k$ and $\bar{\succ}_k$ such that (i)-(iv) hold. Suppose, by contradiction, for every k and every pair of individual preferences $\hat{\succ}_k$ and $\bar{\succ}_k$, either

P_1: $\hat{\succ}_k \notin \psi_k(\succ_{-k})$ or $\bar{\succ}_k \notin \psi_k(\succ_{-k})$; or

P_2: $\tau(\bar{\succ}_k) \neq \tau(\succ_k)$; or

P_3: $f(\hat{\succ}_k, \succ_{-k}) = x$; or

P_4: $f(\succ_k, \succ_{-k}) = f(\hat{\succ}_k, \succ_{-k})$ or $f(\succ_k, \succ_{-k}) \,\bar{\succ}_k\, f(\hat{\succ}_k, \succ_{-k})$.

Let us consider each possibility in order to get a contradiction. By Lemma 1, every agent $k \in I$ has at least two alternative preferences, say $\hat{\succ}_k$ and $\bar{\succ}_k$, such that $\hat{\succ}_k, \bar{\succ}_k \in \psi_k(\succ_{-k})$. Moreover, it is easy to see that at least one of these orderings, say $\bar{\succ}_k$, must satisfy $\tau(\bar{\succ}_k) = \tau(\succ_k)$. Therefore,

[17] Under single-crossing, the individual indexed by i is not necessarily at the i-th position in the distribution of tops. Therefore, it is important to distinguish between the index of the agent and the position of its peak. For notational simplicity we omit this distinction whenever it is not relevant.

P_1 and P_2 cannot be true. Next we prove that at least one of these agents, say k, has also a deviation $\hat{\succ}_k \in \psi_k(\succ_{-k})$ such that $f(\hat{\succ}_k, \succ_{-k}) = y \neq x$.

P_3: Suppose, by contradiction, for every $k \in I$ and every $\hat{\succ}_k \in \psi_k(\succ_{-k})$, $f(\hat{\succ}_k, \succ_{-k}) = x$. In words, this means no individual deviation from $(\succ_k, \succ_{-k})$ matters. Denote $\hat{\succ} = (\hat{\succ}_k, \succ_{-k})$. We claim $f(\succ_j^{'}, \hat{\succ}_{-j}) = x$ for all $\succ_j^{'} \in \psi_j(\hat{\succ}_{-j})$, and all $j \in I$. Suppose not. Then, there must exists $j \in I$ and $\succ_j^{'} \in \psi_j(\hat{\succ}_{-j})$ such that $f(\succ_j^{'}, \hat{\succ}_{-j}) = y \neq x$. This means $j \neq k$. Therefore, we can write $f(\succ_j^{'}, \hat{\succ}_{-j}) = f(\succ_j^{'}, \hat{\succ}_k, \succ_{-\{j,k\}})$. Since f is CSP, we require $x \succ_j y$ and $y \succ_j^{'} x$.

Suppose j deviates from $\succ_j^{'}$ to $\hat{\succ}_j = \hat{\succ}_k$. Clearly, $(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}}) \in SC(X)$. Moreover, $f(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}}) = y$. Suppose not. That is, assume $f(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}}) = z \neq y$. If $z \neq x$, then by taken $y \hat{\succ}_j z$ (i.e., $y \hat{\succ}_k z$) we get a contradiction, since j manipulates f at $(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}})$ via $\succ_j^{'}$. On the other hand, if $z = x$, then fixing $y \hat{\succ}_j x$ (i.e., $y \hat{\succ}_k x$) we have that j manipulates $(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}})$ via $\succ_j^{'}$. Notice that $y \hat{\succ}_j z$ (respectively, $y \hat{\succ}_j x$) is perfectly possible, since the fact that k cannot modify $f(\succ_k, \succ_{-k})$ allows us to choose any $\hat{\succ}_k \in \psi_k(\succ_{-k})$. Further, it is easy to see that this preference always exists. If there is $i \neq k$ such that $y \succ_i z$ (respectively, $y \succ_i x$), then we can simply set $\hat{\succ}_k = \succ_i$. Contrary, if $z \succ_i y$ (respectively, $y \succ_i x$) for all $i \neq k$, then it is trivial to define an ordering with the desired preference. Thus, $f(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}}) = y$.

