QUANTIFYING CHAOS WITH PREDICTIVE FLOWS AND MAPS: LOCATING UNSTABLE PERIODIC ORBITS

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ABSTRACT

Several authors have suggested methods for constructing "predictors" from time series data. It is shown that using one type of predictor, constructed with radial basis functions, for some known chaotic systems, the existence of unstable periodic orbits may be established using much less data than that required by alternative methods. The general question of quantifying the error in a predictor is also addressed. Considering the fraction of the data that can be predicted as a function of the accuracy of the prediction provides a method of distinguishing different sources of error in the predictor and, in doing so, yields an estimate of the magnitude and distribution of the observational noise in the system.

INTRODUCTION

In the past, low dimensional dynamical systems have been represented by points generated directly from an observed trajectory; coverage of the relevant regions of phase space has required data records both long (in terms of the rate at which information appears in the system) and finely sampled. Recent proposals to quantify systems not with the data directly, but with reconstructed flows (Broomhead, 1988, Castagli,1989, Crutchfield and McNamara,1987, Farmer and Sidorowich, 1988, and Mees,1989) offer a new approach to predicting and quantifying nonlinear dynamical systems. While a wide variety of reconstruction techniques have been proposed, all are based on utilizing the information contained in the time ordering of the data points to construct an M-dimensional interpolation scheme. The techniques differ in the method used to form the interpolation from the known data points. Hereafter, a reconstructed flow (or map) will be referred to as a predictor. Since the information in the time ordering of the points is not used in the geometrical analysis (Grassberger and Procaccia,1983), constructing a good predictor may require less data than the direct calculation of generalized dimensions (Smith, 1988).

In this paper, predictors constructed from time series data are used to find and quantify the unstable periodic orbits of the chaotic system which generated the series. New difficulties which arise in locating the periodic orbits of continuous systems are discussed. The interpolation technique employed uses the radial basis functions (see Powell, 1985) originally applied to nonlinear time series data by Castagli(1989). When the system is known, the convergence of true and reconstructed flows is examined by comparing the unstable periodic orbits present in each system.

To quantify the quality of prediction, the fraction of the time series which can be predicted within an accuracy a is determined. The behavior of this "predictor error profile", $P_f(a)$, is related to changes in the number of points used in constructing the predictor and the time scale of the prediction. It can be used to quantify the noise in a deterministic system. The variation of $P_f(a)$ with the time scale of the prediction provides a tool for distinguishing stochasticity from determinism and suggests

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a numerically efficient method for estimating an optimal delay time for reconstructing the system.

Predictors for several systems have been studied including the Henon map (Henon, 1976), the Moore-Spiegel system (Moore and Spiegel, 1966), the Duffing oscillator (either periodically or stochastically forced, see Stone and Holmes (1989)), an Ornstein-Ulhembeck process and experimental results of baroclinic flow (Read, 1989), boundary layer turbulence (Gaster, 1989) and an electrical oscillator (King, 1989). In addition real world data records of climate and solar activity have been considered. In general, predictors for the numerical systems give excellent results; those for low dimensional laboratory systems provide good predictions and successfully locate unstable periodic orbits. The correct interpretation of the results for the real world systems is not yet clear. This short contribution will concentrate on the numerical models with a few remarks on the laboratory experiments. A more detailed report is in preparation.

RADIAL BASIS FUNCTION PREDICTORS

An introduction to the prediction of chaotic dynamical systems using radial basis functions is given by Castagli (1989). Only a very brief summary is provided here. Consider a deterministic system with phase space dimension M_s. A trajectory, x(t), of this system may be constructed in M_s dimensions from a time series of single observable, o(t), by the method of delays (Packard et al. 1980) to yield $x(t) = (o(t), o(t+\tau_d), \ldots, o(t+(m-1)\tau_d))$. Each point on the trajectory has an image $s(t) = o(t+t_p)$, where t_p is determined either by a fixed prediction time, τ_p (i.e. $t_p = (m-1)\tau_d + \tau_p$) or through some geometric constraint (as when taking a surface of section of the trajectory).

