Dynamical Systems: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/cdss20

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Version of record first published: 16 Jan 2013.

To cite this article: Paul Glendinning & Leonard A. Smith (2013): Lacunarity and period-doubling, Dynamical Systems: An International Journal, DOI:10.1080/14689367.2012.755496

To link to this article: http://dx.doi.org/10.1080/14689367.2012.755496

Dynamical Systems, 2013
http://dx.doi.org/10.1080/14689367.2012.755496

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(*Received 8 November 2012; final version received 23 November 2012*)

We show that the deviation from power laws of the scaling of chaotic measures, such as Lyapunov exponents and topological entropy, is periodic in the logarithm of the distance from the accumulation of period doubling. Moreover, this periodic function is asymptotically universal for each measure (for functions in the appropriate universality class). This is related to the concept of lacunarity known to exist for scaling functions describing the mass distribution of self-similar fractal sets.

Keywords: lacunarity; fractal; period-doubling; universality class

1. Lacunarity

In self-similar sets, lacunarity describes the way in which voids or lacunae are distributed at all scales. The presence of these voids is reflected in scaling functions, which characterize the spatial distribution of points. In self-similar sets, the regular appearance of voids results in a periodic component of the scaling functions; such oscillations, however, are often found in scaling quantities, an observation going back at least to DeBruin ([1], see [2] for a historical review). Guckenheimer [3] suggests that the oscillations in scaling functions seriously hamper numerical attempts to approximate scaling exponents, such as the capacity of fractal sets. These arguments are supported by observations [4–7]. For a more recent discussion, see [8] and references therein. In these studies, the effect of lacunarity is confined to problems involving spatial structures; here we show how the same ideas are useful in describing scaling behaviour in families of dynamical systems, where the (self-similar) fractal structure occurs in parameter space rather than in state space. In particular, we show that universal functions describe the asymptotic structure of quantities such as the topological entropy of a map just after the accumulation of period-doubling.[1]

The source of the lacunarity described here is the existence of self-similar scaling relations of the form $af(x) = f(bx)$ for the quantities we describe. This equation has general solution,

$$f(x) = \psi(\log x)x^{\frac{\log a}{\log b}},$$

where $\psi(\log x)$ is periodic with period $\log b$, and the lacunarity of the set is reflected in the oscillation due to ψ.

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As an introduction to lacunarity, we shall consider a simple example in which all the functions involved can be calculated explicitly (see [9] for additional discussion). The 'middle thirds' Cantor set, Λ, is perhaps the most easy fractal set to work with, and the periodic oscillation of the scaling function can be calculated explicitly. The set,

$$\Lambda = \left\{ x \in [0,1] | x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, a_i \in \{0,2\} \right\}, \tag{1}$$

is a Cantor set since it is uncountable, closed, contains no intervals, and every point is an accumulation point of the set. The standard construction of Λ is inductive: take the interval $[0, 1]$ and remove the middle third, $(\frac{1}{3}, \frac{2}{3})$, leaving two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Now remove the middle third of each of these intervals and repeat the construction, removing the middle third of each remaining interval at each stage. In the limit, this process defines the Cantor set, Λ. Let $N(\epsilon)$ be the smallest number of ϵ-balls needed to cover Λ. The dimension or capacity of Λ, $C(\Lambda)$, is then defined as

$$C(\Lambda) = \lim_{\epsilon \to 0} -\frac{\log N(\epsilon)}{\log \epsilon}. \tag{2}$$

At the nth stage of its construction, there are 2^n intervals, each of length 3^{-n} and Λ is contained in these intervals. Hence,

$$N(\epsilon) = 2^n \text{ if } \frac{1}{3^n} \le \epsilon < \frac{1}{3^{n-1}}. \tag{3}$$

