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LACUNARITY AND INTERMITTENCY IN FLUID TURBULENCE ☆

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Oscillations in the high-order moments of turbulent velocity fields are inherent to the fractal character of intermittent turbulence. Such oscillations are a feature of the lacunarity of fractal sets.

Introduction. The fine structure of intermittent, turbulent flows may be probed by high moments of the velocity field. They have been studied experimentally by Anselmet et al. [1] who report that "oscillations, which are only weakly manifested for $n = 10$ and 12, are rapidly amplified at larger n," where n is the order of the moment. We suggest that such oscillations are to be expected in consequence of the fractal structure of developed turbulence [2,3]. They are related to oscillations arising in the statistics of fractal sets.

We shall discuss the connection between oscillations in fractals and in the turbulent velocity correlations using phenomenological arguments like those of the β model of intermittent turbulence [4]. But first, we briefly describe the oscillations in simple Cantor sets by way of illustration. We shall elaborate on the theory of the oscillatory structure of fractal sets elsewhere; here, our main concern is the application to turbulence.

Lacunarity. Consider a statistical moment, C, on a fractal set that depends on a separation scale l. A self-similar fractal set may be expected to obey the scaling law

$$C(l) = \sigma^{-1} C(\rho l) , \tag{1}$$

where ρ and σ are numbers. The usual power law

$$C_0(l) = A l^d , \tag{2}$$

is a particular solution of (1), where A is a constant and

$$d = \ln \sigma / \ln \rho . \tag{3}$$

The general solution is

$$C(l) = l^d \chi(\ln l/P) , \quad P = \ln \rho , \tag{4}$$

where χ is a periodic function of period 1. The general solution predicts oscillations that are observed in numerical experiments. If we identify $1/\rho$ with the similarity ratio of the sets, we find agreement with the observed periodicities in simple sets.

To illustrate these oscillations, we construct some numerical approximations to Cantor sets with the usual recipe: (1) Divide the unit interval in R^1 into r equal subintervals. (2) Delete all but s of the subintervals. (3) Repeat the process on each of the remaining $r - s$ subintervals. (4) Repeat again on the remaining intervals. And so on. Each set constructed in this way can be designated by an r-digit binary number with s ones and $r - s$ zeroes. The fractal dimension of the

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corresponding Cantor set is $D = \ln s/\ln r$ and its similarity ratio is $1/r$.

Consider three such sets: Set I is 101 (the most familiar Cantor set), set II is 101010001 and set III is 101001001. All three sets have the fractal dimension $D = \ln 2/\ln 3$. To calculate D numerically, we follow Grassberger and Procaccia [5] and define the correlation integral

$$C(l) = \lim_{N\to\infty} N^{-2} \sum_{\substack{i,j=1\\ j\neq i}}^{N} \theta(l - r_{ij}) , \tag{5}$$

where N is the number of points in the set, r_{ij} is the distance between the ith and jth points, and θ is the Heaviside function. As our notation indicates, the correlation integral is typical of the quantities we have in mind. For fractal sets, $C(l) \simeq l^{\nu}$, for small l, where ν is called the correlation exponent and $\nu \leqslant D$ [5].

In fig. 1, we plot $\ln C$ versus $\ln l$ for set I. This plot, as well as those for sets II and III, is well approximated by a straight line of slope $\nu = \ln 2/\ln 3$. For sets constructed by the removal algorithm just described, we find $\nu = D$, which indicates that the construction procedure gives equal weights to all regions of the set.

In each case, there are regular oscillations in $\ln C(l)$ about the line $\ln C = \nu \ln l$. These oscillations are seen

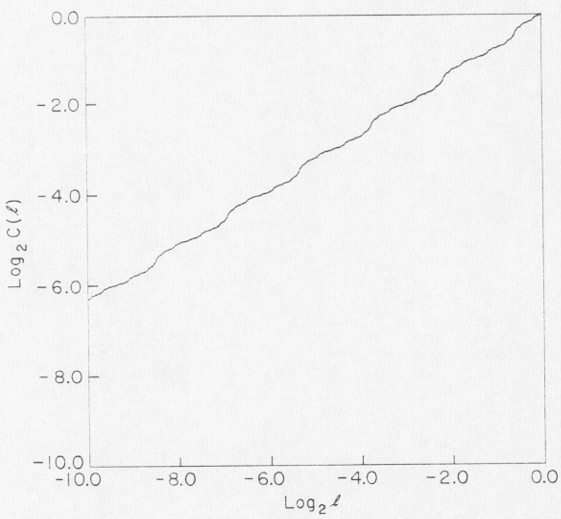


Fig. 1. The correlation integral of (5) as a function of the separation. The original interval is the unit of length. The logarithms are base 2.

over the full range of $\ln l$ and have constant amplitude and form. In fig. 2, we plot $\ln[l^{-\nu}C(l)]$ versus $\ln l$ for the three sets. The oscillatory residuals are very different for the three sets, though the periods of sets II and III are both close to $\ln 9$.