Consider now a deviation for k, from $\hat{\succ}_k$ to $\succ_k^{'} = \succ_j$. Notice that $(\hat{\succ}_j, \succ_k^{'}, \succ_{-\{j,k\}}) \in SC(X)$, since $(\succ_j, \hat{\succ}_k, \succ_{-\{j,k\}}) \in SC(X)$. Suppose $f(\hat{\succ}_j, \succ_k^{'}, \succ_{-\{j,k\}}) = z \neq x$. Then, by selecting $x \hat{\succ}_j z$ ($\Leftrightarrow x \hat{\succ}_k z$), we have that j wants to manipulate f at $(\hat{\succ}_j, \succ_k^{'}, \succ_{-\{j,k\}})$ via $\succ_j$, since by hypothesis $f(\succ_j, \succ_k^{'}, \succ_{-\{j,k\}}) = x$.[18] Therefore, $f(\hat{\succ}_j, \succ_k^{'}, \succ_{-\{j,k\}}) = x$. But then, by choosing $x \hat{\succ}_k y$, we get that k wants to manipulate f at $(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}})$ via $\succ_k^{'}$. Contradiction. Hence, it must be $f(\hat{\succ}_j, \hat{\succ}_k, \succ_{-\{j,k\}}) = x$. But then CSP requires $f(\succ_j^{'}, \hat{\succ}_{-j}) = x$. Contradiction.

By induction, assume now $f(\hat{\succ}_{I'}, \succ_{I''}) = f(\succ_k, \succ_{-k})$, where $(I^{'}, I^{''})$ is any partition of the set of agents. In effect, the case in which $I^{'} = \{k\}$ and $I^{''} = I \setminus \{k\}$ is in fact our hypothesis: $f(\hat{\succ}_k, \succ_{-k}) = f(\succ_k, \succ_{-k})$. Following the same reasoning than before, it follows that $f(\hat{\succ}_{I'+j}, \succ_{I''-j}) = f(\succ_k, \succ_{-k})$ for every partition $(I^{'}, I^{''})$ of the set of agents and every $j \in I^{''}$. In the limit, $f(\hat{\succ}) = f(\succ)$.

Once achieved this limit, consider the case in which preferences are identical for all agents. Concretely, define the permutation $\sigma : I \to I$ such that, for every $i, l \in I$, $\sigma(i) = \sigma_i < \sigma_l = \sigma(l)$ if $\theta_i < \theta_l$; and, if

[18]By hypothesis, any admissible deviation of k from $(\succ_k, \succ_{-k})$ produces an outcome equal to x.

$\theta_i = \theta_l$ and $l < i$, set $\sigma(l) > \sigma(i)$. To avoid to work explicitly with the permutation, suppose the index of each individual refers to his number under the permutation. Now choose sequentially $\hat{\succ}_k = \succ_{i^*}$ for each agent $k = i+1, i+2, \ldots, n$, and then for $k = i-1, i-2, \ldots, 1$. By $(*)$, $f(\succ) \neq \tau(\succ_{i^*})$. Therefore, $\tau(\succ_{i^*}) \,\hat{\succ}_k\, f(\hat{\succ})$ for every $k \in I$. But this contradicts the fact that f is Pareto efficient.

Finally, we disprove P_4 in the following way. Let I^* be the set of agents that satisfy (i), (ii) and (iii):

