

## Storage Costs in Commodity Option Pricing\*

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**Abstract.** Unlike derivatives of financial contracts, commodity options exhibit distinct particularities owing to physical aspects of the underlying. An adaptation of no-arbitrage pricing to this kind of derivative turns out to be a stress test, challenging the martingale-based models with diverse technical and technological constraints, with storability and short selling restrictions, and sometimes with the lack of an efficient dynamic hedging. In this work, we study the effect of storability on risk neutral commodity price modeling and suggest a model class where arbitrage is excluded for both commodity futures trading and simultaneous dynamical management of the commodity stock. The proposed framework is based on key results from interest rate theory.

**Key words.** commodity options, theory of storage, futures markets, LIBOR model

**AMS subject classifications.** 91G20, 91G30, 91G70

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**1. Introduction.** Prospering economies are highly dependent on commodities. As a consequence, sustainable commodity supply is a key factor for their future growth. Thus, the commodity price risk becomes increasingly important. In the past, the price outbursts for oil, biofuels, and agricultural products have clearly demonstrated that the commodity price modeling and hedging deserve particular attention.

Despite the success of financial mathematics in many fields, we believe that the quantitative understanding of commodity price risk is behind the state of the art and needs further research. In the area of commodities, the models are less sophisticated, flexible, and consistent than, for instance, in the theory of fixed income markets. Not surprisingly, many important questions in commodity risk management cannot be addressed accordingly. For the sake of concreteness, let us consider two commodities, electricity and gold, which are very different in their nature and in their price behavior. Gold is, as a precious metal, a perfectly storable good. Furthermore, gold is considered as an appreciated investment opportunity, particularly during critical times. As a result, the price behavior of gold shows many similarities to a foreign currency. For instance, for gold loans, an interest rate (paid in gold) is available. On the contrary, electricity is not economically storable. Strictly speaking, electrical energy delivered at different points in time must be considered as different commodities. Electricity spot price

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spikes occur regularly; each price jump is followed by a relatively fast price decay, returning back to the normal price level. Such a pattern is not possible for the gold spot price. Consider now a calendar spread call option, which can be viewed as a regular call written on the price spread between commodity futures with different maturities. Such a contract is obviously sensitive to spot price spikes. Evidently, the pricing and hedging of such an instrument must depend on whether it is written on electricity or on gold. However, such a differentiation is hardly possible within common risk neutral commodity price models. At the present level of the theory, the practitioner is essentially left alone with the problem of how to adapt a given commodity price model to account for a perfect storability or for an absolute nonstorability of the underlying good.

Apparently, the lack of storage parameter in the common commodity price models is traced to the very philosophy of commodity risk hedging. Physical commodities are cumbersome: Their storage could be difficult and expensive, the quality may be deteriorated by storage, and the supply may require a costly transportation. Furthermore, short positions in commodities are almost impossible. Contrary to this, futures are clean financial instruments, predestined to hedge against undesirable price changes. Accordingly, futures are frequently considered as prime underlyings. Thus, the generic approaches in commodity modeling attempt to exclude merely the *financial arbitrage* (achieved by futures and options trading), losing sight of the *physical arbitrage*, which may result from the trading of financial contracts in addition to an appropriate *inventory management*. At the present stage, the valuation of commodity options sticks to the calculation of prices which exclude arbitrage within a given futures market, thus neglecting the existing real storage opportunities. According to this, there is a need for a unified model which encompasses all commodities, distinguishing particular cases by their storability degree. Here, we are confronted with complex situations. The variety of storage cost structures ranges from a simple quality deterioration (agricultural products) and dependence on related commodities (fodder price may depend on livestock prices) to the availability of the inventory capacities. Furthermore, economists argue that negative storage costs are useful for describing the benefit or premium associated with holding an underlying product or physical good rather than a financial contract (convenience yield arguments). This benefit may depend on the inventory levels since the marginal yield of the physical stock decreases as the quantity approaches a level larger than the business requires. To complete the perplexity, we should mention that the inventory levels, in turn, are interrelated with commodity spot prices (the inventories are full when the commodity is cheap) and also could exhibit seasonalities (harvest times for agricultural products). The bottom line is that there is no simple approach to facing the entire range of storage particularities. However, we hope that a simplified cost structure is able to capture those storability aspects which are quantitatively essential for derivatives pricing. An empirical study presented in this contribution supports this assumption.

The connection between spot and futures prices for commodities with restricted storability and the valuation of storage opportunities have attracted research interest for a long time. In this work, we emphasize, among others, the works [3], [5], [7], [8], [9], and [10]. Moreover, the comprehensive book [6] presents a state-of-the-art exposition in the commodity derivatives pricing. More specifically, commodity spread options are discussed in [4], in the recent works [1], [2], and in the literature cited therein.

In what follows, we present an approach where a single parameter controls the maximally

possible slope of contango, thus giving a storability constraint. This should yield commodity option prices more realistic than those obtained from traditional models, especially when the instrument under valuation explicitly addresses the storability aspects (like a calendar spread option, a virtual storage, or a swing-type contract).

**2. Risk neutral modeling.** Common approaches to the valuation of commodity derivatives (see [7]) are based on the assumptions that the commodity trading takes place continuously in time without transaction costs and taxes and that no arbitrage exists for all commodity-related trading strategies. In the class of *spot price models*, the evolution  $(S_t)_{t \in [0, T]}$  of the commodity spot price is described by a diffusion dynamics

$$(2.1) \quad dS_t = S_t((- \mu_t)dt + \sigma_t dW_t)$$

realized on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t, P)_{t \in [0, T]})$  with the prespecified drift  $(\mu_t)_{t \in [0, T]}$  and volatility  $(\sigma_t)_{t \in [0, T]}$ . The process  $(W_t)_{t \in [0, T]}$  stands for a Brownian motion under the so-called spot martingale measure  $Q$ . This measure represents a vehicle for excluding arbitrage opportunities for the trading of commodity-related financial contracts, predominantly of futures. The assumption therefore is that at any time  $t \in [0, \tau] \subset [0, T]$  the price  $E_t(\tau)$  of the futures contract written on the price of a commodity delivered at  $\tau$  is given by the  $Q$ -martingale

$$(2.2) \quad E_t(\tau) = E^Q(S_\tau | \mathcal{F}_t) \quad \text{for all } t \in [0, \tau] \text{ for each futures maturity } \tau \in [0, T]$$

whose terminal value equals the spot price  $S_\tau$ . Beyond this property, futures dynamics has to fulfill a series of reasonable assumptions. First of all, the dynamics (2.1), (2.2) has to be consistent with the futures curve  $(E_0^*(\tau_i))_{i=0}^{n+1}$  initially observed at the market in the sense that  $E(S_{\tau_i} | \mathcal{F}_0) = E_0^*(\tau_i)$  for all listed maturity dates  $\tau_0, \dots, \tau_{n+1}$ . Next, one has to ensure a certain flexibility of the futures curve, at least in terms of the feasibility for changes between backwardation and contango, which evidently occur in commodity markets. Moreover, some authors have argued that the correct choice of the spot price process has to reflect the frequently noticed mean-reverting property. However, we believe that this observation is disputable since there is no obvious reason why a risk neutral dynamics must inherit the statistical properties evident from the perspective of the objective measure. Overall, the correct choice of the commodity price dynamics turns out to be a challenging task, more so because the following storability requirement needs to be considered:

$$(2.3) \quad \begin{array}{l} \text{Given a storage cost structure, the dynamics} \\ (2.2) \text{ should exclude arbitrage opportunities} \\ \text{for futures and physical commodity trading.} \end{array}$$

Our approach aims to give an appropriate implementation of this principle such that particular commodities may be distinguished by a single parameter which stands for their specific storage costs.

In our approach, we utilize a connection between commodity and money market models (see [8]), which needs to be briefly outlined next. Given the dynamics (2.1), the diffusion parameter  $(\sigma_t)_{t \in [0, T]}$  obviously reflects the fluctuation of the spot price, whereas the drift term

$(\mu_t)_{t \in [0, T]}$  needs to be adjusted accordingly, in order to match the observed initial futures curve  $(E_0^*(\tau_i))_{i=0}^{n+1}$  and to reflect some of its typical changes. It turns out that by an appropriate change of measure these questions can be naturally carried out in the framework of short rate models. Namely, observe that the solution

$$S_\tau = S_0 e^{-\int_0^\tau \mu_s ds} e^{\int_0^\tau \sigma_s dW_s - \frac{1}{2} \int_0^\tau \sigma_s^2 ds}, \quad \tau \in [0, T],$$

to (2.1) satisfies

$$S_\tau = S_t e^{-\int_t^\tau \mu_s ds} \Lambda_\tau \Lambda_t^{-1}, \quad 0 \leq t \leq \tau \leq T,$$

where, under appropriate assumptions on  $(\sigma_t)_{t \in [0, T]}$ , the martingale

$$\Lambda_\tau = e^{\int_0^\tau \sigma_s dW_s - \frac{1}{2} \int_0^\tau \sigma_s^2 ds}, \quad \tau \in [0, T],$$

provides a measure change to a probability measure  $\tilde{Q}$  which is equivalent to  $Q$  and is given by

$$d\tilde{Q} = \Lambda_T dQ.$$

Using the measure  $\tilde{Q}$ , we obtain

$$E_t(\tau) = \mathbb{E}_t^Q(S_\tau) = \mathbb{E}_t^Q(S_t e^{-\int_t^\tau \mu_s ds} \Lambda_\tau \Lambda_t^{-1}) = S_t \mathbb{E}_t^{\tilde{Q}}(e^{-\int_t^\tau \mu_s ds})$$

with the proportion between the futures price and the spot price

$$E_t(\tau)/S_t = \mathbb{E}_t^{\tilde{Q}}(e^{-\int_t^\tau \mu_s ds}) =: B_t(\tau), \quad t \leq \tau.$$

Obviously, all desired properties of the futures curve evolution can be addressed in terms of the dynamics of  $(B_t(\tau))_{t \in [0, \tau]}$ ,  $\tau \in [0, T]$ . This observation shows that by modeling  $(\mu_t)_{t \in [0, T]}$  as a short rate of an appropriate interest rate model (with respect to  $\tilde{Q}$ ) one obtains a commodity price model which inherits futures curve properties from the zero bond curve of the underlying interest rate model. More generally, [8] argues that any commodity futures price model can be constructed as

$$E_t(\tau) = S_t B_t(\tau), \quad 0 \leq t \leq \tau \leq T,$$

by a separate realization of spot price  $(S_t)_{t \in [0, T]}$  and an appropriate zero bond  $(B_t(\tau))_{0 \leq t \leq \tau \leq T}$  dynamics. Although such a rigid framework is not ideal for addressing storage cost issues, we utilize an analogy between commodity and money markets and borrow ideas from LIBOR markets to introduce storage cost restrictions into commodity modeling.