Furthermore,

$$N(\epsilon) = 2N(3\epsilon). \tag{4}$$

This functional equation has solutions,

$$N(\epsilon) = \phi(\log \epsilon)\epsilon^{-\frac{\log 2}{\log 3}}, \tag{5}$$

where

$$\phi(\log \epsilon) = \phi(\log \epsilon + \log 3), \tag{6}$$

which gives $C(\Lambda) = \frac{\log 2}{\log 3}$. Since $N(\epsilon)$ is known explicitly from Equation (3), we can calculate ϕ to obtain the saw-tooth function (for $\epsilon < 1$, i.e. $\log \epsilon < 0$),

$$\log \phi(\log \epsilon) = \frac{\log 2}{\log 3} \log \epsilon + \log 2, \qquad 0 > \log \epsilon \ge -\log 3 \tag{7}$$

and $\phi(\log \epsilon - \log 3) = \phi(\log \epsilon)$. This function reflects two aspects of the Cantor set. First, the self-similarity of the set is expressed in the periodicity of ϕ, and second, the non-uniformity or bunched aspect of the set is expressed by deviations of ϕ from a constant (hence the term lacunarity, referring to the holes or non-uniformity of the set). Numerically computed estimates of $\log N(\epsilon)$ against $-\log \epsilon$ show periodic variations from a straight line of slope $\frac{\log 2}{\log 3}$ and any attempt to measure the capacity of a set must take this into account. Plots of the lacunarity oscillations of related scaling functions for a number of different Cantor sets are given in [5]. Note that the modulation function, ϕ, of Equation (7) is *not* what we would find in the standard numerical computations of ϕ. $N(\epsilon)$ is the smallest number of ϵ-balls needed to cover Λ, whilst numerical computations do not, in general, compute this minimal cover (see [11], and references thereof).

Similar effects arise in the fractal sets, which arise in deterministic dynamical systems [7]. For example, the scaling behaviour of strange attractors near hyperbolic periodic points can be studied in this way (examples from the Hénon map [12] are given [9] and [13]). In this case, the stable eigenvalue of the periodic point determines the period of oscillation in the (local) lacunarity.

The periodic oscillation observed here is due to the fact that there is only one scaling between each level of the Cantor set. If there are two or more irrationally related scalings as, for example, in some iterated function systems, the oscillation could be quasi-periodic. If the scaling from one level to the next was random, but still resulting in a Cantor set with a well-defined dimension, then there would still be oscillations about the line $-d \log \epsilon$ (on a logarithmic scale), but these would not be periodic.

2. Universality and period-doubling

The work mentioned above concentrates on the spatial structure of complex sets. It is equally possible to find systems with complex dependence upon parameters and it is this side of the problem that we wish to comment upon. To do this, we shall concentrate on scaling behaviour associated with the cascades of period-doubling bifurcations, although at the end of this note we suggest some further situations where the idea of parametric lacunarity may be useful.

Period-doubling cascades in unimodal (or one-hump) maps such as the logistic map, $f_\mu(x) = \mu x(1-x)$, have been studied extensively, both from a topological and a functional point of view [14–16]. It has been observed (and understood theoretically) that the envelope of quantities representing the complexity of solutions, such as the Lyapunov exponents, scale in a characteristic manner. Indeed, the Lyapunov exponents of the attractor above the point of accumulation of period-doubling in a typical family of unimodal maps, $\{f_\mu\}$, scales like

$$\lambda \sim |\mu - \mu_\infty|^{\frac{\log 2}{\log \delta}} \tag{8}$$

as shown in Figure 1(a), where μ_∞ is the accumulation value and δ is a constant (the Feigenbaum constant [17], depending only upon the university class of $\{f_\mu\}$). In Figure 1(b), the underlying power law behaviour of the Lyapunov exponents has been factored out of the signal. The resulting curve certainly looks periodic, and it is this feature of the problem that will be explained below.

The Lyapunov exponent of a point x_0 under the map f is the average of the logarithm of the modulus of the derivative of the map along the orbit (when it exists), i.e.

