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THE BOY WHO CRIED WOLF REVISITED: THE IMPACT OF FALSE ALARM INTOLERANCE ON COST-LOSS SCENARIOS

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1 Introduction

In Aesop's fable about the "The Boy who Cried Wolf", a young shepherd boy guarding the village flock cries that there is a wolf worrying the sheep. The villagers rush out to protect the sheep, but there is no wolf. This event is repeated two or three times before a wolf actually does show up on the hillside. The boy cries "Wolf!", but to no avail, the villagers, no longer regarding the warning as credible, fail to act and the wolf decimates the flock. The traditional moral of this tale is that liars are not believed, even when telling the truth. But, are we being too harsh on the shepherd boy? Perhaps his mistake was to overestimate the rationality of the villagers. This traditional tale can be expressed in terms of a cost-loss problem, used in the analysis of the utility of weather forecasts (Murphy 1966; Richardson 2000). The villagers' cost-loss matrix is

	NO WOLF	WOLF
VILLAGERS RESPOND	C	C
VILLAGERS DON'T RESPOND	0	L

where C is the price the villagers paid when they ran to aid the shepherd boy. This price was largely an opportunity cost, the value of the goods and services they could have been producing had they not been running about the hills. L is the loss associated with losing part of the flock to a lupine predator. If the shepherd boy believed the probability that there was a wolf about was p then the expected cost to the village was

$$E[COST] = \begin{cases} pL & \text{if you don't cry wolf} \\ C & \text{if you cry wolf} \end{cases} \tag{1}$$

To minimize the expected cost the boy should have cried wolf if $p > C/L$. If we assume that the flock of sheep was one of the village's most important assets it seems reasonable to say that the villagers' cost-loss ratio, C/L, was quite modest. An estimate of C/L using contemporary prices gives

$$\frac{C}{L} \approx \frac{\text{6 villagers for 1 hour at \$10/hour}}{\text{3 sheep at \$200 each}} = \frac{\$60}{\$600} = 0.1 \tag{2}$$

So the cost-loss analysis implies that the shepherd boy should have issued a warning even if he believed there was only a 10% probability of lupine activity. While this is a rational strategy, it would obviously have resulted in a high rate of "false alarms"—possibly as high as 90%. The fact the villagers were apparently unprepared to tolerate a false alarm rate of around 67% seems to suggest that they had not performed a cost-benefit analysis of responding to the cries of "Wolf!". Unfortunately, such behavior is not confined to the characters in stories written over two millenia ago. Psychological factors influence the people's perception of risk, and this perception affects the way that they interpret meteorological forecasts. In this paper, we examine how the public's compliance with warnings might affect the value of the forecasts, and whether the forecast value can be increased by taking the user's response into account. We do this in the context of the familiar cost-loss scenario. The standard cost-loss scenario is a *prescriptive* decision model, it specifies how a user *should* respond. In contrast, *descriptive* decision models describe how users actually do respond (Stewart 1997). The modified cost-loss model we present is a somewhat idealized descriptive model, but it provides a quantitative illustration of the importance of the descriptive-prescriptive distinction. To be relevant, evaluations of the value of forecasts to society cannot ignore this descriptive element.

2 Cost-loss Scenario with Imperfect Compliance

To model imperfect compliance with warnings in the cost-loss scenario we introduce a compliance rate, q. This is the probability that action will be taken if a warning is given. The expected cost, $E[COST]$, if the forecast probability is p is thus

$$E[COST] = \begin{cases} pL & \text{when no warning is given} \\ qC + (1-q)pL & \text{when a warning is given} \end{cases} \quad (3)$$

The first term on the lower line of the right-hand-side (RHS) of Eq. 3 denotes the expected cost of protective action being taken, while the second term is the expected cost of a loss if no protective action is taken. If warnings are issued if the forecast probability exceeds some threshold, p_w, then the expected cost averaged over time is given by

$$\langle E[COST]\rangle = \int_0^{p_w} \rho(p)pLdp + \int_{p_w}^1 \rho(p)[qC + (1-q)pL]dp \tag{4}$$

where $\rho(p)$ is the frequency distribution of forecasted probabilities. The first term on the RHS of Eq. 4 represents days when no warning is issued, and the second term times when a warning is issued. Equation 4 can be rewritten as

$$\langle E[COST]\rangle = L\int_0^1 \rho(p)pdp + qC\int_{p_w}^1 \rho(p)dp - qL\int_{p_w}^1 \rho(p)pdp \tag{5}$$