**3. A risk neutral approach to storage costs.** Let us agree that a commodity market on the time horizon  $[0, T]$  is modeled by adapted processes

$$(3.1) \quad (S_t)_{t \in [0, T]}, \quad (E_t(\tau_i))_{t \in [0, \tau_i]}, \quad i = 0, \dots, n+1,$$

realized on  $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$  with the interpretation that  $(S_t)_{t \in [0, T]}$  is the spot price process and  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  denotes the price evolution of the futures maturing at  $\tau_i \in \{\tau_0, \dots, \tau_{n+1}\} \subset [0, T]$ . For simplicity, we assume that  $\tau_0 = 0$ ,  $\tau_{n+1} = T$  and that all maturity times differ by a fixed tenor  $\Delta = \tau_{i+1} - \tau_i$  for all  $i = 0, \dots, n$ . We shall agree on the following.

**Definition 3.1.** The price processes (3.1) define a commodity market (which excludes arbitrage for futures trading and simultaneous commodity stock management) if the following conditions are satisfied:

- (C0)  $(S_t)_{t \in [0, T]}$ ,  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  for  $i = 0, \dots, n+1$  are positive-valued processes.
- (C1) There is no financial arbitrage in the sense that there exists a measure  $Q^E$  which is equivalent to  $P$  and such that, for each  $i = 0, \dots, n+1$ , the process  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  follows a martingale with respect to  $Q^E$ .
- (C2) The initial values of the futures price processes  $(E_t(\tau_i))_{t \in [0, \tau_i]}$ ,  $i = 0, \dots, n+1$ , fit the observed futures curve  $(E_0^*(\tau_i))_{i=0}^{n+1} \in ]0, \infty[^{n+1}$ ; i.e., it holds almost surely that  $E_0(\tau_i) = E_0^*(\tau_i)$  for all  $i = 0, \dots, n+1$ .
- (C3) The terminal futures price matches the spot price:  $E_{\tau_i}(\tau_i) = S_{\tau_i}$  for  $i = 0, \dots, n+1$ .
- (C4) There exists  $\kappa > 0$  such that

$$(3.2) \quad E_t(\tau_{i+1}) - \kappa \leq E_t(\tau_i) \quad \text{for all } t \in [0, \tau_i], i = 1, \dots, n.$$

Let us explain why assumption (C4) is a convenient description of storage cost. For any time  $t \leq \tau_i$ , consider the commodity forward prices  $F_t(\tau_i)$ ,  $F_t(\tau_{i+1})$  and the prices  $p_t(\tau_i)$ ,  $p_t(\tau_{i+1})$  of zero bonds (whose face value is normalized to one) maturing at  $\tau_i$  and  $\tau_{i+1}$ , respectively. To exclude arbitrage from physical storage facilities, we derive a relation between these prices and the price  $k_t(\tau_{i+1})$  of a contract which serves as a storage facility for one commodity unit within  $[\tau_i, \tau_{i+1}]$ . Thereby, we assume that the price  $k_t(\tau_{i+1})$  is agreed at time  $t$  and is paid at  $\tau_{i+1}$ . It turns out that to exclude the cash and carry arbitrage the prices must satisfy

$$(3.3) \quad F_t(\tau_{i+1}) - k_t(\tau_i) - \left( \frac{p_t(\tau_i)}{p_t(\tau_{i+1})} - 1 \right) F_t(\tau_i) \leq F_t(\tau_i), \quad t \in [0, \tau_i], \quad i = 1, \dots, n+1.$$

This relation follows from the no-arbitrage assumption by examining a strategy, which fixes the prices at time  $t$ , buys at time  $\tau_i > t$  one commodity unit, stores it within  $[\tau_i, \tau_{i+1}]$ , and sells it at  $\tau_{i+1}$ . Let us investigate in more detail the revenue from such a strategy.

Time	$\tau_i$ -future	$\tau_{i+1}$ -future	Storage	$\tau_i$ -bond	$\tau_{i+1}$ -bond
$t$	1 long	1 short	1 long	$F_t(\tau_i)$ long	$\frac{p_t(\tau_i)}{p_t(\tau_{i+1})} F_t(\tau_i)$ short
$\tau_i$	supply	1 short	store	cash flow $F_t(\tau_i)$	$\frac{p_t(\tau_i)}{p_t(\tau_{i+1})} F_t(\tau_i)$ short
$\tau_{i+1}$	expired	delivery	pay $k_t(\tau_{i+1})$	expired	cash flow $-\frac{p_t(\tau_i)}{p_t(\tau_{i+1})} F_t(\tau_i)$

Obviously, our agent starts with no initial capital since entering forward positions at time  $t$  does not require any cash flow and both bond positions are balanced. Furthermore, the strategy is self-financed. Namely, the capital required to buy one commodity unit at time  $\tau_i$  is financed by a cash flow from the expiring long bond position. At the end of this strategy, the agent requires a capital  $k_t(\tau_{i+1})$  to pay for the storage and  $\frac{p_t(\tau_i)}{p_t(\tau_{i+1})} F_t(\tau_i)$  to close the short bond position. However, our agent earns a revenue  $F_t(\tau_{i+1})$  from the delivery of the stored commodity unit. Hence, the terminal capital is known with certainty in advance, at the initial time  $t$ , and is equal to

$$F_t(\tau_{i+1}) - k_t(\tau_{i+1}) - \frac{p_t(\tau_i)}{p_t(\tau_{i+1})} F_t(\tau_i).$$

In order to exclude arbitrage, we have to suppose that this terminal wealth cannot be positive. Thus, we obtain (3.3). Now, let us elaborate on the approximation (3.2) of (3.3). Consider the cumulative effect

$$(3.4) \quad k_t(\tau_i) + \left( \frac{p_t(\tau_i)}{p_t(\tau_{i+1})} - 1 \right) F_t(\tau_i) \quad \text{for all } t \in [0, \tau_i] \text{ and } i = 1, \dots, n$$

of the storage costs and interest rates. If we propose a model which satisfies

$$(3.5) \quad F_t(\tau_{i+1}) - \kappa \leq F_t(\tau_i) \quad \text{for all } t \in [0, \tau_i] \text{ and } i = 1, \dots, n,$$

then the arbitrage by cash and carry is excluded at least in those situations where (3.4) is bounded from below by the parameter  $\kappa$ . In this context, the accuracy of the estimation

$$(3.6) \quad k_t(\tau_i) + \left( \frac{p_t(\tau_i)}{p_t(\tau_{i+1})} - 1 \right) F_t(\tau_i) \geq \kappa \quad \text{for all } t \in [0, \tau_i] \text{ and } i = 1, \dots, n$$

is critical. Whether such an estimate is possible in practice and whether it yields models which capture storability aspects of the commodity price evolution must be explicitly verified in any particular situation. In any case, we believe that for certain commodities the left-hand side of (3.6) can be reasonably approximated by a constant and deterministic parameter  $\kappa$ , which justifies the assumption (3.5). Finally, we pass from (3.5) to (3.2) by the approximation of the forward prices  $F_t(\tau_i)$ ,  $F_t(\tau_{i+1})$  by futures prices  $E_t(\tau_i)$ ,  $E_t(\tau_{i+1})$ .

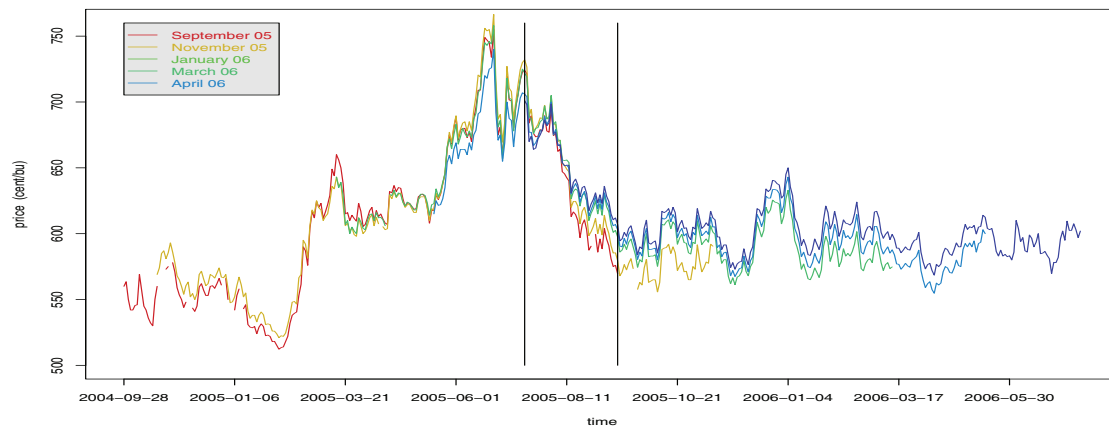
**4. Storage costs as contango limit.** In principle,  $\kappa$  can be estimated from the actual physical storage costs and the interest rate effects. Consequently, for certain commodities there exists a maximally possible steepness of the futures curve in contango situations, which is known among traders as the *contango limit*. That is, such a rough estimate of  $\kappa$  could also be obtained from historical data by inspecting the maximal increase of the historical futures curves

$$\kappa = \max\{E_t(\tau_{i+1})(\omega) - E_t(\tau_i)(\omega) : \text{for all } t \leq \tau_i < \tau_{i+1}\}$$