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{0}^{n-1} \log |f'(f^k(x_0))|, \tag{9}$$

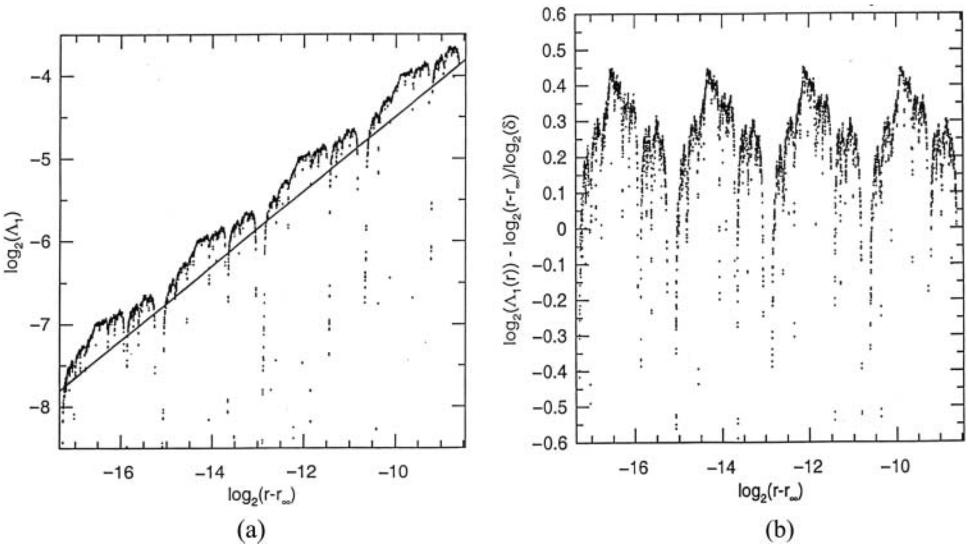

Figure 1. The Lyapunov exponent for the logistic map: (a) base 2 logarithm of the Lyapunov exponent against logarithm $r - r_\infty$. The line reflects the power law $(r - r_\infty)^{\frac{1}{\log_2(\delta)}}$. (b) Modulation function $\psi(\log_2(r - r_\infty))$ obtained by factoring out the power law.

provided the limit exists. It provides a measure of the speed of separation of nearby orbits. If a solution is in the basin of attraction of an attractor with an ergodic invariant measure then λ is independent of x_0 for almost all initial points x_0 and so we can talk about the Lyapunov exponent of the attractor. Clearly, the Lyapunov exponent is undefined if the attractor is a superstable periodic orbit, i.e. a periodic orbit containing a turning point of the map, since $f'(x) = 0$ implies that λ equals 'minus infinity'. So for families such as the logistic map, for which there are infinitely many parameters having superstable periodic orbits, the Lyapunov function is not continuous.

To be able to calculate more robust (i.e. continuous) quantities, we will also work with the topological entropy of a map. For unimodal maps, the topological entropy is essentially the asymptotic growth rate (limsup) of the number of periodic points of period p as $p \to \infty$. This characterization makes it appear rather hard to compute numerically, however, some fundamental results of Milnor and Thurston [18] show that the topological entropy can be quite easy to calculate. Moreover, it varies continuously for C^1 families of unimodal maps, making it a more regular function to use (see e.g. [16] chapter II.9, for technical details). A similar argument to that presented below should hold (with certain extra assumptions) for the Lyapunov exponents except that the modulating functions cannot be continuous.