The problem is analytically simplified if we assume that $\rho(p) = 1$ on the interval $[0,1]$. In this case, Eq. 5 can be rewritten

$$J = \frac{\langle E[COST]\rangle}{L} = \frac{1}{2} + qc(1-p_w) - q\left(\frac{1}{2} - \frac{p_w^2}{2}\right) \tag{6}$$

where J is the expected cost-per-unit-loss and $c = C/L$ is the *cost-loss* ratio. If the compliance rate, q, is independent of p_w then the minimum of J occurs when $p_w = c$, as in the standard cost-loss scenario.

A more interesting situation arises if we assume that the compliance rate is a function of the false alarm rate. The false alarm rate is a monotonically decreasing function of the warning threshold, p_w, so we can write $q = q(p_w)$. As p_w increases, the false alarm rate decreases and we can expect the compliance rate to rise. Thus, a reasonable form of $q(p_w)$ is a monotonically increasing function of p_w. The simplest function of this form is $q = p_w$. Assuming such a relationship and differentiating J with respect to p_w gives

$$\frac{dJ}{dp_w} = c - 2p_wc - \frac{1}{2} + \frac{3}{2}p_w^2 \tag{7}$$

Thus, the value of the warning threshold, p_w, that minimizes the cost-per-unit-loss is

$$p_w^* = \frac{4}{3}c \pm \sqrt{\frac{4c^2 - 6c + 3}{9}} \tag{8}$$

If the cost of acting is small relative to potential loss if no action is taken then $c \ll 1$ and Eq. 8 gives

$$p_w^* \approx \frac{1}{\sqrt{3}} \approx 58\% \tag{9}$$

This threshold is close to the 60% threshold the U.K. National Severe Weather Warning Service (NSWWS) uses for its "Early Warnings" (Mylne and Legg 2002). Given the low cost-loss ratios often associated with severe weather, this threshold would appear too high with respect to the standard cost-loss analysis. Imperfect compliance on the part of the public could be a possible justification for the high value of the operational threshold.

3 Imperfect Compliance when Warnings are Rare

While analytically tractable, the assumption of a uniform distribution of forecasted probabilities is not really appropriate when considering extreme and rare events. For such events, an exponential distribution of forecasted probabilities is a better model. Such a distribution is given by

$$\rho(p) = \frac{\alpha e^{-\alpha p}}{1 - e^{-\alpha}} \qquad \alpha > 0 \tag{10}$$

where α parameterizes the rarity of higher forecast probabilities. The normalization of $\rho(p)$ is such that $\int_0^1 \rho(p)dp = 1$. When the distribution of forecast probabilities is given by Eq. 10 then the frequency with which the forecast probability is equal to, or exceeds r is

$$\text{Prob}(p \geq r) = \frac{e^{-\alpha r} - e^{-\alpha}}{1 - e^{-\alpha}} \tag{11}$$

As expected as α increases the frequency of probability forecasts exceeding r decreases. Figure 1 shows the form of $\rho(p)$ for three different values of α. Values of α close to zero correspond to an effectively uniform distribution on the interval [0,1]. If the forecasted probabilities are reliable, then the climatological probability of the event is the expected value of p, that is

$$\int_0^1 \rho(p)pdp = \frac{1}{\alpha} - \frac{e^{-\alpha}}{1 - e^{-\alpha}} \tag{12}$$