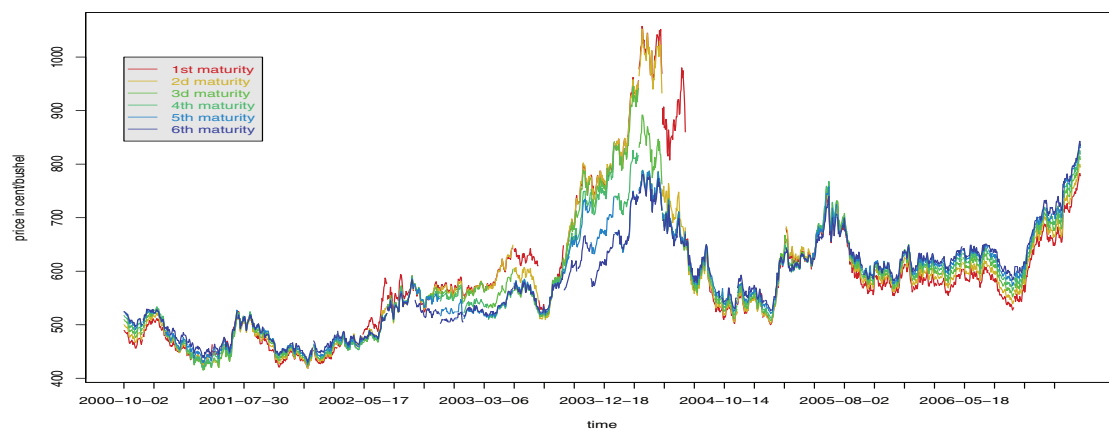
based on a representative data record. Let us illustrate this method.

Consider the history of soybean trading at the Chicago Board of Trade (CBOT). At this exchange, the soybean futures expire in January, March, May, July, August, September, and November. Each contract is listed one year prior to expiry. Let us suppose a fixed tenor  $\Delta$  of two months. Thus, all prices of futures maturing in August are not considered. Within each period  $[\tau_{i-1}, \tau_i]$ , six futures prices with delivery dates  $\tau_i, \tau_{i+1}, \dots, \tau_{i+5}$  are available. Figure 1 shows a typical price evolution of six futures and the period where all six contracts are listed. Moreover, Figure 2 illustrates the entire data set we use in this study. It encompasses the end of the day futures prices ranging from 2000-10-02 to 2007-02-23. Figure 3 shows the behavior of the difference of consecutive contracts in the entire data record. Note that this picture clearly supports our viewpoint since there is a clear contango limit, represented by a price which has never been hit by the difference  $E_t(\tau_{i+1}) - E_t(\tau_i)$ . At the same time, there is no limitation on the backwardation side since the differences  $E_t(\tau_{i+1}) - E_t(\tau_i)$  tilt seemingly arbitrarily far downwards. To estimate the storage cost parameter, we use the historical data depicted in Figure 2 and calculate  $\kappa$  by

$$(4.1) \quad \max\{(E_t(\tau_{i+1}) - E_t(\tau_i))(\omega) : t, \tau_i, \tau_{i+1} \text{ where price observations are available}\}$$



**Figure 1.** The price evolution of six consecutive futures contracts. Vertical lines separate a two month period where all six contracts are traded.

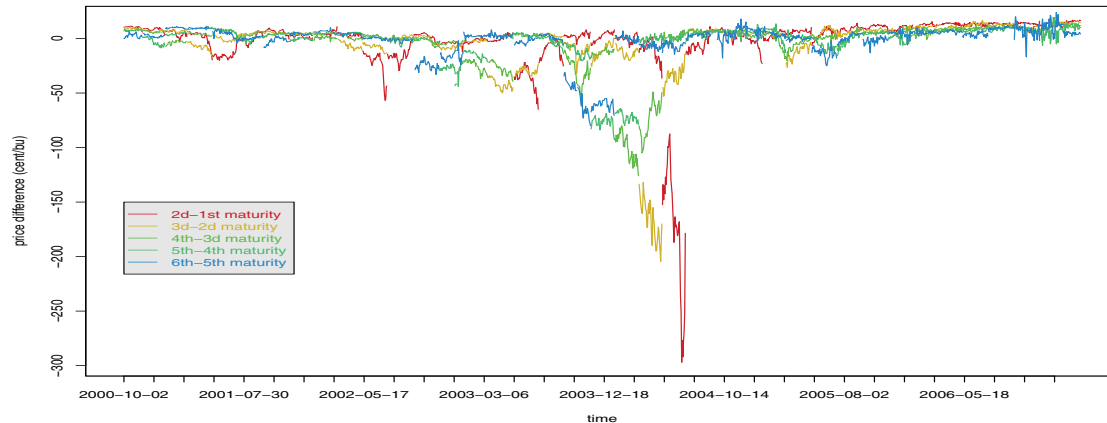


**Figure 2.** Soybean closing daily prices from CBOT in cents per bushel.

giving 24 US cents per bushel for two months. Thus, setting  $\kappa \approx 26$  could give a reasonable futures price model which excludes cash and carry arbitrage for soybeans. Still, there is no guarantee why the difference  $E_t(\tau_{i+1}) - E_t(\tau_i)$  in a future trajectory does not exceed 26. As discussed before, a reliable estimation of  $\kappa$  should be based on a study of storage costs and on bond prices. However, we believe that (4.1) may serve as a reasonable approximation.

Not surprisingly, similar analysis on other commodities shows that a clear historical contango limit can also be observed for other storable agricultural products and for precious metals but not for assets with limited storability (oil, gas, and electricity) and for perishable goods (like livestock).





**Figure 3.** The difference  $E_t(\tau_{i+1}) - E_t(\tau_i)$  shows an upper bound at 24 US cents per bushel for two months.

**5. Modeling commodity dynamics.** This section is devoted to the construction of commodity markets which satisfy the axioms formulated in Definition 3.1. Here the main task is to establish a dynamics for martingales  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  ( $i = 1, \dots, n+1$ ) which obeys the storage restriction (C4) and, at the same time, possesses a certain flexibility in the movements of the futures curve. Fortunately, similar problems occurred in the theory of fixed income markets and have been treated successfully. As a paradigm, we use the forward LIBOR market model, also known as the BGM approach, named after A. Brace, D. Gatarek, and M. Musiela. In their context, the dynamics of zero bonds  $(p_t(\tau_i))_{t \in [0, \tau_i]}$  with the fixed tenor  $\Delta = \tau_{i+1} - \tau_i$ ,  $i = 1, \dots, n$ , is described in terms of the so-called simple rates  $(L_t(\tau_i))_{t \in [0, \tau_i]}$  defined by

$$(5.1) \quad p_t(\tau_{i+1}) = \frac{p_t(\tau_i)}{1 + \Delta L_t(\tau_i)} \quad \text{for } i = 1, \dots, n, \quad t \in [0, \tau_i],$$

whose dynamics is modeled by stochastic differential equations

$$(5.2) \quad dL_t(\tau_i) = L_t(\tau_i)(\beta_t(\tau_i)dt + \gamma_t(\tau_i)dW_t), \quad i = 1, \dots, n,$$

where the deterministic volatilities  $(\gamma_t(\tau_i))_{t \in [0, \tau_i]}$  are freely chosen for  $i = 1, \dots, n$ , whereas the drifts  $(\beta_t(\tau_i))_{t \in [0, \tau_i]}$  for  $i = 1, \dots, n$  are determined by this choice. The importance of the BGM formulation is that each simple rate  $(L_t(\tau_i))_{t \in [0, \tau_i]}$  follows a geometric Brownian motion with respect to the forward measure corresponding to the numeraire  $(p_t(\tau_{i+1}))_{t \in [0, \tau_i]}$ . This fact yields explicit formulae for Caplets and therefore provides an important tool, *implicit calibration*, for this fixed-income model class. We suggest transferring this successful concept to commodity markets by proposing a similar framework, where futures are consecutively inter-related by stochastic exponentials, similarly to simple rates in (5.1). As in the BGM setting, it turns out that this concept provides appropriate tools for the implicit model calibration. The idea is to link the dynamics  $(E_t(\tau_i), E_t(\tau_{i+1}))_{t \in [0, \tau_i]}$  by

$$(5.3) \quad E_t(\tau_{i+1}) = \frac{E_t(\tau_i) + \kappa}{1 + Z_t(\tau_i)}, \quad t \in [0, \tau_i], \quad i = 1, \dots, n,$$



where, likewise to the simple rate (5.2), the *simple ratio*  $(Z_t(\tau_i))_{t \in [0, \tau_i]}$  follows a diffusion

$$(5.4) \quad dZ_t(\tau_i) = Z_t(\tau_i)(\alpha_t(\tau_i)dt + \sigma_t(\tau_i)dW_t), \quad t \in [0, \tau_i], \quad i = 1, \dots, n,$$

where  $(\sigma_t(\tau_i))_{t \in [0, \tau_i]}$  and  $(\alpha_t(\tau_i))_{t \in [0, \tau_i]}$  denote the volatilities and the drifts, respectively. We will see later that the drifts follow from the choice of simple ratio volatilities and other model ingredients. Before entering the details of the construction, let us emphasize that (5.3) indeed ensures (3.2) by the nonnegativity of  $(Z_t(\tau_i))_{t \in [0, \tau_i]}$ , which is a consequence of (5.4), under appropriate conditions.