Milnor and Thurston [18] developed a symbolic description of orbits of unimodal maps called kneading theory. If the critical point of the map is at $x = c$, then define $a_0(x) = 1$ if $x \leq c$ and $a_0(x) = -1$ if $x > c$. Now define $a_i(x)$ inductively by the relation $a_i(x) = a_{i-1}(x)a_0(f^i(x))$, $i > 0$. The kneading sequence (or itinerary) of a point is the sequence $I(x) = a_0(x)a_1(x)a_2(x)\ldots$ and the kneading invariant of a map, f, is the sequence

$$k(f) = \lim_{x \uparrow c} I(x). \tag{10}$$

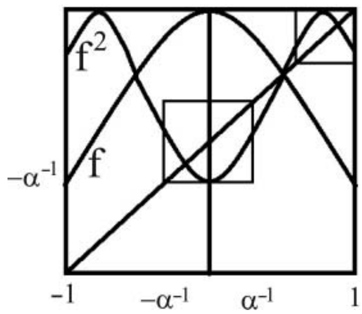

Figure 2. The function f and f^2 showing the geometry of renormalization and the map $\mathcal{T}f$.

Now, if $k(f) = k_0k_1k_2\ldots$, then we can form the formal power series,

$$K(t) = \sum_{i=0}^{\infty} k_i t^i. \tag{11}$$

Milnor and Thurston [18] prove that if $h(f) > 0$ and s is the smallest positive zero of $K(t)$, i.e. $K(s) = 0$ and $K(t) > 0$ for all $0 \le t < s$, then

$$h(f) = \log(s^{-1}). \tag{12}$$

The first symbols of the kneading invariant of a map are easy to calculate numerically, and we can solve the polynomial approximation of the power series $K(t)$ to find an approximate value of the topological entropy of f. This method will be used to compute the equivalent of Figure 1 for the variation and modulation of the topological entropy (the reader may wish to look forward to Figure 4).

To understand the scaling of the Lyapunov exponents shown in Figure 1 and to extend it to the topological entropy we need to recall some basic facts about period-doubling cascades. In what follows, we assume that the family $\{f_\mu\}$ has a quadratic maximum for each μ and that there exists a parameter value μ_∞ for which f_{μ_∞} is at the accumulation of period-doubling. It is also convenient to choose a normalization condition: we assume that the maximum is at $x = 0$ and that $f_\mu(0) = 1$ for all μ. Furthermore, we assume that the maps are even ($f_\mu(-x) = f_\mu(x)$), so the maps are defined on the interval $[-1, 1]$ as shown in Figure 2. The analysis of the period-doubling cascades of unimodal maps uses the idea of renormalization [15–17,19]: if $f_\mu(1) = -\alpha^{-1} < 0$ then define an induced map on the interval $[-\alpha^{-1}, \alpha^{-1}]$ by taking the second iterate of f in that interval. This new map is again a unimodal map and maps the interval $[-\alpha^{-1}, \alpha^{-1}]$ into itself provided the map is sufficiently close to the accumulation of period-doubling. This new map can be rescaled to give a unimodal map of $[-1, 1]$, $\mathcal{T}f$, defined by

$$\mathcal{T}f(x) = -\alpha f \circ f(-x/\alpha). \tag{13}$$

The operator $\mathcal{T}$ can now be thought of as a map on a suitably defined function space (see [16], for full details). In the universality class, we are considering, $\mathcal{T}$ has a fixed point $f_*(x)$,

$$\mathcal{T}f_* = f_*,$$

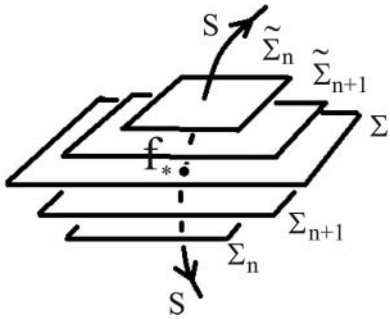

Figure 3. The structure of function space near the fixed point f_* of $\mathcal{T}$, after [20].

with a one-dimensional unstable manifold, S, with associated eigenvalue $\delta > 1$ and a co-dimension one stable manifold Σ as shown in Figure 3. The fixed point, f_* also fixes a value of α: $-\alpha_*^{-1} = f_*(1)$. For unimodal maps with quadratic maxima $\delta \approx 4.6692$ and $\alpha_*^{-1} \approx 0.3995$.