We interpret the observed oscillations as first corrections to the $d \ln l$ term in the expansion for $\ln C(l)$; that is, as l approaches zero, we have

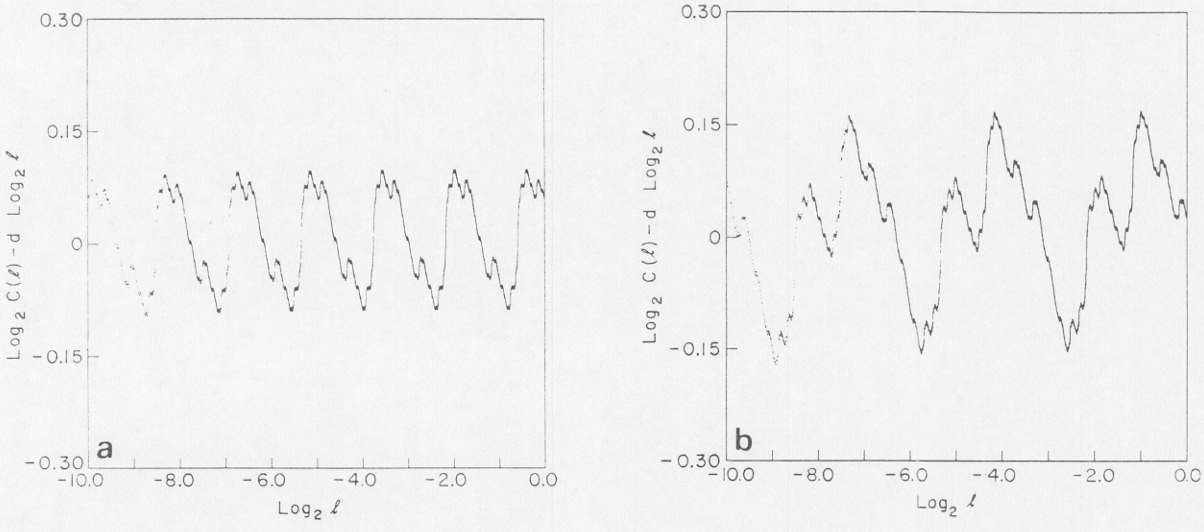
$$\ln C(l) = d \ln l + \psi(P^{-1} \ln l) + \ldots , \tag{6}$$

where ψ is a periodic function of period 1. Here, the units of l have been chosen so that ψ has zero mean. In the numerical studies of oscillations that we have so far performed, the amplitudes of oscillation of ψ are O(1). For example, in set I, 101, ψ is periodic to within the accuracy of the numerical procedures with an amplitude of about 0.1.

We suggest that the function ψ characterizes the textural property of fractal objects that Mandelbrot has called lacunarity [3]. It is difficult to capture the qualitative character implied by this term with a single parameter, and Mandelbrot considers several in his book. If we were to seek to identify a single parameter to measure this property, we would choose a functional of ψ.

Oscillations are inherent to lacunar fractals. Bessis et al. [6] proved their existence for certain fractal Julia sets (asymptotically, near the boundary of the support of the measure). Badii and Politi [7] found them in their analysis of the Zaslavsky attractor [8] using a method based on mean nearest-neighbor distances. We have observed them in many sets including Koch islands [3], the Hénon attractor [9], the Zaslavsky attractor [8], the Feigenbaum attractor [10], and Cantor dusts [3] in two and three dimensions, using diverse algorithms including both point-counting and pair-counting as in (5). Apart from their intrinsic interest, the oscillations can produce the fluctuations in the measurements of dimension that Guckenheimer [11] has noted. Such bias in a measurement of dimension may be especially significant when the oscillations have a large amplitude or a long period compared to the range of scales covered by the data.