Now, let us construct a model which fulfills the axioms from Definition 3.1. We begin with a complete filtered probability space  $(\Omega, \mathcal{F}, Q^E, (\mathcal{F}_t)_{t \in [0, T]})$  where the filtration is the augmentation (by the null sets in  $\mathcal{F}_T^W$ ) of the filtration  $(\mathcal{F}_t^W)_{t \in [0, T]}$  generated by the  $d$ -dimensional Brownian motion  $(W_t)_{t \in [0, T]}$ . All processes are supposed to be progressively measurable. For equidistant futures maturity dates

$$0 = \tau_0 < \tau_1 < \dots < \tau_{n+1} = T \in [0, T], \quad \Delta = \tau_{i+1} - \tau_i, \quad i = 0, \dots, n,$$

and the initial futures curve  $(E_0^*(\tau_i))_{i=1}^{n+1} \in ]0, \infty[^{n+1}$ , we construct a commodity market where futures prices follow

$$(5.5) \quad dE_t(\tau_i) = E_t(\tau_i)\Sigma_t(\tau_i)dW_t, \quad t \in [0, \tau_i], \quad E_0(\tau_i) = E_0^*(\tau_i), \quad i = 1, \dots, n+1,$$

and obey (C0)–(C4) with a given storage cost parameter  $\kappa > 0$ . In a separate section, we discuss how the volatility term structure

$$(\Sigma_t(\tau_i))_{t \in [0, \tau_i]}, \quad i = 1, \dots, n+1,$$

and its dimension  $d \in \mathbb{N}$  are determined from a model calibration procedure.

First, we outline the intuition behind our construction. Given the local  $Q^E$ -martingale  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  as in (5.5), the Itô formula shows how, given  $(\sigma_t(\tau_i))_{t \in [0, \tau_i]}$ , to settle the drift  $(\alpha_t(\tau_i))_{t \in [0, \tau_i]}$  in (5.4) such that (5.3) becomes a martingale. With this principle, we construct  $(E_t(\tau_{i+1}))_{t \in [0, \tau_i]}$  from given  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  and  $(\sigma_t(\tau_i))_{t \in [0, \tau_i]}$ . To proceed, we need to extend this price process by  $(E_t(\tau_{i+1}))_{t \in [\tau_i, \tau_{i+1}]}$  to the expiry date. This is effected by

$$dE_t(\tau_{i+1}) = E_t(\tau_{i+1})\Sigma_t(\tau_{i+1})dW_t, \quad t \in [\tau_i, \tau_{i+1}],$$

where the volatility in front of delivery  $(\Sigma_t(\tau_{i+1}))_{t \in [\tau_i, \tau_{i+1}]}$  is exogenously given by

$$(5.6) \quad \Sigma_t(\tau_{i+1}) := \psi_t \quad \text{for all } t \in [\tau_i, \tau_{i+1}], \quad i = 0, \dots, n,$$

with a prespecified process  $(\psi_t)_{t \in [0, T]}$ . Having thus established  $(E_t(\tau_{i+1}))_{t \in [0, \tau_{i+1}]}$ , the same procedure is applied iteratively to determine all remaining futures  $(E_t(\tau_{j+1}))_{t \in [0, \tau_{j+1}]}$  with  $j = i+1, \dots, n$ .

In the following lemma, we call a  $d$ -dimensional process  $(X_t)_{t \in [0, \tau]}$  bounded if  $\|X_t\| < C$  holds for all  $t \in [0, \tau]$  almost surely, for some  $C \in [0, \infty[$ .

**Lemma 5.1.** *Let  $(E_t)_{t \in [0, \tau]}$  be a positive-valued martingale following  $dE_t = E_t \Sigma_t dW_t$  with a bounded volatility process  $(\Sigma_t)_{t \in [0, \tau]}$ . If  $(\sigma_t)_{t \in [0, \tau]}$  is bounded, then there exists a unique strong solution to*

$$(5.7) \quad \frac{dZ_t}{Z_t} = -\sigma_t \left( \left( \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1} \right) dt - dW_t \right), \quad Z_0 := Z_0^* > 0.$$

Moreover,

$$(5.8) \quad E'_t = \frac{E_t + \kappa}{1 + Z_t}, \quad t \in [0, \tau],$$

follows the martingale dynamics

$$(5.9) \quad dE'_t = E'_t \Sigma'_t dW_t, \quad t \in [0, \tau],$$

with bounded

$$(5.10) \quad \Sigma'_t = \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1}, \quad t \in [0, \tau].$$

*Proof.* Write (5.7) as

$$(5.11) \quad dZ_t = F(Z)_t dt + Z_t \sigma_t dW_t, \quad Z_0 = Z_0^* > 0,$$

where the functional  $F$  acts on the processes  $Z = (Z_t)_{t \in [0, \tau]}$  by

$$(5.12) \quad F(Z)_t = -Z_t \sigma_t \left( \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1} \right), \quad t \in [0, \tau].$$

To avoid technical difficulties in (5.7) occurring when the denominator  $Z_t + 1$  vanishes, we first discuss a stochastic differential equation similar to (5.11)

$$(5.13) \quad dZ_t = \tilde{F}(Z)_t dt + Z_t \sigma_t dW_t, \quad Z_0 = Z_0^* > 0,$$

where the functional  $\tilde{F}$  acts by  $\tilde{F}(Z)_t := F(Z)_t 1_{\{Z_t \geq 0\}}$  for  $t \in [0, \tau]$  on each process  $Z = (Z_t)_{t \in [0, \tau]}$ . Since  $\sup_{t \in [0, \tau]} \|\sigma_t\| \leq C \in [0, \infty[$  by assumption, the diffusion term in (5.13) is Lipschitz continuous:

$$\|Z_t \sigma_t - Z'_t \sigma_t\| \leq C |Z_t - Z'_t| \quad \text{for all } t \in [0, \tau].$$

Thus, to ensure the existence and uniqueness of the strong solution to (5.13) it suffices to verify the Lipschitz continuity of  $\tilde{F}$  in the sense that there exists  $\tilde{C} \in [0, \infty[$  such that

$$(5.14) \quad \|\tilde{F}(Z)_t - \tilde{F}(Z')_t\| \leq \tilde{C} |Z_t - Z'_t| \quad \text{for all } t \in [0, \tau].$$

The decomposition  $\tilde{F}(Z)_t = \tilde{f}_1(t, Z_t) + \tilde{f}_2(t, Z_t) + \tilde{f}_3(t, Z_t)$  with

$$\begin{aligned} \tilde{f}_1(t, z) &= -1_{\{z \geq 0\}} z \frac{E_t}{E_t + \kappa} \sigma_t \Sigma_t, \\ \tilde{f}_2(t, z) &= -1_{\{z \geq 0\}} z \frac{1}{z + 1} \sigma_t \sigma_t, \\ \tilde{f}_3(t, z) &= 1_{\{z \geq 0\}} z \sigma_t \sigma_t \end{aligned}$$

for all  $t \in [0, \tau]$ ,  $z \geq 0$  shows that  $\tilde{C} \geq \sup_{t \in [0, \tau]} (|\sigma_t \Sigma_t| + 2|\sigma_t \sigma_t|)$  yields a Lipschitz constant in (5.14); here  $\tilde{C} \in [0, \infty[$  holds since both  $(\Sigma_t)_{t \in [0, \tau]}$  and  $(\sigma_t)_{t \in [0, \tau]}$  are bounded processes by assumption.

Let  $(Z_t)_{t \in [0, \tau]}$  be the unique strong solution to (5.13). In order to show that this process also solves (5.7), it suffices to verify that  $Z_t \in ]0, \infty[$  holds almost surely for all  $t \in [0, \tau]$ . Indeed, the positivity follows from the stochastic exponential form

$$dZ_t = Z_t \left( \frac{\tilde{F}(Z)_t}{Z_t} dt + \sigma_t dW_t \right), \quad Z_0 = Z_0^* > 0,$$

with bounded drift coefficient

$$(5.15) \quad \frac{\tilde{F}(Z)_t}{Z_t} = -\sigma_t \left( \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1} \right) 1_{\{Z_t \geq 0\}}, \quad t \in [0, \tau].$$

To show the uniqueness, we argue that any solution  $(Z'_t)_{t \in [0, \tau]}$  to (5.11) coincides with  $(Z_t)_{t \in [0, \tau]}$  on the stochastic interval prior the first entrance time of  $(Z'_t)_{t \in [0, \tau]}$  into  $]-\infty, 0]$  since on this interval  $(Z'_t)_{t \in [0, \tau]}$  solves (5.13). Furthermore,  $(Z'_t)_{t \in [0, \tau]}$  is a continuous process, being a strong solution to (5.11) by assumption. The continuity of  $(Z'_t)_{t \in [0, \tau]}$  shows that  $(Z'_t)_{t \in [0, \tau]}$  matches  $(Z_t)_{t \in [0, \tau]}$  on the entire interval  $[0, \tau]$ . Finally, (5.9) is verified by a straightforward application of the Itô formula. ■

Using the common stopping technique, Lemma 5.1 extends in a straightforward way from bounded to continuous processes.

**Proposition 5.2.** *Let  $(E_t)_{t \in [0, \tau]}$  be a positive-valued martingale following  $dE_t = E_t \Sigma_t dW_t$  with a continuous volatility process  $(\Sigma_t)_{t \in [0, \tau]}$ . If  $(\sigma_t)_{t \in [0, \tau]}$  is continuous, then there exists a unique strong solution to*

$$(5.16) \quad \frac{dZ_t}{Z_t} = -\sigma_t \left( \left( \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1} \right) dt - dW_t \right), \quad Z_0 := Z_0^* > 0.$$

Moreover,

$$(5.17) \quad E'_t = \frac{E_t + \kappa}{1 + Z_t}, \quad t \in [0, \tau],$$

follows the martingale dynamics

$$(5.18) \quad dE'_t = E'_t \Sigma'_t dW_t, \quad t \in [0, \tau],$$

with continuous

$$(5.19) \quad \Sigma'_t = \frac{E_t \Sigma_t}{E_t + \kappa} - \frac{Z_t \sigma_t}{Z_t + 1}, \quad t \in [0, \tau].$$

*Proof.* Introduce a sequence of stopping times

$$\vartheta_k = \inf\{t \in [0, \tau] : \max(\|\sigma_t\|, \|\Sigma_t\|) \geq k\}, \quad k \in \mathbb{N}.$$