The standard argument for the universality of period-doubling in quadratic families assumes that there exists a co-dimension one manifold, Σ_n, beneath Σ and intersecting S transversely, such that if $f \in \Sigma_n$ then f has a non-hyperbolic periodic orbit of period 2^{n-1}, which satisfies the conditions for a period-doubling bifurcation. Hence, just below Σ_n maps have a stable periodic orbit of period 2^{n-1}, and just above Σ_n this orbit has lost stability and there is a stable periodic orbit of period 2^n. Now consider maps in the manifold $\mathcal{T}^{-1}\Sigma_n = \Sigma_{n+1}$, which lies between Σ_n and Σ (distances having been contracted by a factor δ^{-1} in the direction of S and expanded in the other directions). If $\mathcal{T} f$ period-doubles from period 2^{n-1} to 2^n, then f period-doubles from period 2^n to 2^{n+1}, so Σ_{n+1} consists of maps with non-hyperbolic periodic orbits of period 2^n, about to period-double. Repeated application of $\mathcal{T}^{-1}$ produces a sequence of manifolds accumulating on Σ at the rate δ^{-1}. By a similar argument, there is a sequence of surfaces $\tilde{\Sigma}_n$ accumulating on Σ from above at the same rate, such that if $f \in \tilde{\Sigma}_n$, then f^{2^n} restricted to a suitable interval is a unimodal map, which maps the interval onto itself twice (as is the case for $x \to 4x(1-x)$), and hence f has topological entropy $2^{-n}\log 2$ [15].

Now consider a family of maps, $\{f_\mu\}$, which intersects Σ transversely at $\mu = 0$, with period-doubling cascades occurring as μ tends to zero from below. This family is represented by a curve in function space as shown in Figure 3, and in terms of the description of the previous paragraph, the sequence of period-doubling bifurcations occurs at the parameter values $\mu_n < 0$ on which f_{μ_n} intersects Σ_n, and $\mu_n \to 0$ as $n \to \infty$. Similarly there is a sequence of parameters $\tilde{\mu}_n > 0$, with $\tilde{\mu}_n \to 0$ as $n \to \infty$ such that $f_{\tilde{\mu}_n}$ intersects $\tilde{\Sigma}_n$. Assuming that μ is a reasonable measure of distance between maps in function space, this immediately gives the famous result [17, 19],

$$\lim_{n\to\infty} \frac{\mu_{n-1} - \mu_n}{\mu_n - \mu_{n+1}} = \lim_{n\to\infty} \frac{\tilde{\mu}_{n-1} - \tilde{\mu}_n}{\tilde{\mu}_n - \tilde{\mu}_{n+1}} = \delta. \tag{14}$$

If $\mu \in (\mu_n, \tilde{\mu}_n)$, then f_μ can be renormalized n times, i.e. $\mathcal{T}^n f_\mu$ is well defined, and the curve of maps $\mathcal{T}^n f_\mu$ has been contracted towards S and stretched in the direction of S. For sufficiently large n, the curve representing $\mathcal{T}^n f_\mu$, $\mu_n < \mu < \tilde{\mu}_n$, is thus arbitrarily close to S. Thus, again assuming that μ is a reasonable measure of distance in function space and