It is worth noting that if the forecasts are reliable, and the warning threshold is p_w, then the expected false alarm rate is *not* $1 - p_w$. The expected false alarm rate depends on the function $\rho(p)$ as well as on the value of p_w. The expected false alarm rate, $E[FAR]$, is given by

$$E[FAR] = \int_{p_w}^1 (1-p)\rho(p)dp = 1 - \int_{p_w}^1 p\rho(p)dp \tag{13}$$

when checking the reliability of probability forecasts Eq. 13 should be used to determine the expected false alarm rate. Given the warning threshold is p_w and the compliance rate is q, the time averaged expected cost is given by substitution of Eq. 10 into Eq. 5. Integration gives

$$J = \frac{\langle E[COST] \rangle}{L} = \frac{1}{\alpha} - \frac{\beta}{\gamma} + qc\frac{\beta^{p_w} - \beta}{\gamma} + q\frac{\beta}{\gamma}\left(\frac{1}{\alpha} + 1\right) - q\frac{\beta^{p_w}}{\gamma}\left(\frac{1}{\alpha} + p_w\right) \quad (14)$$

where $\beta = e^{-\alpha}$ and $\gamma = 1 - e^{-\alpha}$. We seek the value of p_w that minimizes J. If the compliance rate, q, is independent of p_w then setting the derivative of dJ/dp_w equal to zero gives the standard cost-loss scenario result, $p_w = c$. We can now quantify how user intolerance to false alarms impacts the forecast value.

Parameterize the intolerance of the user to false alarms with the following model

$$q = p_w^{\lambda} \quad \lambda \geq 0 \quad (15)$$

The parameter λ is effectively a quantification of the strength of the "cry wolf effect". If $\lambda = 0$ then the compliance rate is 1 and the standard cost-loss scenario is recovered. If λ is small then the compliance rate remains quite high until the warning threshold, p_w, is set very low (the case of of frequent warnings). In the case of $p_w = 0$, warnings occur all the time, and the compliance rate falls to zero. In this case, there is a major difference between ideal rational users of the traditional cost-loss scenario who will always take protective action if $c = 0$, and the intolerant users of the modified scenario who will never take action. This unwillingless to protect against improbable, but potentially catastrophic events, has been by studied in the context of attitudes to insurance (Slovic et al. 1977). If p_w is set to one, warnings are only issued when there is certainty and total compliance is achieved. The extreme cases of $p_w = 0, 1$ are the same for all values of $\lambda > 0$ but higher values λ correspond to less tolerance of false alarms. The higher the value of λ the more rapidly the compliance rate falls as the warning threshold is reduced from near-certain. Figure 2 shows the compliance rate as a function of the warning threshold for four different values of λ.

In this, more general case, finding the global minimum value of the function

$J(p_w)$ is not analytically tractable for arbitrary values of λ. The minimum value of J, and the optimum value of p_w that gives this minimum value can be found numerically. For cost-loss ratios of $c = 0.01$, $c = 0.1$ and $c = 0.5$ the results are shown in Figs. 3, 4 and 5 respectively. Figure 3a shows the optimum value of p_w as a function of α and λ. Figure 3b gives the percent reduction in the average expected cost when this optimum is used, rather than a value of $p_w = c = 0.01$, which is what the conventional cost-loss analysis would suggest. The maximum cost reduction is over 50% and is obtained for $\alpha \ll 1$ and $\lambda \approx 0.2$. This corresponds to an almost uniform frequency distribution of forecast probabilities and a relatively compliant, but not perfectly compliant user. The optimal warning threshold in this case is $p_w \approx 0.30$, which is considerably higher than the cost-loss ratio of 0.01. Inspection of Fig. 3a suggests that the value of 60% used by the U.K. NSWWS implies a value of $\lambda > 1$, and since the effective value of α for extreme events is likely to be at the high end of the range shown, the implied value of λ may be more than 3. Such a value would correspond to a user that is quite intolerant of false alarms. Reference to Fig. 3b indicates that in this case the reduction in the average expected cost obtained by inflating the warning threshold is quite low, less than 5%. Figure 4 shows the same analysis as Fig. 3, except for the case of a cost-loss ratio of $c = 0.1$. Again, the 60% contour corresponds to relatively high values of λ, and again the cost reductions obtained for high values of α and λ are below 5%. In this region of the $\alpha - \lambda$ plane, the high value of λ means that the user tend to only respond to almost certain forecasts, but the high value of α means that such forecasts are highly uncommon, hence the overall impact of accounting for imperfect compliance is small. Figure 5b indicates that the cost reductions possible by accounting for imperfect compliance when the cost-loss ratio is 0.5 are not very large for any combination of α and λ. In Figs. 3 and 4 it can be seen that the biggest reductions in expected cost, for modest values of α, are obtained for values for $0 < \lambda < 1$. In this range, the user deviate from the perfect compliance enough so that taking this deviation into account matters, yet they don't deviate so much that it is infeasible to compensate for their intolerance of false alarms.