P_4: Suppose for all $k \in I^*$, and all $\hat{\succ}_k, \bar{\succ}_k \in \psi_k(\succ_{-k})$, such that $\tau(\bar{\succ}_k) = \tau(\succ_k)$ and $y = f(\hat{\succ}_k, \succ_{-k}) \neq f(\succ_k, \succ_{-k}) = x$, we have $f(\succ_k, \succ_{-k}) \,\bar{\succ}_k\, f(\hat{\succ}_k, \succ_{-k})$. Without loss of generality, assume $f(\succ_k, \succ_{-k}) < f(\hat{\succ}_k, \succ_{-k})$. By single-crossing, $f(\succ_k, \succ_{-k}) \,\bar{\succ}_k\, f(\hat{\succ}_k, \succ_{-k}) \Rightarrow f(\succ_k, \succ_{-k}) \succ_i f(\hat{\succ}_k, \succ_{-k})$ for all $\theta_i \leq \bar{\theta}_k$. On the other hand, $f(\hat{\succ}_k, \succ_{-k}) \,\hat{\succ}_k\, f(\succ_k, \succ_{-k})$. Otherwise, k can manipulate f at $(\hat{\succ}_k, \succ_{-k})$ via $\succ_k$. Combining this with the previous claim, it follows that $\hat{\theta}_k > \bar{\theta}_k$.

Then, if $\tau(\bar{\succ}_k) \neq x$, there exists a type $\theta_k^{'} \in \Theta$, $\bar{\theta}_k < \theta_k^{'} < \hat{\theta}_k$, such that $\tau(\succ_k^{'}) = \tau(\bar{\succ}_k)$, but $f(\hat{\succ}_k, \succ_{-k}) \succ_k^{'} f(\succ_k, \succ_{-k})$, where $\succ_k^{'} = \phi(\theta_k^{'})$. In this case, the pair $\hat{\succ}_k$ and $\succ_k^{'}$ contradicts P_4. On the other hand, if $\theta_k^{'}$ does not exist, then $\tau(\bar{\succ}_k) = x$ for all $k \in I^*$. That is, $\tau(\succ_k) = x$ for all $k \in I^*$. But this could only happens if $i^* \notin I^*$, since by assumption $\tau(\succ_{i^*}) \neq x$. That is, it must be that $f(\hat{\succ}_{i^*}, \succ_{-i^*}) = x$ for all $\hat{\succ}_{i^*} \in \psi_{i^*}(\tilde{\succ}_{-i^*})$. (Remember that (i) and (ii) hold for every agent.) Applying the same reasoning than in P_3, it is easy to show that this leads to the same kind of contradictions. □

Now we strengthen our previous characterization by relaxing the assumption of Pareto efficiency:

Theorem 2 *f, $|r_f| > 2$, is ND, TO and CSP on $SC(X)$ if and only if $f \in \mathcal{D}$.*[19]

Proof ($\Leftarrow$): Immediate from the definition of positional dictators choice rules and Proposition 1.

($\Rightarrow$): The proof is similar to the proof of Theorem 1, but we use $P_3^{'}$ below, instead of P_3, in order to show (iii). Given a profile $(\succ_1, \ldots, \succ_n) \in SC(X)$, let $T(X, \succ) = \{x \in X : \exists i \in I \text{ such that } \tau(\succ_i) = x\}$ be the set of the individual peaks generated by the profile $\succ = (\succ_1, \ldots, \succ_n)$.

$P_3^{'}$: For expositional simplicity, suppose that i^* in $(*)$ is the agent that has the *first* ranked peak in the sequence $\tau(\succ_1), \ldots, \tau(\succ_n)$. Rename the

[19] Of course, any rule f with $|r_f| \leq 2$ is trivially CSP on $SC(X)$. If $|r_f| < 2$, it is also tops-only. Moreover, it can be represented by the extended median rule, with its $n+1$ parameters ranging freely over $\mathbf{R}_+^*$.

agents, using the permutation σ defined in P_3. Assume that for all k, and all $\hat{\succ}_k \in \psi_k(\succ_{-k})$, $f(\hat{\succ}_k, \succ_{-k}) = f(\succ) = x$. By the reasoning applied before, it follows that $f(\hat{\succ}_{I'_\sigma}, \succ_{I''_\sigma}) = f(\succ)$ for every partition (I'_σ, I''_σ) of the set of agents.