Choosing n distinct points $(x_i, i = 1, 2, ..., n)$ and their n images $(s_i, i = 1, 2, ..., n)$, the goal is to determine a predictor $f(x_i) : R^{M_s} \rightarrow R$ such that

$$f(x_i) = s_i \tag{1}$$

Following Powell (1985), consider $f(x)$ of the form

$$f(x) = \sum_{i=1}^{n} \lambda_i \phi(||x - x_i||) \tag{2}$$

where $\phi(r)$ are radial basis functions and the λ_i are constants which are uniquely determined by equations (1) provided the matrix

$$A_{ij} = \phi(||x_i - x_j||) \tag{3}$$

is nonsingular. This is always the case (Micchelli,1985, Powell,1985) for $\phi(r)$ of the form

$$\phi(r) = (r^2 + c^2)^{-\beta} \tag{4}$$

for $\beta > -1$ and $\beta \neq 0$. The particular cases $\phi(r) = r$ and $\phi(r) = (r^2 + c^2)^{1/2}$ (referred to as the "multiquartic") will be considered here.

When the predictor is formed in delay coordinates, a single map $f(x) : R^{M_s} \rightarrow R$ is sufficient to predict the system. Prediction in singular value decomposition (SVD) coordinates (Broomhead and King, 1987), or in the true phase space, requires M_s such maps $f^j(x)$, $(j = 1, \ldots, M_s)$, one for each coordinate.

In order to both construct and test a predictor with the same data set, the trajectory is divided into two parts, a "base" portion from which the base points, x_i, are chosen, and the remaining portion on which the predictor is tested. The simplest method for choosing base points is to distribute them uniformly along the base portion of the trajectory. Since determining the λ_i involves manipulating an n x n matrix and each prediction requires computing n distances in an M_s dimensional space, the computer power required to implement this procedure increases rapidly with n; it is therefore desirable to choose the base points efficiently by sparsely sampling regions in which the flow is smooth and densely sampling the flow elsewhere. A simple way to accomplish this is to build a predictor using $\frac{n}{2}$ points and choose additional points from the base portion depending upon how well the images of these points are predicted. To do this, distribute half the base points uniformly on the base portion and then predict the base portion of the trajectory. Half the remaining number of base points (one quarter the total) are chosen from those points whose images are predicted least accurately (with the condition that no two new points are too close to each other) and the procedure is repeated.

Finally, the initial (uniformly spaced) points may be removed and the entire procedure repeated. This approach can significantly reduce the number of points required to make good predictions; it does not, of course, lessen the data requirements because the base portion must be "long enough" to explore the entire attractor. The amount of data required for a "long enough" base portion will depend on the macroscopic structure of the attractor in question (as well as its dimension). Also note that data from the transient should be included in the base portion when possible.

Two dynamical systems are used as examples below. The Henon map (Henon,1976) is given by

$$x' = 1 - ax^2 + y \tag{5}$$

$$y' = bx \tag{6}$$

with $a = 1.4$ and $b = 0.3$. The Moore-Spiegel system (Moore and Spiegel, 1966) is given by

$$\frac{d^3z}{dt^3} + \frac{d^2z}{dt^2} + (T - R + Rz^2)\frac{dz}{dt} + Tz = 0. \tag{7}$$

This system displays chaotic behavior for $R = 100.0$ and $T = 36.0$, the case considered here. A $z = 0$ surface of section of this attractor is shown in Figure 1.