The introduction of imperfect compliance increases the average costs incurred, relative to the standard cost-loss scenario. The value of the forecast is the difference between the average cost incurred when the forecasts are used, compared to the average cost incurred in the absence of forecasts. The ratio of the values of the forecasts in the imperfect compliance case, with an optimal warning threshold, and the standard perfect compliance case is given by the following expression.

$$\frac{\langle \mathrm{COST}(\text{zero compliance}) \rangle - \langle \mathrm{COST}(\text{imperfect compliance, optimal } p_w) \rangle}{\langle \mathrm{COST}(\text{zero compliance}) \rangle - \langle \mathrm{COST}(\text{perfect compliance, } p_w = c) \rangle} \tag{16}$$

Figure 6 shows the value of the ratio in Eq. 16 for different values of λ and α. The cost-loss ratio used was $c = 0.1$. From Fig. 6 it can be seen that as the rarity of events increases, and the user's intolerance for false alarms increases, the actual potential value of the forecasts falls in relation to the value that the standard cost-loss analysis predicts. Replacing COST(zero compliance) in Eq. 16 with the average cost when decisions are based on climatological probabilities, COST(climatology), will reduce the value of the forecasts and reduce the ratio defined by Eq. 16. That is, if we assume that users, in the absence of a forecast, make better decisions than zero compliance, then the value of forecasts with imperfect compliance will be *further* reduced relative to the forecasts with perfect compliance.

4 Discussion

We have shown that introducing a compliance rate, which is a function of the false alarm rate, in the cost-loss model can have a substantial impact on the optimal choice of warning threshold and the value of forecasts. The extent to which false alarm intolerance modifies the results of the cost-loss analysis depends upon the frequency of forecasted probabilities (α in our model), the cost-loss ratio (c), and the intolerance of the users to false alarms (λ in our model). The modification is most pronounced for low cost-loss ratios ($c \ll 0.1$), relatively high frequency events ($\alpha < 1$), and users who are moderately intolerant of false alarms ($0 < \lambda < 1$). For such situations the optimal warning threshold can be many times the cost-loss ratio, and the savings obtained by changing the warning threshold can exceed 25%.

Establishing the value of α for a particular type of forecast is a problem that lies comfortably inside the domain of meteorologists. Past forecasts can be used to estimate this parameter of the exponential distribution. [1] Determination of the cost-loss ratio, c, is more a problem of economics than meteorology. Nevertheless, the cost-loss type of decision model is now well established in the field of weather forecasting (Katz and Murphy 1997; Richardson 2000). The value of the parameter, λ, indeed the functional relationship $q(p_w)$, is poorly understood. At its simplest, q can be interpreted as the probability that a random individual will comply with forecast warning. The compliance rate can also be interpreted as the fraction of individuals who will comply with a warning. These two interpretations are equivalent if each individual's decision is independent of the decisions made by other individuals. In practice, this is unlikely to be the case. For example, an individual's decision on whether to evacuate their home may depend on what their neighbors are doing. This means that the form of $q(p_w)$ will be an emergent property of a system of interacting individual choices. In addition, q may depend on the false alarm rate during a finite period in the past. In such a situation the optimum warning threshold will be time dependent.

The simplified problem addressed in this paper illustrates the importance of including the actual user response in models of forecast value. This descriptive component cannot be neglected if estimates of the true value of forecasting systems to the economy and society are required. The results in this paper also indicate that consideration of users' response by forecasters can increase the realized value of their forecasts.

Acknowledgements

[1]Study of past forecasts may of course indicate that an exponential distribution is inappropriate, in which case the analysis can be performed using a different functional form for $\rho(p)$.