Since $|r_f| > 2$, there exists $y \in X$, $y \neq x$, and $\tilde{\succ} = (\tilde{\succ}_1, \ldots, \tilde{\succ}_n) \in SC(X)$ such that $f(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = y$. We prove that, after a finite number of deviations from $(\succ_1, \ldots, \succ_n)$, it is possible to reach a profile $\hat{\succ} = (\hat{\succ}_1, \ldots, \hat{\succ}_n) \in SC(X)$, where $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ $\forall i \in I_\sigma$, but $f(\hat{\succ}) \neq f(\tilde{\succ})$.

Consider the most *leftist* ordering in X, noted $\check{\succ}$, characterized by the relation $x_1 \check{\succ} x_2 \check{\succ} \ldots \check{\succ} x_{m-1} \check{\succ} x_m$. If each agent $k = 1, \ldots, n$ is sequentially deviated from $\succ_k$ to $\check{\succ}$, we obtain the unanimous profile $(\check{\succ}_1, \ldots, \check{\succ}_n)$. By the argument employed in P_3, $f(\check{\succ}_1, \ldots, \check{\succ}_n) = x$. Then, consider agent $n \in I_\sigma$, with his top in the n-th position, and define a sequence of deviations $\succ_n^1, \ldots, \succ_n^h$ for this agent, where $\succ_n^1$ is obtained from $\check{\succ}_n$, by raising to the top the greatest alternative in $T(X, \tilde{\succ})$; $\succ_n^2$ is derived from $\succ_n^1$, by moving up to the second position the second higher alternative in $T(X, \tilde{\succ})$; etc. Clearly, the profile $(\check{\succ}_l, \check{\succ}_2, \ldots, \check{\succ}_{n-1}, \succ_n^h) \in SC(X)$ for all h. Moreover, $f(\check{\succ}_l, \check{\succ}_2, \ldots, \check{\succ}_{n-1}, \succ_n^h) = x$. Denote the last deviation $\succ_n^h = \hat{\succ}_n$.

Consider now individual $n-1 \in I_\sigma$. Define as before a sequence of deviations $\succ_{n-1}^1, \ldots, \succ_{n-1}^{h-1}$ for this agent, where $\succ_{n-1}^1$ is obtained from $\check{\succ}_{n-1}$ by raising to the first position the second greatest alternative in $T(X, \tilde{\succ})$; $\succ_{n-1}^2$ is obtained from $\succ_{n-1}^1$ by moving up to the second position the third higher alternative in $T(X, \tilde{\succ})$; etc. After repeating this process for each agent, we finally reach individual 1, with his top in the first position, for which we simply define an alternative ordering $\hat{\succ}_1$ that moves up to the top the smallest element in $T(X, \tilde{\succ})$. By proceeding in this way, we end up with a profile $(\hat{\succ}_1, \ldots, \hat{\succ}_n) \in SC(X)$ such that, $\tau(\hat{\succ}_i) = \tau(\tilde{\succ}_i)$ for all i, but $x = f(\hat{\succ}_1, \ldots, \hat{\succ}_n) \neq f(\tilde{\succ}_1, \ldots, \tilde{\succ}_n) = y$, contradicting the assumption that f is tops-only. $\square$

Corollary 2 *If f, $|r_f| > 2$, is TO and CSP on $SC(X)$, then f is P on $SC(X)$.*

Proof By contradiction, suppose f satisfies the hypothesis of Corollary 2, but f is not Pareto efficient. Then, there exists $\succ \in SC(X)$ and a pair of alternatives x, $y \in X$, $x \neq y$, such that $f(\succ) = x$, while $y \succ_i x$ for every $i \in I$. Therefore, $f(\succ) \notin T(X, \succ)$, which contradicts Theorem 2.[20] $\square$

Finally, we relax the assumption that f is tops-only:

Theorem 3 *f is ND, U and CSP on $SC(X)$ if and only if $f \in \mathcal{D}$.*

Proof ($\Leftarrow$): Immediate from Proposition 1.

[20] Of course, if f is dictatorial, then it is efficient on $SC(X)$.