$|\mu|$ is sufficiently small,

$$\mathcal{T} f_\mu \sim f_{\delta\mu}. \tag{15}$$

But if $h(f)$ is the topological entropy of f, then (e.g. [14])

$$h(\mathcal{T} f) = h(f^2) = 2h(f),$$

so, using Equation (15) and writing $h(\mu)$ for $h(f_\mu)$,

$$h(\delta\mu) \sim 2h(\mu). \tag{16}$$

Solving this equation gives

$$h(\mu) \sim \psi(\log\mu)\mu^{\frac{\log 2}{\log\delta}}, \tag{17}$$

where the function $\psi(\log\mu)$ is periodic with period $\log\delta$. We have already noted that the curve in function space given by $\{\mathcal{T}^n f_\mu | \mu \in (0, \tilde{\mu}_n)\}$ converges to the part of S (the unstable manifold of f_*) above Σ. Hence, as $\mu \downarrow 0$, we can expect the modulation ψ in Equation (17) to tend to the modulation on S. In this sense, then, the function ψ in Equation (17) is asymptotically universal within the universality class of the renormalization operator (so a map with a different order maximum would have a different asymptotic modulation and, of course, a different δ).

The asymptotically universal function describes the fine structure of the entropy between $\tilde{\mu}_{n+1}$ and $\tilde{\mu}_n$; since we know that there are intervals of μ values in this range on which the entropy is constant, $\psi(\log\mu)$ is not constant and reflects the variation of $h(\mu)$ from the simple power law $\mu^{\frac{\log 2}{\log\delta}}$.

As mentioned earlier, the entropy, $h(f)$, is a continuous function of parameters, whereas the Lyapunov exponent is not. Although the Lyapunov exponent is easier to calculate directly from iterates of the map numerically using Equation (9), the topological entropy can be calculated using kneading theory as described above: we computed an approximation of $K(t)$ in Equation (11) at each parameter value and used Equation (12) to determine the topological entropy. The results are shown in Figure 4. Figure 4(a) shows a graph of the entropy of f_μ against μ for the logistic map, and Figure 4(b) shows the (asymptotically universal) function $\psi(\log\mu)$ over three periods. Figure 1(b) shows the corresponding function for the Lyapunov exponents of the map over four periods (that is, the horizontal extent of the graph corresponds to $4\log_2(\delta)$). We consider this figure to be strong evidence that a similar argument holds for the Lyapunov exponents, or at least that it holds to a good approximation (for the argument to work for the Lyapunov exponents, Equation (15) must hold with differentiable conjugacy; we do not know whether this is the case).

Of course, this same universal function should exist near the accumulation of period-doubling in other types of dynamical systems, in differential equations for example. To illustrate this, we have calculated the largest Lyapunov exponent for the Lorenz equations near a window of stable periodic orbits identified by Sparrow [21]. The Lorenz equations are

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy \tag{18}$$

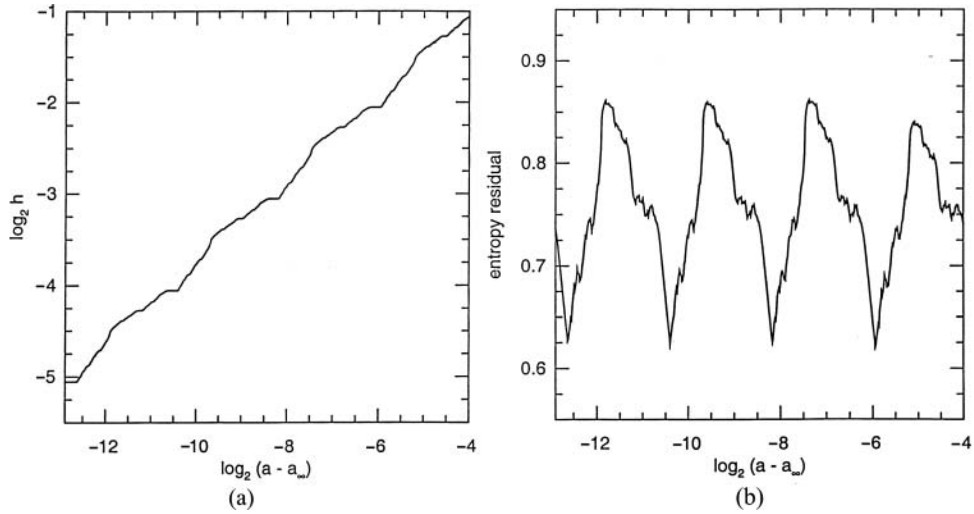

Figure 4. The topological entropy for the logistic map: (a) logarithm of the topological entropy against logarithm of the parameter showing power law behaviour and (b) modulation function obtained by factoring out the power law.