Since both processes  $(\sigma_t)_{t \in [0, \tau]}$  and  $(\Sigma_t)_{t \in [0, \tau]}$  are continuous, we have  $\lim_{k \rightarrow \infty} \vartheta_k = \tau$ ; hence the monotonically increasing sequence of stochastic intervals  $[0, \vartheta_k]$ ,  $k \in \mathbb{N}$ , covers the entire time horizon:

$$(5.20) \quad \bigcup_{k \in \mathbb{N}} [0, \vartheta_k] = \Omega \times [0, \tau].$$

Now, by stopping  $(\sigma_t)_{t \in [0, \tau]}$  and  $(\Sigma_t)_{t \in [0, \tau]}$  at the time  $\vartheta_k$ , one obtains bounded processes

$$(\sigma_t^{(k)} := \sigma_{t \wedge \tau_k})_{t \in [0, \tau]}, \quad (\Sigma_t^{(k)} := \Sigma_{t \wedge \tau_k})_{t \in [0, \tau]}.$$

Further, define  $(E_t^{(k)})_{[0, \tau]}$  as the solution to

$$dE_t^{(k)} = E_t^{(k)} \Sigma_t^{(k)} dW_t, \quad E_0^{(k)} = E_0.$$

It follows that for each  $k \in \mathbb{N}$  the processes  $(E_t^{(k)})_{t \in [0, \tau]}$ ,  $(\Sigma_t^{(k)})_{t \in [0, \tau]}$ , and  $(\sigma_t^{(k)})_{t \in [0, \tau]}$  satisfy the assumptions of Lemma 5.1, which yields the corresponding processes  $(Z_t^{(k)})_{t \in [0, \tau]}$ ,  $(E_t'^{(k)})_{t \in [0, \tau]}$ , and  $(\Sigma_t'^{(k)})_{t \in [0, \tau]}$ . By construction, the next set of processes coincides with the previous one on the common stochastic interval

$$\left. \begin{aligned} Z_t^{(k)}(\omega) &= Z_t^{(k+1)}(\omega) \\ \Sigma_t'^{(k)}(\omega) &= \Sigma_t'^{(k+1)}(\omega) \\ E_t'^{(k)}(\omega) &= E_t'^{(k+1)}(\omega) \end{aligned} \right\} \quad \text{for all } t \in [0, \vartheta_k(\omega)], \omega \in \Omega, k \in \mathbb{N}.$$

Thus, their limits

$$Z_t := \lim_{k \rightarrow \infty} Z_t^{(k)}, \quad \Sigma_t' := \lim_{k \rightarrow \infty} \Sigma_t'^{(k)}, \quad E_t' := \lim_{k \rightarrow \infty} E_t'^{(k)}, \quad t \in [0, \tau],$$

are well defined on  $[0, \tau]$  because of (5.20) and satisfy the assertions (5.16)–(5.19). ■

*Remark.* For later use, let us point out that the volatility process  $(\Sigma_t')_{t \in [0, \tau]}$  resulting from (5.19) can be written as a function

$$(5.21) \quad \Sigma_t' = s(E_t', E_t, \Sigma_t, \sigma_t) := \frac{E_t \Sigma_t + E_t' \sigma_t}{E_t + \kappa} - \sigma_t, \quad t \in [0, \tau].$$

The representation (5.21) follows directly from (5.19) by using

$$\frac{Z_t}{Z_t + 1} = 1 - \frac{1}{1 + Z_t} = 1 - \frac{E_t'}{E_t + \kappa}, \quad t \in [0, \tau],$$

where the last equality is a consequence of (5.17). Let us point out that in the limiting case, where the contango limit is almost reached, i.e.,  $E_t' \approx E_t + \kappa$ , the volatility  $\Sigma_t'$  can be approximated as

$$\Sigma_t' \approx \frac{E_t \Sigma_t}{E_t + \kappa} \approx \frac{E_t \Sigma_t}{E_t'};$$

thus the dynamics  $(E'_t)_{t \in [0, T]}$  follows that of  $(E_t)_{t \in [0, T]}$ , because of

$$dE'_t = E'_t \Sigma'_t dW_t \approx E'_t \frac{E_t \Sigma_t}{E'_t} dW_t = E_t \Sigma_t dW_t = dE_t.$$

This observation shows that the restriction  $E_t + \kappa \geq E'_t$  necessarily causes strong correlations of the process increments if the prices  $E_t$  and  $E'_t$  come close to the contango limit. On this account the sensitivity of the model to the choice of storage cost parameter  $\kappa$  can be significant. Now consider the entire construction of futures prices. Starting with

$$(5.22) \quad \begin{cases} E_0^*(\tau_i) \in ]0, \infty[ \text{ for } i = 1, \dots, n+1, \text{ initial futures curve,} \\ (\psi_t)_{t \in [0, T]}, \text{ in-front-of-maturity futures volatility (continuous),} \\ (\sigma_t(\tau_i))_{t \in [0, \tau_i]}, i = 1, \dots, n, \text{ simple ratio volatilities (continuous),} \end{cases}$$

we apply the following procedure.

*Initialization.* Start with  $E_0(\tau_1) = E_0^*(\tau_1), \dots, E_0(\tau_{n+1}) = E_0^*(\tau_{n+1})$ .

*Recursion.* Given initial values  $E_{\tau_{i-1}}(\tau_i), \dots, E_{\tau_{i-1}}(\tau_{n+1})$ , define

$$E_t(\tau_i), \dots, E_t(\tau_{n+1}) \quad \text{for all } t \in [\tau_{i-1}, \tau_i]$$

successively for all  $i = 1, \dots, n$  by the following recursive procedure started at  $i := 1$ :

(i) Extend the next maturing futures price to its delivery date  $\tau_i$  by

$$(5.23) \quad \begin{aligned} \Sigma_t(\tau_i) &= \psi_t, \\ dE_t(\tau_i) &= E_t(\tau_i) \Sigma_t(\tau_i) dW_t, \end{aligned} \quad t \in [\tau_{i-1}, \tau_i];$$

then proceed with the other futures.

(ii) For  $j = i, \dots, n$ , starting with the initial condition

$$Z_{\tau_{i-1}}(\tau_j) = \frac{E_{\tau_{i-1}}(\tau_j) + \kappa}{E_{\tau_{i-1}}(\tau_{j+1})} - 1$$

solve the stochastic differential equation

$$\frac{dZ_t(\tau_j)}{Z_t(\tau_j)} = -\sigma_t(\tau_j) \left( \frac{E_t(\tau_j) \Sigma_t(\tau_j)}{E_t(\tau_j) + \kappa} - \frac{Z_t(\tau_j) \sigma_t(\tau_j)}{Z_t(\tau_j) + 1} \right) dt + \sigma_t(\tau_j) dW_t$$

for  $t \in [\tau_{i-1}, \tau_i]$  and define for all  $t \in [\tau_{i-1}, \tau_i]$

$$\begin{aligned} \Sigma_t(\tau_{j+1}) &= \frac{E_t(\tau_j) \Sigma_t(\tau_j)}{E_t(\tau_j) + \kappa} - \frac{Z_t(\tau_j) \sigma_t(\tau_j)}{Z_t(\tau_j) + 1}, \\ E_t(\tau_{j+1}) &= \frac{E_t(\tau_j) + \kappa}{1 + Z_t(\tau_j)}. \end{aligned}$$

(iii) If  $i < n+1$ , we set  $i := i+1$  and proceed with the recursion onto  $[\tau_i, \tau_{i+1}]$ ; otherwise we finish the loop.

To see that this procedure is well defined, we apply the results of Lemma 5.1 to the recursion step replacing  $E_t$ ,  $\Sigma_t$ ,  $\sigma_t$ ,  $Z_t$  by  $E_t(\tau_i)$ ,  $\Sigma_t(\tau_i)$ ,  $\sigma_t(\tau_i)$ ,  $Z_t(\tau_i)$ , respectively. The presented construction yields futures prices  $(E_t(\tau_i))_{t \in [0, \tau_i]}$  for  $i = 1, \dots, n+1$  which obviously satisfy (C0), (C1), (C2), (C4) from Definition 3.1. The last requirement (C3) follows from

$$E_t(\tau_{i+1}) - \kappa < E_t(\tau_i) \iff Z_t(\tau_i) > 0,$$

where the existence of  $Z_t(\tau_i) \geq 0$  for all  $t \in [0, \tau_i]$  is ensured by the assumptions (5.22) and Lemma 5.1. Note that we do not consider (C3) since the spot price is not covered by the above construction. If required, the spot price can be constructed in accordance with assumption (C3). (See (5.24) below.)

*Remark.* Let us elaborate on simplifying assumptions, which we adopted in the present approach in order to highlight the limitations and possible extensions of the model. First, our framework is easily extendable to a nonequidistant maturity grid. By assuming that the tenors  $\Delta_i := \tau_{i+1} - \tau_i$  depend on  $i = 0, \dots, n$ , we have to introduce different storage costs  $(\kappa_i)_{i=0}^n$ , supposing that each  $\kappa_i$  is valid for the corresponding interval  $[\tau_i, \tau_{i+1}]$ . At this point, let us mention that our construction also works if the parameter  $\kappa_i$  is random, provided that it is known with certainty prior to  $\tau_i$ , just before the beginning of corresponding interval  $[\tau_i, \tau_{i+1}]$ . Next, note that we address the futures prices directly, without reference to the spot price. More precisely, the spot price occurs from the construction merely at the grid points  $(\tau_i)_{i=1}^{n+1}$ , being the terminal futures price  $S_{\tau_i} = E_{\tau_i}(\tau_i)$ ,  $i = 1, \dots, n+1$ . Since the spot price is not quoted for most commodities, we believe this gives almost no limitation of model applicability. However, if for some reason a spot price model is required, then the prices  $(S_{\tau_i})_{i=1}^{n+1}$  can be interpolated accordingly, for instance, linearly:

$$(5.24) \quad S_t = \frac{t - \tau_i}{\tau_{i+1} - \tau_i} E_t(\tau_{i+1}) + \frac{\tau_{i+1} - t}{\tau_{i+1} - \tau_i} E_{\tau_i}(\tau_i), \quad t \in [\tau_i, \tau_{i+1}], \quad i = 1, \dots, n.$$

Such a choice comes close to the frequently used approximation of the spot price by the price of the futures contract with nearest maturity. Finally, let us mention that by the reconstruction of the spot price through interpolation the model can be extended towards a continuous system of maturities by defining futures price evolution

$$E_t(\tau) = \mathbb{E}^{Q^E}(S_\tau | \mathcal{F}_t) \quad \text{for all } t \in [0, \tau], \tau \in [0, T].$$

Obviously, such an extension provides at any time  $t$  a futures curve  $(E_t(\tau))_{\tau \in [t, T]}$  which is given by the underlying discrete curve  $(E_t(\tau_i))_{\tau_i \geq t}$  interpolated by the procedure used in the construction of the spot price. In particular, the linear interpolation (5.24) yields a continuous and piecewise linear futures curve which respects the same contango limit as the discrete model.