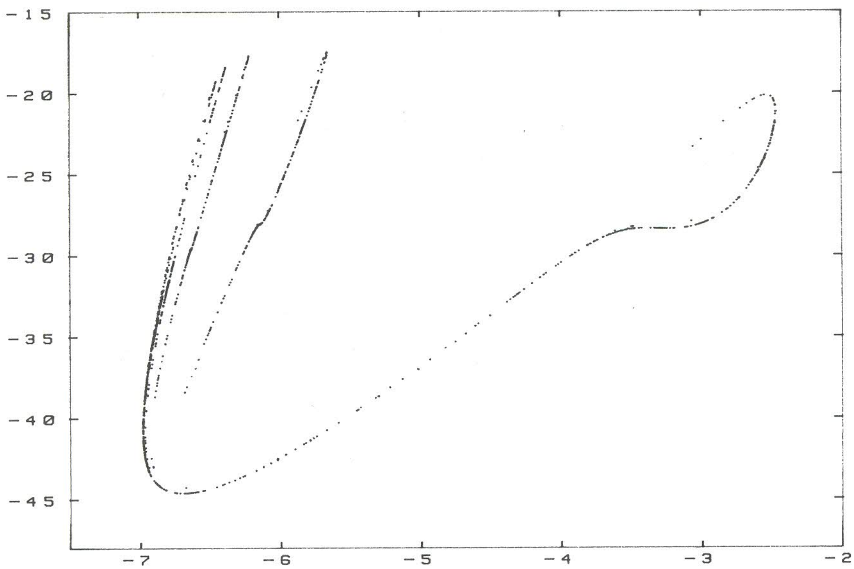

Figure 1. A $z = 0$ surface of section for the Moore-Spiegel system consisting of 2^{12} points.

EVALUATION OF A PREDICTOR

Intrinsic limits to very long term prediction of chaotic systems are imposed by the exponential divergence of nearby initial conditions, the average rate of which is quantified by the Lyapunov exponents of the system. It is misleading, however, to think of the Lyapunov exponents as limiting short term prediction at macroscopic length scales. The level of organization of the flow required to give exponential divergence (on average) is very high; there may be regions in which the flow is contracting, making prediction relatively easy. The predictability of a chaotic system should be expected to vary with location in phase space; there often exist regions of phase space in which chaotic systems are easier to predict than stochastic systems which, in the mean, diverge more slowly than exponentially and have Lyapunov exponents equal to zero.

The quality of a prediction varies for several reasons. Consider the error in predicting the image of some initial condition, x. In addition to the error introduced by the sensitive dependence to initial condition, error arises from failure of the base points to sufficiently cover the attractor; either when x is located in a region of the attractor not explored by the base portion of the trajectory or when

x is relatively close to base points, but is located in a region where the flow possesses significant fine structure. Predictors are often evaluated by computing the average difference between the predicted and observed values. This approach does not distinguish different sources of error. Comparing this average error to a "random guess" is also misleading, especially in continuous systems with persistence (or seasonality). In continuous systems, predicting the motion on a surface of section reduces the effects of persistence. To further distinguish these sources of error, consider a plot of the observed error in a one step prediction versus the distance from x to the nearest base point, d_n.

Such a plot for a predictor constructed for the Henon system with $n = 2^6$ consecutive base points is shown in Figure 2. In general, the observed error grows with increasing d_n; exceptions are the finely structured lines in regions of large error and small d_n. Points in these regions can be divided into groups which have the same nearest base point. The structures in Figure 2 are then seen to reflect the structure of the attractor near these base points; as the number of base points used in constructing the predictor increases, these structures become less apparent while the general trend of increasing predictor error with increasing d_n remains.