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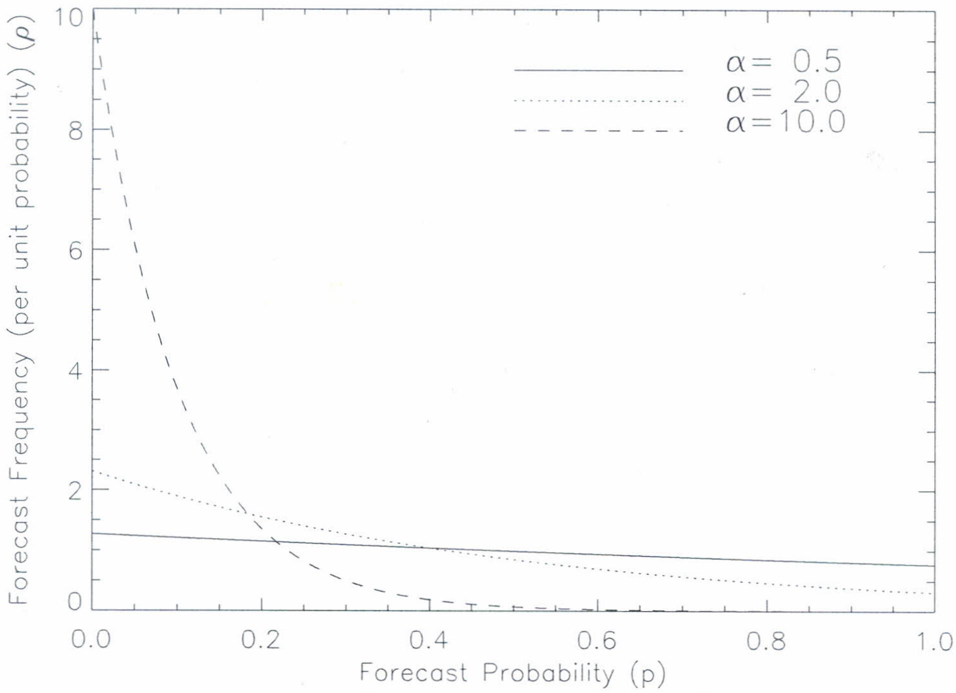

Figure 1: The frequency with which probabilistic forecasts of an event with a particular probability are issued, as a function of the parameter α. Low values of α correspond to events for which high probability forecasts are relatively common. High values of α correspond to events for which high probability forecasts are rare.

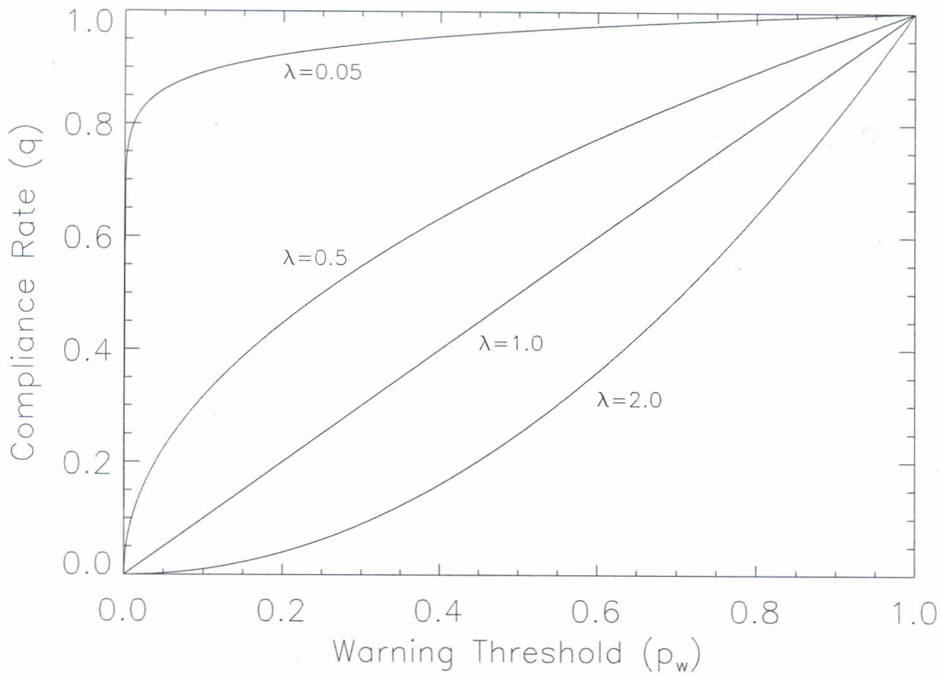

Figure 2: The compliance rate, as a function of the probability threshold at which warnings are issued, for four different values of the parameter λ. The model of compliance, q, as a function of warning threshold, p_w, is $q = p_w^\lambda$. Thus a value of $\lambda = 0$ corresponds to full compliance regardless of the warning threshold.