Finally, we emphasize that one of the main advantages of our model is a perfect time consistency of the futures curve evolution, whereas to the best of our knowledge all approaches in commodity modeling existing so far suffer from an inconsistency. For instance, defined by few parameters, common spot price based commodity models (see [10]) are not able to match an arbitrary initial futures curve. From this perspective, the futures price based models (discussed in, among others, [4], [8]) are more appropriate since they provide an exact fit

to the futures curve at the beginning. However, starting from such an initial curve, the model-based futures curve evolution is in general not able to capture the real-world futures curve change. This does not occur in our model since we describe a finite number of futures contracts, in accordance with the common market practice. Namely, starting from the initial curve  $(E_0(\tau_i))_{i=1}^{n+1}$ , the futures curve  $(E_t(\tau_i))_{i=1}^{n+1}$  at a later time  $t \in [\tau_0, \tau_1]$  can be arbitrary, with the only restriction that it respect the contango limit. To see this, observe that the distribution of

$$E_t(\tau_1), \quad \frac{E_t(\tau_2)}{E_t(\tau_1) + \kappa} = \frac{1}{Z_t(\tau_1) + 1}, \quad \dots, \quad \frac{E_t(\tau_{n+1})}{E_t(\tau_n) + \kappa} = \frac{1}{Z_t(\tau_n) + 1}$$

is equivalent to the Lebesgue measure on  $]0, \infty[ \times ]0, 1[^n$ , which is ensured by our construction of the next to maturity future and of the simple ratio processes from geometric Brownian motions, using appropriate volatility structures.

**6. Model calibration.** This section is devoted to the calibration of the parameters of our model. In the case that an appropriate type of calendar spread is actively traded on the market, an implicit calibration is possible. Otherwise one has to rely on a historical calibration based on principal component analysis. Alternatively, an approximation of calendar spread option prices which is described in section 8 could also be used for an implicit calibration.

*Implicit calibration.* As mentioned previously, the model inherits the implied calibration features from the BGM paradigm. Namely, for the case where interest rates and simple ratio volatilities are deterministic, the fair prices  $(C_s)_{s \in [0, t]}$  of the calendar spread option maturing at  $t$  with the terminal payoff

$$(6.1) \quad C_t = (E_t(\tau_i) + \kappa - (1 + K)E_t(\tau_{i+1}))^+ \quad (t \leq \tau_i < \tau_{i+1})$$

are given by

$$(6.2) \quad C_s = e^{-r(t-s)} E_s(\tau_{i+1}) \text{BS} \left( \frac{E_s(\tau_i) + \kappa - E_s(\tau_{i+1})}{E_s(\tau_{i+1})}, K, t, s, 0, \frac{D(s, t, \tau_i)}{\sqrt{t-s}} \right), \quad s \in [0, t],$$

where  $\text{BS}(x, k, t, s, \rho, v)$  stands for the standard Black-Scholes formula

$$\begin{aligned} \text{BS}(x, k, t, s, \rho, v) &:= x\mathcal{N}(d_+) - e^{-\rho(t-s)} k\mathcal{N}(d_-), \\ d_+ &= \frac{1}{v\sqrt{t-s}} \left[ \log \left( \frac{x}{k} \right) + \left( \rho + \frac{1}{2}v^2 \right) (t-s) \right], \\ d_- &= d_+ - v\sqrt{t-s} \end{aligned}$$

and  $D(s, t, \tau_i) = \int_s^t \|\sigma_u(\tau_i)\|^2 du$ . Consider the measure  $Q^{\tau_{i+1}}$ , given by

$$dQ^{\tau_{i+1}} = \frac{E_{\tau_{i+1}}(\tau_{i+1})}{E_0(\tau_{i+1})} dQ^E.$$

By the change of measure technique, the process

$$Z_t(\tau_i) = \frac{E_t(\tau_i) + \kappa - E_t(\tau_{i+1})}{E_t(\tau_{i+1})}, \quad t \in [0, \tau_i],$$



follows a martingale with respect to  $Q^{\tau_{i+1}}$  with stochastic differential

$$dZ_t(\tau_i) = Z_t(\tau_i)\sigma_t(\tau_i)dW_t^{\tau_{i+1}}$$

driven by the process  $(W_t^{\tau_{i+1}})_{t \in [0, \tau_i]}$  of Brownian motion with respect to  $Q^{\tau_{i+1}}$ . Since the simple ratio volatility  $(\sigma_t(\tau_i))_{t \in [0, \tau_i]}$  is deterministic by assumption,  $(Z_t(\tau_i))_{t \in [0, \tau_i]}$  follows a geometric Brownian motion with respect to  $Q^{\tau_{i+1}}$ , which we use to derive

$$\begin{aligned} C_s &= e^{-r(t-s)} \mathbb{E}^{Q^E}((E_t(\tau_i) + \kappa - (1+K)E_t(\tau_{i+1}))^+ | \mathcal{F}_s) \\ &= e^{-r(t-s)} \mathbb{E}^{Q^E} \left( E_{\tau_i}(\tau_{i+1}) \left( \frac{E_t(\tau_i) + \kappa - E_t(\tau_{i+1})}{E_t(\tau_{i+1})} - K \right)^+ | \mathcal{F}_s \right) \\ &= e^{-r(t-s)} E_s(\tau_{i+1}) \mathbb{E}^{Q^{\tau_{i+1}}}((Z_t(\tau_i) - K)^+ | \mathcal{F}_s). \end{aligned}$$

Note that the observation of the implied volatilities through (6.2) yields information on the term structure of the simple ratio volatilities. This kind of implicit calibration is possible if the market lists a sufficient number of calendar spread calls with appropriate parameters  $(K+1)$  and  $\kappa$  as in (6.1). Realistically, one cannot assume that there is always trading in such specific instruments. Hence, the identification of the volatility structure from historical data may become unavoidable.

*Historical calibration.* Next, we present a method for the historical model calibration based on principal component analysis (PCA). This methodology has been applied for calibration of fixed income market models (see [4]) and has been successfully adapted to the estimation of futures volatilities in commodity and energy markets. In general, this technique requires appropriate assumptions on time homogeneity. A typical hypothesis here is that the volatility is time dependent through the time to maturity only. Under this condition, the volatility term structure is identified by measuring the quadratic covariation of appropriate processes. Let us adapt this technique to our case.

In our approach, the model is defined by the volatility processes  $(\psi_t)_{t \in [0, T]}$  and  $(\sigma_t(\tau_i))_{t \in [0, \tau_i]}$ ,  $i = 1, \dots, n$ , giving next to maturity futures prices and the simple ratio processes:

$$\begin{aligned} dE_t(\tau_i) &= E_t(\tau_i)\psi_t dW_t, & t \in [\tau_{i-1}, \tau_i], & i = 1, \dots, n+1, \\ dZ_t(\tau_i) &= Z_t(\tau_i)(\alpha_t(\tau_i) + \sigma_t(\tau_i)dW_t), & t \in [0, \tau_i], & i = 1, \dots, n. \end{aligned}$$

We now consider the following assumption:

$$(6.3) \quad \begin{aligned} &\text{There exist } v^0, v^1, \dots, v^m \in \mathbb{R}^d \text{ such that } \psi_t = v^0 \text{ for all } t \in [0, T] \text{ and} \\ &\sigma_t(\tau_i) = \sum_{k=1}^m v^k 1_{\Delta(k-1), \Delta(k)}(\tau_i - t) \text{ holds for all } t \in [0, T] \text{ and } i = 1, \dots, m. \end{aligned}$$

In other words, (6.3) states that the in-front-of-maturity futures follow constant and deterministic volatility and that the deterministic simple ratio volatility is piecewise constant and time dependent through the time to maturity only. Under the assumption (6.3), the vectors  $v^0, v^1, \dots, v^m \in \mathbb{R}^d$  are recovered from the quadratic covariation

$$(6.4) \quad v^k v^l \Delta = [X^k(i), X^l(i)]_{\tau_i} - [X^k(i), X^l(i)]_{\tau_{i-1}},$$

where the processes  $(X_t^k(i))_{t \in [\tau_{i-1}, \tau_i]}$ ,  $i = 1, \dots, n+1$ ,  $k = 0, \dots, m$ , are given by

$$(6.5) \quad X_t^0(i) := \ln E_t(\tau_i),$$

$$(6.6) \quad X_t^k(i) := \ln Z_t(\tau_{i-1+k}) = \ln \left( \frac{E_t(\tau_{i-1+k}) + \kappa}{E_t(\tau_{i+k})} - 1 \right), \quad k = 1, \dots, m,$$

for  $t \in [\tau_{i-1}, \tau_i]$ . Consider historical futures prices, where within each trading period  $[\tau_{i-1}, \tau_i]$  futures prices for  $m+1$  subsequent maturity dates  $\tau_i, \dots, \tau_{i+m}$  are available. For such data, calculate the observations

$$X_{t_j}^k(i)(\omega), \quad t_j \in ]\tau_{i-1}, \tau_i], \quad k = 0, \dots, m, \quad i = 1, \dots, m,$$

from (6.5), (6.6) at discrete times  $t_j$  where the corresponding futures prices are available. With this data set, approximate the quadratic covariation (6.4) by

$$V^{k,l}(i) = \sum_{(t_j, t_{j+1}) \in ]\tau_{i-1}, \tau_i]^2} (X_{t_{j+1}}^k(i) - X_{t_j}^k(i))(X_{t_{j+1}}^l(i) - X_{t_j}^l(i)), \quad k, l = 0, \dots, m.$$