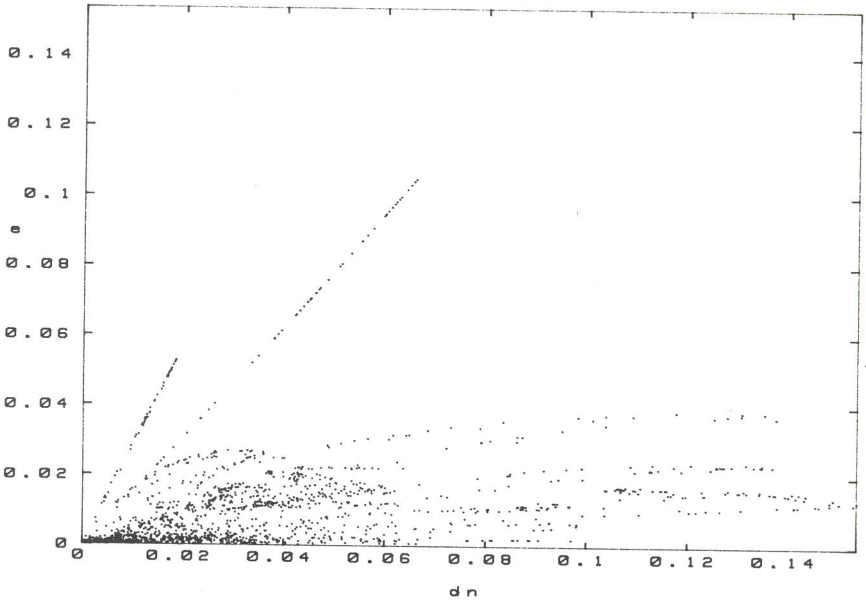

Figure 2. The observed error in the one step prediction of the Henon map versus the distance to the nearest base point.

An alternative approach to evaluating the predictor error is to consider the fraction of the data, $P_f(a)$, which can be predicted within a certain accuracy, a. Figure 3 shows this curve for predictors constructed from 2^5, 2^6, 2^7 and 2^8 base points for the Moore-Spiegel system using $\phi(r) = r$. The distribution of base points on the attractor and structure of the flow determine the shape of $P_f(a)$. As the number of base points is increased, there is improvement in the high end of the $P_f(a)$ curve (say, $P_f(a) > 0.9$) corresponding to new regions becoming predictable. The low end of the curve (say, $P_f(a) < 0.1$) is determined by the fine scale structure of the data; the details of the flow if the data is clean, the details of the noise otherwise. Given a perfect predictor of the system, predictor errors will remain due to observational effects (noise and truncation error) in the data. These effects impose a lower bound on the predictor error below which $P_f(a)$ is approximately zero. The shape and location of this transistion may be used to identify the distribution and magnitude of the observational noise. The $P_f(a)$ profiles of Gaussian, Poisson and uniformly distributed white noise are distinguishable. For well sampled attractors generated from clean experimental signals the resolution of the A/D converter is apparent as a sharp drop in $P_f(a)$. Data from numerical systems tends to show a slower transition which is dependent on the distribution of the base points; the prediction error for points located near a base point may be very low. For the case shown in Figure 3, the error in this data is off the scale, primarily arising from taking the surface of section. If noise of greater magnitude were added to the data, a sharper drop in $P_f(a)$ would be observed.

Comparing $P_f(a)$ curve for one step prediction with that for a many step prediction displays the change in predictability with increasing prediction time. When the underlying process is random, the $P_f(a)$ profile will not change (assuming persistence effects have been accounted for). For chaotic systems, the P_f profile should degrade toward some limiting case equivalent to making a random guess from an appropriate distribution. Finally, note that the quality of the predictor will vary with the details of the reconstruction. This suggests that observing the changes in $P_f(a)$ provides an efficient method for choosing the delay time, τ_d. Evidence that this is the case will be presented elsewhere.