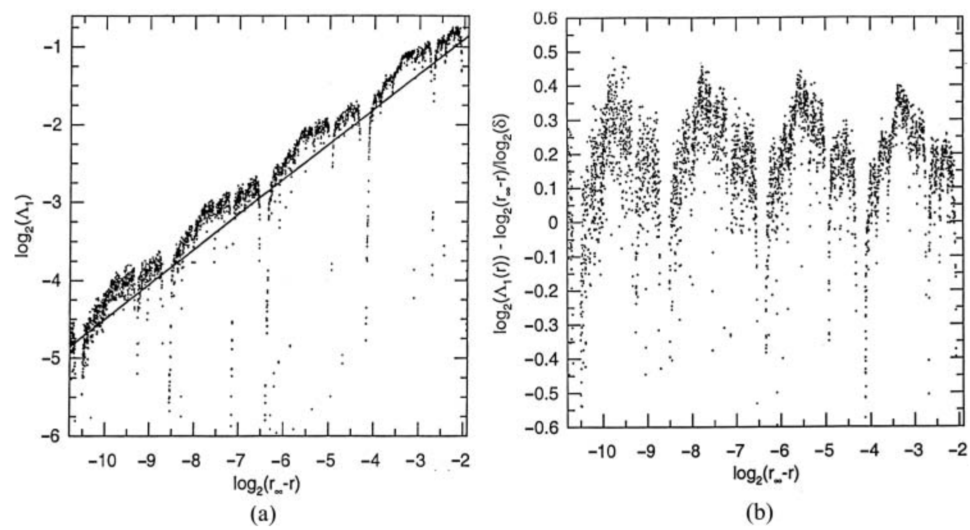

Figure 5. The leading Lyapunov exponent, Λ_1, for the Lorenz equations near $r_\infty \approx 99.5247$: (a) logarithm of the Lyapunov exponent against logarithm of $|r_\infty - r|$ and (b) modulation function obtained by removing the power law $(r_\infty - r)^{\frac{1}{\log_2(\delta)}}$. There are no free parameters in extracting this oscillation. Note the similarity to the result of Figure 1(a).

where we take the standard parameter values $\sigma = 10$ and $b = \frac{8}{3}$ and treat r as the bifurcation parameter. The system shows a series of period-doubling bifurcations as r decreases through $r_\infty \approx 99.5247$. Figure 5 shows the results for the Lorenz equations, which parallel those of the logistic map shown in Figure 1. Although the estimated[2] Lyapunov exponents for the

Lorenz equations are more uncertain than those of the logistic map, one can still discern the similarity of the oscillation in Figures 1(b) and 5(b).

3. The period-doubling Cantor set

We began this note by discussing lacunarity in the context of spatial structure and have ended with an approximate function associated with intermittency. At the accumulation of period-doubling, a unimodal map has an invariant Cantor set on which the motion is not chaotic. Once again, there is an approximately self-similar structure here.

The invariant Cantor set at the accumulation of period-doubling is not self-similar. As Figure 2 suggests, the Cantor set can be constructed by successively dividing an interval into two smaller intervals of different length. One of these is a factor of α times the parent interval, whilst the other (the right-hand invariant interval in Figure 2) can be calculated approximately using a quadratic approximation to f_* to be roughly a factor of α^2 times smaller than the parent interval. So each interval at the nth generation splits into two intervals whose lengths are, respectively, α and α^2 times the length of the parent interval. From this observation, it is easy to see that the at the nth generation there are 2^n intervals of which $\binom{n}{k}$ are of length $\alpha^k \alpha^{2n-2k}$. Thus we have

$$N(\alpha^n) = N(\alpha^{n-1}) + N(\alpha^{n-2}), \tag{19}$$

(cf. $F_n = F_{n-1} + F_{n-2}$, which generates the Fibonacci numbers), and so we obtain the approximation

$$N(\alpha_*^2 \epsilon) = N(\alpha_* \epsilon) + N(\epsilon), \tag{20}$$

where α has been replaced by the asymptotic value α_* defined above.