For a representative history and sufficiently high data frequency, we can suppose that the empirical quadratic covariation

$$(6.7) \quad V^{k,l} := \frac{1}{\Delta(n+1)} \sum_{i=1}^{n+1} V^{k,l}(i), \quad k, l = 0, \dots, m,$$

estimates

$$V^{k,l} = v^k v^l, \quad k, l = 0, \dots, m,$$

the Gram's matrix of  $v^0, \dots, v^m$ . The orthonormal eigenvectors of  $V = (V^{k,l})_{k,l=0}^m$  give the diagonalization  $V = \Phi \Lambda \Phi^\top$  as follows: The columns  $\Phi = (\Phi_{j,k})_{j,k=0}^m$  are given by eigenvectors  $\phi^0, \dots, \phi^m$  and the corresponding eigenvalues  $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_m$  (which we agree to place in descending order) are the diagonal entries of  $\Lambda$ . Following the philosophy of PCA, the model based on

$$(6.8) \quad \hat{v}^k := (\lambda_j^{1/2} \Phi_{j,k})_{j=0}^m, \quad k = 0, \dots, m,$$

in (6.3) (instead of  $v^k$ ) fits the historical observations since the quadratic covariation in (6.4) is correctly reflected due to

$$(6.9) \quad \hat{v}^k \hat{v}^l = V^{k,l} \quad \text{for } k, l = 0, \dots, m.$$

An important problem is whether the observed historical data could be approximatively described by the same type of model but with a reduced number  $d' < d$  of stochastic factors. In other words, the question is whether a model driven by  $d' < d$  Brownian motions would be sufficient to reproduce the observed empirical quadratic covariance. Note that if the user decides to reduce the dimension to  $d' < d$ , then a reasonable approximation of the measured quadratic covariance is attained by dropping  $\lambda_{d'}, \dots, \lambda_m$ , the smallest eigenvalues. In this

case, (6.8) reduces to  $\hat{v}^k := (\lambda_j^{1/2} \Phi_{j,k})_{j=0}^{d'}$  for  $k = 0, \dots, m$  with  $\hat{v}^k \hat{v}^l \approx V^{k,l}$  for  $k, l = 0, \dots, m$  instead of (6.9). Clearly, the dimension reduction is a trade-off between the accuracy of the covariance approximation and the dimension of underlying Brownian motion. For instance, a rule of thumb could be to reduce the dimension to  $d'$  if the remaining factors describe 95% of the covariance:

$$\sum_{j=0}^{d'-1} \lambda_j \geq 0.95 \sum_{j=0}^m \lambda_j.$$

**7. Empirical results.** Let us return to the soybean trading from section 4. There we estimated the historical contango limit  $\kappa = 26$ , which we will use in the following calibration. With this parameter, the realization of the logarithmic simple ratios can be computed and their quadratic covariations can be estimated as explained above.

For the entire soybean data set, from 2000-10-02 to 2007-02-23, the Gram matrix  $V$  from (6.7) is obtained as

$$(7.1) \quad V \approx \begin{bmatrix} 0.06 & -0.01 & 0.00 & 0.02 & 0.04 & 0.04 \\ -0.01 & 1.45 & -0.09 & 0.04 & 0.05 & -0.18 \\ 0.00 & -0.09 & 0.98 & -0.16 & 0.07 & -0.04 \\ 0.02 & 0.04 & -0.16 & 0.96 & -0.40 & 0.10 \\ 0.04 & 0.05 & 0.07 & -0.40 & 2.37 & -1.79 \\ 0.04 & -0.18 & -0.04 & 0.10 & -1.79 & 5.86 \end{bmatrix}.$$

For this symmetric matrix one obtains the following eigenvalue decomposition,

$$(7.2) \quad \Phi \approx \begin{bmatrix} 0.00 & 0.02 & 0.00 & -0.02 & -0.05 & 1.00 \\ -0.04 & -0.18 & 0.96 & 0.21 & -0.01 & 0.01 \\ -0.01 & 0.15 & -0.16 & 0.84 & -0.49 & -0.01 \\ 0.04 & -0.39 & 0.00 & -0.41 & -0.82 & -0.04 \\ -0.39 & 0.81 & 0.19 & -0.28 & -0.27 & -0.04 \\ 0.92 & 0.36 & 0.12 & -0.08 & -0.08 & -0.02 \end{bmatrix},$$

where the orthonormal eigenvectors are displayed as the columns and the corresponding eigenvalues are given by

$$(7.3) \quad \lambda_0 \approx 6.63, \quad \lambda_1 \approx 1.78, \quad \lambda_2 \approx 1.45, \quad \lambda_3 \approx 1.01, \quad \lambda_4 \approx 0.74, \quad \lambda_5 \approx 0.05.$$

As shown in the previous section, the volatility vectors are calculated by (6.8). Based on (7.2) and (7.3) one obtains the following volatility vectors for our soybean market:

$$(7.4) \quad \begin{aligned} v^0 &= [ 0.01 & 0.03 & 0.00 & -0.02 & -0.04 & 0.23 ]^\top, \\ v^1 &= [ -0.09 & -0.24 & 1.16 & 0.21 & -0.01 & 0.00 ]^\top, \\ v^2 &= [ -0.03 & 0.20 & -0.19 & 0.84 & -0.43 & 0.00 ]^\top, \\ v^3 &= [ 0.11 & -0.53 & 0.01 & -0.41 & -0.71 & -0.01 ]^\top, \\ v^4 &= [ -1.00 & 1.08 & 0.23 & -0.28 & -0.23 & -0.01 ]^\top, \\ v^5 &= [ 2.37 & 0.48 & 0.14 & -0.08 & -0.07 & 0.00 ]^\top. \end{aligned}$$

**8. Option pricing.** This section is devoted to the pricing and hedging of European options written on storable underlyings. First, we show how to compute a hedging strategy, based on a partial differential equation. However, as this equation is typically high dimensional, a Monte Carlo simulation could be the appropriate method to price the options in our framework. As the Monte Carlo simulation is straightforward, we do not discuss this method in detail but apply it to examine the risk neutral distribution of the spread option payoff. We find out that this distribution is close to lognormal, which suggests that calendar spread option prices could be approximated by a Black–Scholes-type formula. Here, we build on the model we calibrated in the previous sections. Hence we assume the settings of Lemma 5.1, i.e.,

$$(8.1) \quad \left. \begin{array}{l} \text{the volatilities } (\psi_u)_{u \in [0, T]} \text{ and} \\ (\sigma_u(\tau_i))_{u \in [0, T]} \text{ for all } i = 1, \dots, n+1 \\ \text{are bounded and deterministic} \end{array} \right\}.$$

Consider  $k+1$  futures prices

$$(8.2) \quad (E_u(\tau_i), E_u(\tau_{i+1}), \dots, E_u(\tau_{i+k})), \quad u \in [\tau_{i-1}, \tau_i], \quad i+k \leq n+1,$$

listed within the interval  $[\tau_{i-1}, \tau_i]$ . By construction, all of these processes follow

$$dE_u(\tau_{i+j}) = E_u(\tau_{i+j}) \Sigma_u(\tau_{i+j}) dW_u, \quad u \in [\tau_{i-1}, \tau_i], \quad j = 0, \dots, k,$$

where, according to (5.21), the volatility is given by a function

$$\Sigma_u(\tau_{i+j}) = s^{(j)}(u, E_u(\tau_i), \dots, E_u(\tau_{i+j})), \quad u \in [\tau_{i-1}, \tau_i],$$

which can be calculated recursively:

$$(8.3) \quad \begin{aligned} & \text{(i) if } j = 0, \text{ then } s^{(0)}(u, E_u(\tau_i)) = \psi_u; \\ & \text{(ii) if } j > 0, \text{ then} \\ & s^{(j)}(u, (E_u(\tau_{i+l}))_{l=0}^j) = s(E_u(\tau_{i+j}), E_u(\tau_{i+j-1}), s^{(j-1)}(u, (E_u(\tau_{i+l}))_{l=0}^{j-1}), \sigma_u(\tau_{i+j-1})), \end{aligned}$$

where  $s(\cdot)$  is the function introduced in (5.21).

That is, due to Proposition 5.2, the vector of processes (8.2) follows a unique strong solution to

$$dE_u(\tau_{i+j}) = E_u(\tau_{i+j}) s^{(j)}(u, E_u(\tau_i), \dots, E_u(\tau_{i+j})) dW_u, \quad u \in [\tau_{i-1}, \tau_i], \quad j = 0, \dots, k;$$

hence (8.2) is a Markov process.