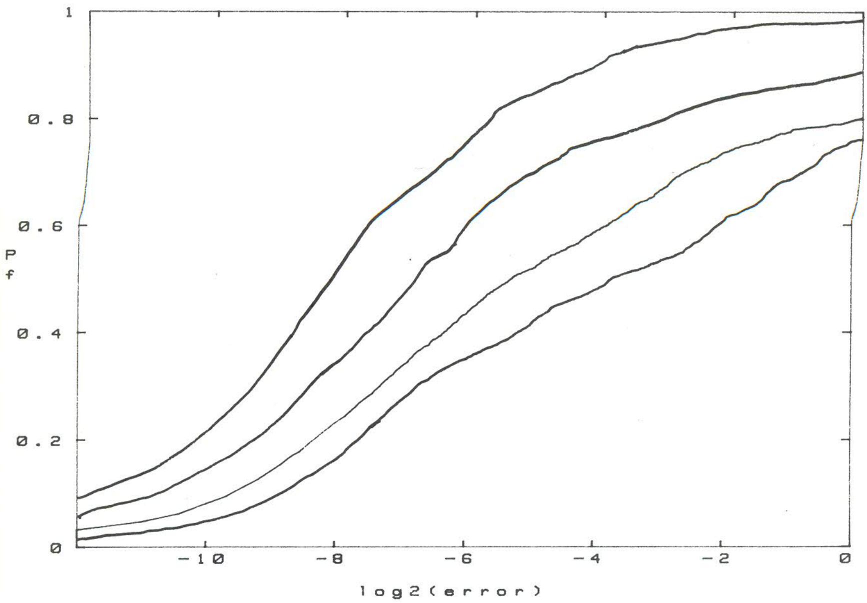

Figure 3. The predictor error profile for 4 predictors of the Moore-Spiegel system. The curves (from lowest for highest) correspond to predictors constructed with $n = 2^5, 2^6, 2^7$, and 2^8 points.

UNSTABLE PERIODIC ORBITS

A catalogue of the unstable periodic orbits within a basin of attraction provides a useful description of chaotic attractors. These orbits may be viewed as a skeleton of the attractor; their properties (number and stability) have been successfully related to the metric properties of the attractor in the case of the Henon map (Cvitanovic et al., 1989). In this section a new method for determining these orbits is presented which consists, in short, of estimating the periodic orbits of the system by those of the predictor. The orbits and their stability are approximated by this method with modest data requirements; some orbits not accessible with other methods can be identified in a predictive flow. A detailed application to experimental data is in preparation (Smith and King, 1989).

When the system is known analytically, the original method proposed by Auerbach et al. (1987) is to select a grid of points on the basin of attraction and, starting at each point, apply a Newton-Raphson iteration scheme to search for an orbit of a given period. An alternative method is to follow a single trajectory and invoke the Newton-Raphson scheme at all near returns. This approach quickly locates the orbits near the attractor but has difficulties with orbits far from the attractor (or in rarely visited regions on the attractor). For the Henon map it finds periodic points up to period 10 in agreement with the results of Cvitanovic et al. (1989). Note however, that even in cases where the "exact" map is known, it is difficult to be certain that all the periodic orbits have been found.

In experimental flows, it is suggested that near returns be considered for each available time lag. A periodic point is then approximated by the center of mass of a ball of near returns and the eigenvalues may be estimated by assuming a linearized dynamics about this point. For the Henon map, Auerbach et al (1988) report data requirements of 2 x 10^5 iterations to locate orbits of period less than or equal to 10. This approach will locate only those orbits which are closely shadowed by a trajectory for at least one complete period. Using a predictive flow requires only that each segment of the orbit is visited at some point in the time series.

Considering continuous systems introduces complications not found in maps. When considering a surface of section of a continuous system, it is necessary to distinguish the number of times an orbit intersects the section from the period of the orbit. The former will be called the passage of the orbit and will, of course, depend on the section taken while the period will not. In maps, the passage and period of a periodic orbit are equal. The method of looking at a set time lag will miss periodic orbits when topologically nearby trajectories have different temporal appearances. The simplest example occurs when a periodic orbit passes close to a fixed point of the flow. The "period" (near return time) of initial conditions near such a periodic orbit will vary greatly, depending sensitively on their path near the fixed point. Two approximations of the same periodic orbit which passes through this region will not look the same when parameterized by time, although they are topologically similar. (This effect will also bias generalized dimension calculations by redistributing the probability density.) Taking a surface of section removes this problem. Points which are in the section will return, in general, after similar periods of time. Initial conditions near a periodic orbit which passes close to a stagnation point of the flow will have a wide distribution of first return times while those near other periodic orbits will have a tight distribution. This is, in fact, observed in the nonlinear electronic oscillator. Comparing the results from several surfaces of section provides an estimate of the error.