In self-similar sets, the oscillations persist at constant amplitude to small l. For some sets, with non-uniform lacunarity, amplitude drops off as l decreases, as for example, in the Hénon attractor. But at fixed l,



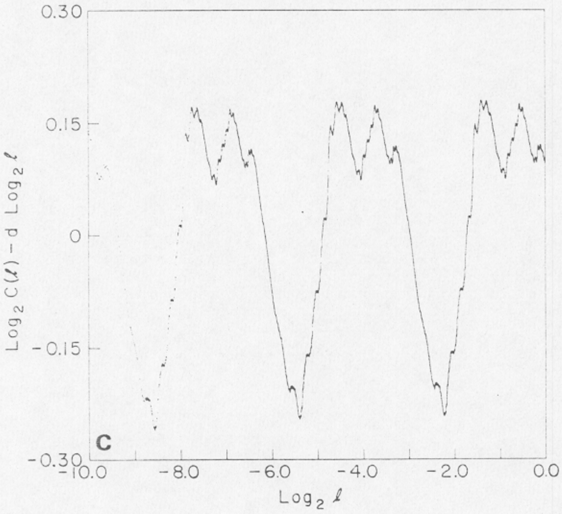


Fig. 2. Oscillatory residuals ψ for the sets (a) 101, (b) 101010001 and (c) 101001001.

the oscillations are amplified when we go to higher moments. This is important for our proposed application to turbulence.

For the purpose of applying these ideas to intermittent turbulence, we neglect the terms indicated by the ellipsis, and write

$$C(l) = \Lambda l^{\delta}, \quad \delta(l) = d + \psi(P^{-1} \ln l)/\ln l\,. \tag{7}$$

Here, we are once again regarding C in its general sense of being any statistical moment on the fractal.

Turbulence. It seems likely that the intermittency of developed turbulence comes about because turbulence is active on a fractal set [3]. Consequent deviations from Kolmogorov's [12] scaling have been qualitatively derived in a phenomenological discussion of the cascade process [4], called the β model. The β model underscores the qualitative feature that the effects of fractal geometry are emphasized by the structure functions of large order, n. These functions are

$$S_n(l) = \langle |\Delta \mathbf{v}(l)|^n \rangle / \langle |\Delta \mathbf{v}(l)|^2 \rangle^{n/2}\,, \tag{8}$$

where $\Delta \mathbf{v}$ is a velocity difference across a separation l and the angular brackets denote volume averages. From the β model comes the estimate that

$$S_n(l) = S_n^{(0)}(l)(l/l_0)^{D_n} , \tag{9}$$

where superscript (0) refers to values obtained by averaging over only the active region. The correction allows for intermittency, through a presumed dependence on the eddy sizes l (loosely identified with $|\boldsymbol{l}|$). The assumption of a geometrical cascade leads, by phenomenological arguments, to

$$D_n = (D - 3)(2 - n)/2 , \tag{10}$$

where D is the fractal dimension of the region of active turbulence and l_0 is the length scale at which energy is fed into the fluid. The model is named for $\beta = (l/l_0)^{D-3}$.

The discussion of the previous section implies that it is possible to extend (9) and (10) in the same way that (4) generalizes (2). Using (7) in this more general context, and applying the arguments of Frisch, Sulem and Nelkin [4], we get

$$S_n(l) = S_n(l_0) X_n(\ln l/P)(l/l_0)^{(3-D)(2-n)/2} ,$$

$$X_n(\ln l/P) = \exp\{\tfrac{1}{2}(n-2)[\psi(\ln l/P) - \psi(\ln l_0/P)]\} . \tag{11}$$

In the β model, the D_n are expressed in terms of only one fractal dimension. Likewise we have only one periodic function, ψ. To the extent that such arguments are qualitatively adequate, we conclude from (11) that the oscillations should be more apparent for larger n, as observed by Anselmet et al. [1]. Moreover, we judge from their fig. 13 that the period of the oscillations is not dependent on n, as our discussion indicates. The occurrence of this period introduces lengths other than l_0 that are relevant for the inertial range and this gives significance to the observed oscillations if they are indeed inherent to turbulence and are due to its fractal nature. Even if the observed oscillations are of another origin, it remains true that oscillations may be expected if the intermittency of turbulence originates in its fractal structure.

We have enjoyed a discussion of fractal matters with D. Bessis and of oscillations with R. Badii and P. Pfeifer.

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