(⇒): As in the proof of Theorem 1, consider a position, say the i-th, and a profile $\succ \in SC(X)$ such that $(*)$ is satisfied. That is, let $f(\succ) = x \neq \tau(\succ_{i^*})$. Following the argument applied before, properties (i) and (ii) hold for every agent k. To show that (iii) is also true, consider $P_3^{'}$ in the proof of Theorem 2. Recall that there exists a profile $(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) \in SC(X)$, which is obtained from $\succ$ by means of a sequence of individual deviations, such that $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = f(\succ)$. In particular, this profile can be such that $\hat{\succ}_{\sigma_j} = \hat{\succ}_{\sigma_l}$ for all $j, l \in I$. Moreover, since $|r_f| > 2$, it is possible to choose $\tau(\hat{\succ}_{\sigma_i}) = y \neq x$ for each $i \in I$. But then, on one hand, $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = x$, while by unanimity $f(\hat{\succ}_{\sigma_1}, \ldots, \hat{\succ}_{\sigma_n}) = y$. Contradiction.

Thus, since the negation of P_4 does not involve the tops-only condition, there must exist $k \in I$ such that (i)-(iv) hold. Moreover, CSP implies $f(\bar{\succ}_k, \succ_{-k}) = x$. Hence, we get the same contradiction as in the proof of Theorem 1, meaning that $f \in \mathcal{D}$. □

Corollary 3 *If f is ND, U and CSP on $SC(X)$, then f is TO on $SC(X)$.*

Proof By contradiction, suppose there exists $\hat{\succ}, \succ \in SC(X)$ such that $\tau(\hat{\succ}_i) = \tau(\succ_i)$ for all $i \in I$, while $f(\hat{\succ}) \neq f(\succ)$. By Theorem 3, there exists $(\alpha_1, \ldots, \alpha_{n-1}) \in \{0, +\infty\}^{n-1}$ such that,

$$f(\succ) = m^{2n-1}(\tau(\succ_1), \ldots, \tau(\succ_n), \alpha_1, \ldots, \alpha_{n-1}),$$

while

$$f(\hat{\succ}) = m^{2n-1}(\tau(\hat{\succ}_1), \ldots, \tau(\hat{\succ}_n), \alpha_1, \ldots, \alpha_{n-1}).$$

But, since $\tau(\hat{\succ}_i) = \tau(\succ_i)\ \forall\, i \in I$, $f(\hat{\succ}) = f(\succ)$. Contradiction. □

Corollary 4 *f is ND, U and CSP on $BSC(X)$ if and only if $f \in \mathcal{D}$.*

Proof (⇐): Immediate from Corollary 1.

(⇒): Suppose, by contradiction, f satisfies the hypothesis of Corollary 4, but $f \notin \mathcal{D}$. That is, assume that for each position $i = 1, \ldots, n$, there exists a profile $\phi(\theta) \in BSC(X)$ such that $f(\phi(\theta)) \neq \tau(\theta_{i^*})$ where i^* is again the individual with peak at the i-th place in the distribution $\tau(\theta_1), \ldots, \tau(\theta_n)$, according to the linear order $>_\theta$. By Corollary 1, $\phi(\gamma_\theta(\theta)) = (\phi(\gamma_\theta(\theta_1)), \ldots, \phi(\gamma_\theta(\theta_n)))$ is single-crossing over X with respect to $>_\theta$, for the family of types $\gamma_\theta(\Theta)$. Hence, by Theorem 3, $f(\phi(\gamma_\theta(\theta))) = \tau(\gamma_\theta(\theta_{i^*}))$. But, γ_θ changes the position of the agents, but not their preferences. Therefore, $f(\phi(\theta)) = \tau(\theta_{i^*})$. Contradiction. □

5 Implementation

In this section, we connect CSP with dominant strategy implementation (DSI). First, we prove that, for any admissible domain of preferences, CSP

is a necessary condition for DSI. Then, we use this and the results of the previous section to characterize the family of non dictatorial and unanimous social choice rules that can be implemented in dominant strategies over the single-crossing domain. As we show, this family coincides with the class of positional dictator choice rules.