It is now easy to check that if

$$d_1 = \frac{\log\left(\frac{\sqrt{5}+1}{2}\right)}{\log \alpha_*} \quad \text{and} \quad d_1 = \frac{\log\left(\frac{\sqrt{5}-1}{2}\right)}{\log \alpha_*}$$

and $\phi_i(\log x)$, $i = 1, 2$, are arbitrary periodic functions with period $2 \log \alpha_*$ then

$$\phi_1(\log \epsilon)\epsilon^{d_1} \cos \frac{2m\pi \log \epsilon}{\log \alpha_*} \quad \text{and} \quad \phi_2(\log \epsilon)\epsilon^{d_2} \cos \frac{(2m+1)\pi \log \epsilon}{\log \alpha_*} \tag{21}$$

are solutions, for each $m \in \mathbf{Z}$. Noting that $d_1 < 0$ and $d_2 > 0$ (as $\alpha_* < 1$) and that the cosines are also periodic with period $2 \log \alpha_*$ we find that

$$N(\epsilon) \sim \chi(\log \epsilon)\epsilon^{d_1} \tag{22}$$

as ϵ tends to zero, where $\chi(\log \epsilon) = \chi(\log \epsilon + 2 \log \alpha_*)$. This is only an approximate result: the invariant Cantor set is not strictly self-similar in the way described and the recursion relation for $N(\epsilon)$ is also an approximation. For a fuller, and more precise treatment, see [22,23].

4. Conclusion

We have given four examples of lacunarity: two for sets in state space and two for sets in parameter space. For the middle third Cantor set, the analysis is exact and the modulation function can be derived explicitly. In the second example, for the scaling of topological entropy, the modulation function was computed numerically, and in this case the theory is asymptotically exact, in the sense that the modulation function is universal sufficiently close to the accumulation of period-doubling. The third example, for the Lyapunov exponents above the accumulation of period-doubling, mirrors the analysis of the topological entropy except the function is much less regular (it is not continuous). The analysis of the final example is approximate. Here we defined a scaling for the Cantor set at the accumulation of period-doubling, which is asymptotically exact at small length scales, but we have used a rough approximation of this scaling law to obtain approximate scaling functions.

There are other situations in which it is possible to obtain approximate self-similarity over some set in parameter space or state space. In these cases, it is possible to repeat the argument used above, but the periodic modulation functions will only be approximately valid. The most obvious example of approximate self-similarity in parameter space is intermittency. The standard analysis (e.g. [24]) uses an approximate renormalization scheme to describe the length of time spent close to the location of a saddlenode bifurcation, and this can be used to derive scaling and *approximate* modulation functions. We have not considered this case in detail, but it would be interesting to know how much relevant information these approximations hold.

Acknowledgements

The work on the logistic map was done at D.A.M.T.P., University of Cambridge, in 1991 supported by a research grant from SERC (now renamed as EPSRC). We are grateful to P. McSharry for computing the Lyapunov exponents for the Lorenz system in finite time.

Notes

1. After completing this work, we learned that Robert MacKay had noted the universal nature of the modulation function for period-doubling in his Ph.D. thesis, though he did not compute the functions [10].
2. These numerical estimates were obtained from using the ϵ–returns algorithm. The relative uncertainty in the estimate increases (visibly) for smaller $r_\infty - r$, as the value of $\Lambda(r) \to 0$.

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