With a predictor, the methods developed for known systems may be applied to time series data. To test the stability of true orbits in the predictor, the known periodic points with periods less than or equal to 10 where used as initial conditions for a Newton Raphson style iteration procedure applied with a predictor constructed from 2^7 consecutive points on the Henon attractor with $\phi(r) = r$. In this case all periodic points of period less than 6 were present, with similar stability, to the full map. Most orbits of higher period were preserved, however their unstable eigenvalue often differed by a factor of 2 from the true orbit. Note however that the quadratic structure of the Henon map makes it a questionable test case. This is most apparent when multiquartics are used as the radial basis functions; in this case a 16 point predictor reproduces the map to astounding accuracy. Such multiquartic predictors are sensitive to noise.

A more general test case is provided by the $z = 0$ section of the Moore-Spiegel system. In this case, near returns on the section were used as initial estimates of periodic orbits in several predictors with $\phi(r) = r$. Predictors using 2^8 or fewer base points often had periodic orbits not present in Moore-Spiegel system. These spurious orbits often contain points far from the attractor. The spurious orbits are not found in a 2^9 point predictor where the location of the passage 9 points tested agree between the predictor and the full system to within 1 percent. The stability of the orbits is not well estimated, however, often differing by a factor of 2.

Before leaving this system, note that other observables may be estimated. For example, the time until the next passage through a section. Such calculations are required if the Lyapunov exponents of the reconstructed flow are to be computed from a section. Consider the series of magnifications an infinitesimal vector experiences as it is iterated along the attractor. The mean of this distribution corresponds to the largest Lyapunov exponent. For the Henon map, there is good agreement between a 2^9 point predictor and the map not only in the mean, but also in the structure of the distribution.

DISCUSSION AND CONCLUSIONS

Constructing a predictor from data when the system is not known presents new complications not considered above. These are most often clarified given "long enough" data records. For example embedding a low dimensional attractor in a large M will result in lower quality predictions. In delay coordinates, separation in the first coordinate direction(s) corresponds to distance in the true phase space at earlier times than separation in the last coordinate directions. For embedding dimensions larger than M_s, base points near the point to be predicted in the first coordinate direction(s) (but which correspond to a path which diverges from the trajectory leading to the point to be predicted) are weighted equally to those which are far away in the first coordinate direction(s) and arbitrarily close to the current position of the point in the true phase space. Conversely, embedding a high dimensional system in a low dimensional space will yield good predictions for those regions which are not saturated by projection effects. Predictions for points which lie in populated regions due to projection will be poor if the base points do not come from the same region of phase space; however the prediction of such points would still be poor in the correct embedding dimension if the region was not well sampled. This is especially true if the reconstruction is determined in SVD coordinates where the first few coordinate directions contain the majority of the signal. A method to identify flows reconstructed in $M < M_s$ needs to be developed. Periodic orbits will, of course, remain periodic in projection.

It has been shown that the predictor error profile is related to the magnitude and distribution of observational noise in a deterministic system. Observing the change of this profile as the prediction time is increased provides information on the observational noise and the chaos in the system. Predictors from reconstructed flows are capable of estimating the spectrum of unstable periodic orbits with significantly less data than other methods. Using the information contained in the time ordering of the data reduces the bulk of data required - the sampling rate and the length of the series may be shortened (nearest neighbors may be farther away) relative to the requirements of dimension calculations. It does not, of course, reduce the need to explore the entire attractor. Nonetheless, the use of reconstructed flows as discussed above should allow a much improved quantitative description of a chaotic attractor from a given series of observations.

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