Definition 10 *A mechanism Γ with consequences in X is a strategic game form $(S_i, g)_{i\in I}$, where S_i is the set of actions of each agent i and $g : \prod_{i\in I} S_i \to X$ an outcome function that associates an alternative with every action profile.*

A mechanism Γ and a profile $\succ$ over X induce a strategic game $G(\Gamma, \succ)$. A dominant strategy equilibrium of $G(\Gamma, \succ)$ is a profile $s^* \in S = \prod_{i\in I} S_i$ such that $g(s_i^*, s_{-i}) \succ_i g(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, all $s_i \in S_i$, and all $i \in I$.

Let $D(X) \subset P(X)$ be the admissible domain of preference profiles:

Definition 11 (DSI) Γ *dominant strategy implements f on $D(X)$ if for every $\succ \in D(X)$ there exists a dominant strategy equilibrium $s^* = (s_1^*, \ldots, s_n^*)$ in $G(\Gamma, \succ)$ such that $g(s^*(G(\Gamma, \succ))) = f(\succ)$.*

Proposition 2 *If there exists a mechanism Γ that DSI f on $D(X)$, then f is CSP on $D(X)$.*

Proof Let $\Gamma = (S_i, g)_{i\in I}$ be a mechanism that implements $f : D(X) \to X$ in dominant strategies. Then, for every $\succ \in D(X)$, there exists a dominant strategy equilibrium in $G(\Gamma, \succ)$, noted $s = (s_1, \ldots, s_n)$, such that $g(s) = f(\succ)$. Moreover, since $(s_i)_{i\in I}$ is a dominant strategy equilibrium, $s_i(\succ) = s_i(\succ_i)$.

Suppose, by contradiction, f is not CSP on $D(X)$. Then, there must exist $i \in I$ and $(\succ_i, \succ_{-i}) \in D(X)$ such that, for some $\hat{\succ}_i \in \psi_i(\succ_{-i})$, $f(\hat{\succ}_i, \succ_{-i}) \succ_i f(\succ_i, \succ_{-i})$. By the previous paragraph, $f(\hat{\succ}_i, \succ_{-i}) = g(s_i(\hat{\succ}_i), \{s_j(\succ_j)\}_{j\neq i})$ and $f(\succ_i, \succ_{-i}) = g(s_i(\succ_i), \{s_j(\succ_j)\}_{j\neq i})$. Therefore,

$$g(s_i(\hat{\succ}_i), \{s_j(\succ_j)\}_{j\neq i}) \succ_i g(s_i(\succ_i), \{s_j(\succ_j)\}_{j\neq i}),$$

which contradicts that $s_i(\succ_i)$ is a dominant strategy for i in $G = (\Gamma, \succ)$. Thus, f is CSP. □

Theorem 4 *f is ND, U and DSI on $SC(X)$ if and only if $f \in \mathcal{D}$.*

Proof ($\Rightarrow$): Suppose $f : SC(X) \to X$ satisfies the hypothesis of Theorem 4. Then, by Proposition 3, f is CSP on $SC(X)$. Therefore, by Theorem 3, $f \in \mathcal{D}$.

($\Leftarrow$): Suppose $f : SC(X) \to X$ belongs to $\mathcal{D}$. Without loss of generality, assume $f = d_j$. We show there exists a mechanism Γ that implements d_j in dominant strategies over $SC(X)$.

Let $\succ = (\succ_1, \ldots, \succ_n) \in SC(X)$ be the profile of true preferences. Consider a mechanism $\Gamma = (S_i, g)_{i \in I}$, where an action for agent $i \in I$ is simply to choose an element of the set $S_i = T(X, \succ)$, and the outcome function $g(s_1, \ldots, s_n) = m^{2n-1}(s_1, \ldots, s_n, \alpha_1, \ldots, \alpha_{n-1})$, with $\alpha_1 = \ldots = \alpha_j = 0$ and $\alpha_{j+1} = \ldots = \alpha_{n-1} = +\infty$. We will show that the action profile $(\tau(\succ_1), \ldots, \tau(\succ_n))$ constitutes a dominant strategy equilibrium of $G(\Gamma, \succ)$. That is, we will prove that there is no $i \in I$ and $\hat{s}_i \neq \tau(\succ_i)$ such that, for some $s_{-i} \in \prod_{j \neq i} S_j$,