Let us now consider the valuation of derivatives, written on commodity futures prices. Given the European option with payoff  $f((E_\tau(\tau_{i+j}))_{j=0}^k)$  at maturity  $\tau \in [\tau_{i-1}, \tau_i]$ , its expected payoff conditioned on time  $t \in [\tau_{i-1}, \tau]$  is given by

$$\mathbb{E}(f((E_\tau(\tau_{i+j}))_{j=0}^k) | \mathcal{F}_t) =: g((E_t(\tau_{i+j}))_{j=0}^k)$$

with an appropriate function  $g(\cdot)$  whose existence follows from the Markov property. Furthermore, Itô's formula shows that this function can be determined as  $g(\cdot) = \phi(t, \cdot)$  from the solution to the partial differential equation

$$(8.4) \quad \frac{\partial}{\partial u} \phi(u, e_0, \dots, e_k) \\ = -\frac{1}{2} \sum_{j,l=0}^k \frac{\partial^2}{\partial e_j \partial e_l} \phi(u, e_0, \dots, e_k) E_u(\tau_{i+j}) E_u(\tau_{i+l}) s^{(j)}(u, e_0, \dots, e_j) s^{(l)}(u, e_0, \dots, e_l)$$

for  $(u, e_0, \dots, e_k) \in ]t, \tau[ \times ]0, \infty[^{k+1}$  subject to the boundary condition

$$\phi(\tau, e_0, \dots, e_k) = f(e_0, \dots, e_k) \quad \text{for } (e_0, \dots, e_k) \in ]0, \infty[^{k+1}.$$

Finally, from the stochastic integral representation one obtains

$$f((E_\tau(\tau_{i+j}))_{j=0}^k) - g((E_t(\tau_{i+j}))_{j=0}^k) = \sum_{j=0}^k \int_t^\tau \frac{\partial}{\partial e_j} \phi(u, E_u(\tau_i), \dots, E_u(\tau_{i+k})) dE_u(\tau_{i+j}),$$

and the hedging strategy for the European contingent claim becomes evident. Holding at any time  $u \in [t, \tau]$  the position

$$h_u(\tau_{i+j}) = \frac{\partial}{\partial e_j} \phi(u, E_u(\tau_i), \dots, E_u(\tau_{i+k})) p_u(\tau)$$

in the futures contract with maturity  $\tau_{i+j}$ , the European option payoff can be perfectly replicated starting with the initial endowment

$$(8.5) \quad p_t(\tau) g((E_t(\tau_{i+j}))_{j=0}^k).$$

Namely, by transferring the initial endowment and all cash flows from futures settlements to  $\tau$  by a zero bond maturing at  $\tau$ , we determine the wealth of this strategy as

$$\sum_{j=0}^k \int_t^\tau \frac{1}{p_u(\tau)} h_u(\tau_{i+j}) dE_u(\tau_{i+j}) + \frac{p_t(\tau) g((E_t(\tau_{i+j}))_{j=0}^k)}{p_t(\tau)} = f((E_\tau(\tau_{i+j}))_{j=0}^k),$$

which matches the contingent claim of our option. In the case that the replication is required from a date  $t$  earlier than  $\tau_{i-1}$ , the same arguments need to be repeated for the previous intervals.

Note that our model is inherently not complete. For instance, on the very last interval  $[\tau_n, \tau_{n+1}]$  only one future is traded, but the number of uncertainty sources is still  $d > 1$ . However, the above considerations show that under the assumption (8.1) a European option, written on futures, can be replicated by appropriate positions in futures traded prior to the expiry date of the option and in zero bonds maturing at this date.

Since the dimension of the partial differential equation (8.4) could be high and there is no evident price approximations even for the simplest plain-vanilla options, we believe that

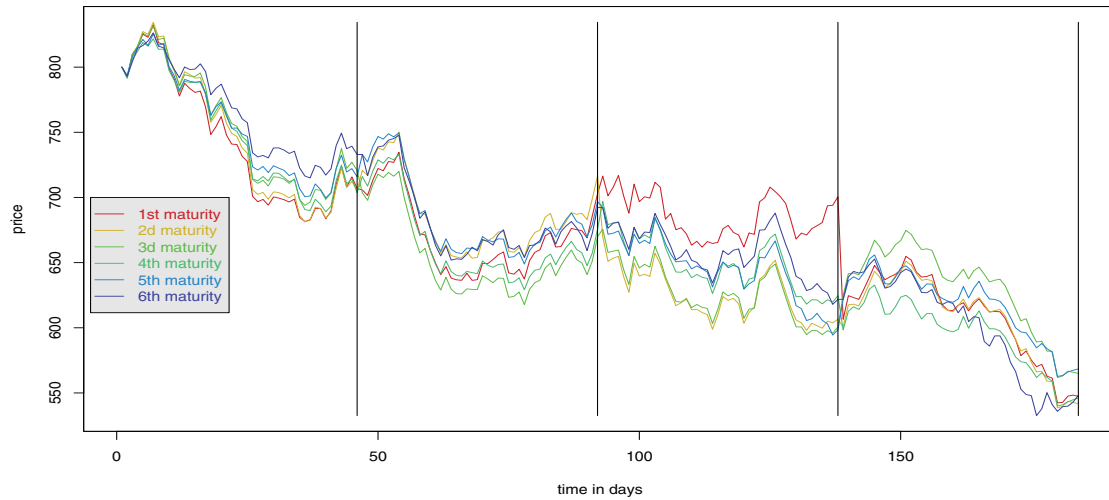


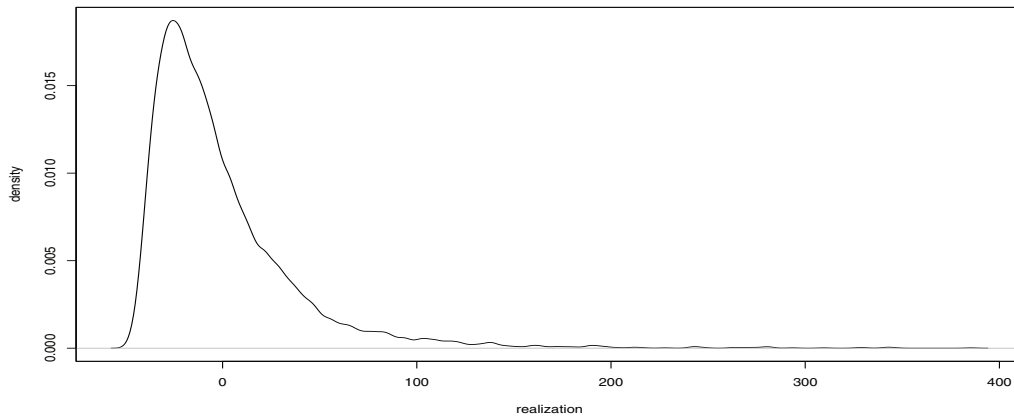
Figure 4. A simulation of futures prices. Vertical lines separate two month periods.

Monte Carlo simulation could be an appropriate way to price derivative instruments within our model. To discuss an application, we turn to the valuation of a calendar spread option. Here, we utilize the parameters obtained from the soybean example presented above. Note that, as previously, futures prices are expressed in US cents per bushel, and  $\Delta$  stands for the two month duration between expiry dates,  $\tau_i = i\Delta$ ,  $i = 1, \dots, n+1$ . The payoff of a calendar spread option depends on the price difference of commodities delivered at different times. For instance, it may provide the holder with the payment  $(E_t(\tau) - E_t(\tau') - K)^+$  at maturity  $t$ , where  $\tau$  and  $\tau'$  are future delivery dates satisfying  $t < \tau < \tau'$ . Based on a Monte Carlo simulation, we examine the distribution of the difference  $E_t(\tau') - E_t(\tau)$  for the parameters  $t = \tau = 4\Delta$ ,  $\tau' = 6\Delta$ . Having supposed that the initial futures curve is flat  $(E_0(\tau_i) = 800)_{i=1}^6$ , 5000 realizations are generated. Figure 4 depicts a typical path resulting from a single run of the Monte Carlo method. The estimated density  $E_t(\tau) - E_t(\tau')$  is shown in Figure 5. Note the clear similarity to the shifted lognormal distribution. Next, we give a detailed discussion of this observation. First, the storage costs for two periods (four months) is recognized as the correct shift parameter. Indeed,  $E_t(\tau) - E_t(\tau') > -2\kappa$  holds by construction, so an appropriate choice of the shift parameter corresponds to the expiry dates difference  $\tau' - \tau$ . To compare now the distribution of  $\ln(E_t(\tau) - E_t(\tau') + 2\kappa)$  to the appropriately scaled normal distribution, we plot in Figure 6 the estimated density of

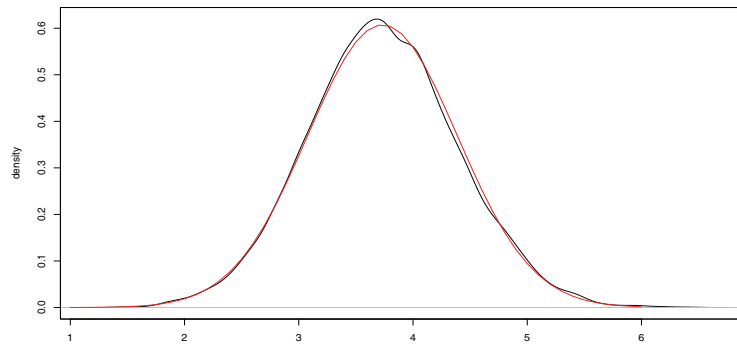
$$(8.6) \quad \ln(E_t(\tau) - E_t(\tau') + 2\kappa)$$

in comparison to the normal density, whose first two moments are chosen to match those estimated for (8.6). That is, a close approximation in distribution

$$(8.7) \quad E_t(\tau) - E_t(\tau') \stackrel{d}{\approx} \exp(X) - 2\kappa, \quad \text{where } X \text{ is Gaussian,}$$



**Figure 5.** The density of  $E_t(\tau) - E_t(\tau')$  exhibits a similarity to the lognormal density.



**Figure 6.** The density of (8.6) (black) in comparison to the normal distribution (red).

seems to be possible. This can be used to derive an approximation for the price of spread options, where by assuming the equality in (8.7) a Black–Scholes-type formula is obtained. This observation is according to the approximative pricing schemes for spread and basket options extensively studied in [1], [2], [4] and in the literature cited therein.

**9. Conclusion.** In this work, we have addressed the role of storage costs in commodity price modeling. By appropriate interpretation of the no-arbitrage principle, we formulate a minimal set of model assumptions which exclude arbitrage opportunities for futures trading and simultaneous management of a stylized storage facility. In the presented commodity model class, the storage cost plays the role of a constant parameter, which bounds the steepness of the futures curve in any contango situation. This bound, well known as the *contango limit* in commodity trading, forms an intrinsic ingredient of the proposed martingale-based futures price dynamics. Following the expertise from the interest rate theory, we demonstrate how



to construct and to calibrate commodity models which correspond to our assumptions. An empirical study of soybean futures trading illustrates this concept. Moreover, we discuss the valuation of calendar spread options. Here, numerical experiments raise the hope that an appropriately shifted lognormal distribution may give an excellent approximation for the payoff distributions of calendar spreads. This issue could be important for efficient pricing and hedging of calendar spread options.

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