$$g(\hat{s}_i, s_{-i}) \succ_i g(\tau(\succ_i), s_{-i}). \tag{$\star$}$$

Since, by definition, g is "similar to" the jth dictator choice rule, we can recast the proof of Proposition 1. Assume there exists a deviation $\hat{s}_i \in S_i$ for i such that ($\star$) holds. For simplicity, denote $g(\tau(\succ_i), s_{-i}) = x$ and $g(\hat{s}_i, s_{-i}) = \hat{x}$. Without loss of generality, assume $\tau(\succ_i) < x$. If $\hat{s}_i \leq x$, then $g(\tau(\succ_i), s_{-i}) = g(\hat{s}_i, s_{-i})$. Contradiction. Thus, suppose $\hat{s}_i > x$. This implies $\hat{x} > x$. By hypothesis, $\hat{x} \succ_i x$. Moreover, since $(\succ_i, \succ_{-i}) \in SC(X)$, $\hat{x} \succ_j x$ for every $\theta_j \geq \theta_i$. Let $j^* \in I$ be such that $\tau(\succ_{j^*}) = x$. This agent exists because g chooses always the top in the jth position. Then, $\theta_{j^*} < \theta_i$. But, since $\tau(\succ_i) < x$ and $x \succ_{j^*} \tau(\succ_i)$, by single-crossing, it follows that $x \succ_i \tau(\succ_i)$. Contradiction. Therefore, $(\tau(\succ_1), \ldots, \tau(\succ_n))$ is a dominant strategy equilibrium. □

Corollary 5 *f is ND, U and DSI on $BSC(X)$ if and only if $f \in \mathcal{D}$.*

Proof As the proof of Theorem 4, but using Corollary 1 and 4. □

The mechanism in the proof of Theorem 4 has a clear interpretation in the case of the median choice rule. It can be seen as the strategic form of a two-stage voting procedure where, first, individuals select a *representative* voter by pairwise majority voting, and then the winner chooses the alternative implemented by the planner. In the last stage each agent has a dominant strategy, which is simply to choose his most preferred alternative in X. Therefore, regarding to the outcome, the extensive game form (i.e., the two-stage voting procedure) is equivalent to an strategic form where individuals choose an alternative from the set of actual ideal points by pairwise majority comparisons.

Besides the interpretation of this mechanism in other situations, the main problem is that it requires the profile of actual preferences being known by all agents. That is, it works under complete information. This problem is related to the fact that our preference domain is not a product set, so we cannot use a direct mechanism to implement the social choice function. Even though there may be cases where it is reasonable to assume complete information, this reduces in part the appeal of the notion of dominant strategy implementation, and it limits the applicability of ours results.

However, as Campbell and Kelly [4] have noted, "there is a sense in which results based on a domain of single-peaked preferences have the same drawback: Although single-peaked domains can be defined as product sets, single-peakedness is characterized by means of a particular linear ordering, and an individual would have to know the linear ordering to which the reported preferences is admissible, before being convinced that his own reported preference is admissible", (pp. 567).

Furthermore, while in some cases this ordering is *natural*, and therefore the assumption that it is commonly known (including by the planner) is not too demanding,[21] in other it is not necessarily obvious. Suppose, for example, that alternatives are political candidates ordered in a left-right scale. Then, the way in which individuals order these candidates over the real line is not immediate.

Moreover, it provides information not only on which preferences can be declared, but also on how the preferences of the rest of the agents may look like. That is, it reveals information on the possible values of the society.[22] This is simply because in these models each preference relation is jointly determined by the ideal point and the common ordering on the set of alternatives.

Following a similar approach, we relax in the next proposition the assumption of complete information. This is done by allowing the planner to know the function ϕ, assuming at the same time that the actual distribution of types, i.e., the realized state $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$, is not observed.[23]

Theorem 5 *Suppose ϕ is common knowledge, but θ_i is private information for all $i \in I$. Then, f is ND, U and DSI on $SC(X)$ if and only if $f \in \mathcal{D}$.*

Proof The "only if" part is exactly as in the proof of Theorem 4. With respect to the "if" part, let $\phi(\theta^*) = (\phi(\theta_1^*), \ldots, \phi(\theta_n^*)) \in SC(X)$ be the true preference profile. Consider a mechanism $\Gamma = (S_i, g)_{i \in I}$, where $S_i = \Theta$ for each $i \in I$, and the outcome function

$$g(\theta_1, \ldots, \theta_n) = m^{2n-1}(\tau(\phi(\theta_1)), \ldots, \tau(\phi(\theta_n)), \underbrace{0, \ldots, 0}_{n-j \text{ times}}, \underbrace{+\infty, \ldots, +\infty}_{j-1 \text{ times}}).$$

We will show that the action profile $(\theta_1^*, \ldots, \theta_n^*)$ constitutes a dominant strategy equilibrium of the game $G(\Gamma, \phi(\theta^*))$. Suppose not. Then, there exists $i \in I$ and $\hat{\theta}_i \neq \theta_i^*$ such that, for some $\theta_{-i} \in \Theta^{n-1}$, $g(\hat{\theta}_i, \theta_{-i}) \, \phi(\theta_i^*) \, g(\theta_i^*, \theta_{-i})$. Since $(\phi(\hat{\theta}_i), \phi(\theta)_{j \neq i})$ and $(\phi(\theta_i^*), \phi(\theta)_{j \neq i})$ are

[21] For instance, this may be the case if alternatives are levels of a public good or different tax rates.

[22] For example, if $X = \{x, y, z\}$, $x > y > z$, and $(\succ_1, \ldots, \succ_n)$ is single-peaked on X, then no $i \in I$ can order $x \succ_i z \succ_i y$.

[23] Of course, everybody must also know that the society has single-crossing preferences over the linear order $\geq$.

derived from ϕ, both profiles belong to $SC(X)$. But then, we can repeat the argument of the proof of Theorem 4, to conclude that such agent i does not exist. Therefore, $(\theta_1^*, \ldots, \theta_n^*)$ is a dominant strategy equilibrium. □

As a final remark, notice that, in contrast to the result of Theorem 4, Theorem 5 cannot be extended to broad single-crossing. This is because its proof assumes that individuals know the linear orderings over X and Θ, over which the profile is single-crossing. But this is impossible if there exists incomplete information about agents' preferences and the orderings over X and Θ are allowed to change from one profile to another.[24]

6 Final Remarks

In this paper, we considered social choice functions over the domain of single-crossing preferences. While this preference domain ensures that the core of the majority rule is nonempty, the literature has assumed that voting is sincere. This naturally raises the issue of potential manipulability, motivating the present paper.

Three main conclusions emerge from our research. First, the set of non dictatorial and unanimous social choice rules that can be implemented in dominant strategies over the single-crossing domain is nonempty. Moreover, it coincides with the class of *positional dictator* choice rules. These social choice functions are obtained from the extended median rule by varying the distribution of $n-1$ parameters at the extremes of the real line. Therefore, they include the median choice rule as a particular case.

This result contrast with the results on single-peakedness, where the extended median rule has been proved to be strategy-proof without any restriction on the distribution of fixed ballots. Moreover, it implies that the family of ND, U and DSI social choice functions on the domain of single-crossing preferences is strictly smaller than the corresponding class over single-peakedness.

Finally, the results derived in this paper also show that the Representative Voter Theorem, i.e., the "single-crossing version" of the Median Voter Theorem, has a well defined strategic foundation, in the sense that its prediction can be implemented in dominant strategies. However, as we discussed at the end of Section 5, the assumption of complete information on individual preferences is difficult to relax in the case of single-crossing. Therefore, it also follows that the RVT would not probably have the same appeal as its counterpart on single-peakedness. That is, it seems that it is not possible to relax very much the assumption that voters know each other preferences

[24] A similar problem occurs in the case of single-peakedness if the order over the set of alternatives is allowed to vary with the profiles.

without affecting at the same time the robustness of the RVT against manipulation.

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