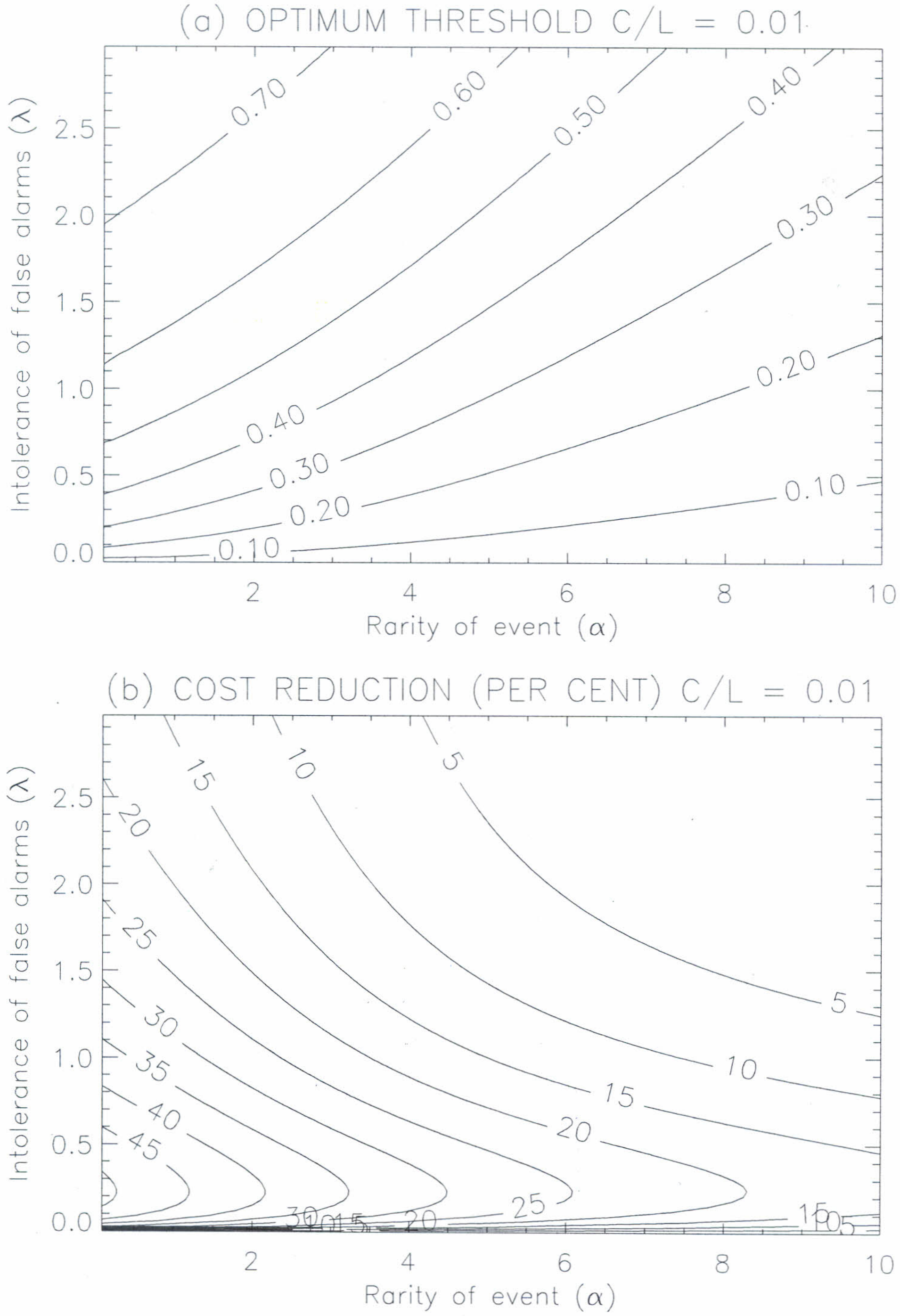

Figure 3: (a) The optimum warning probability threshold as a function of event rarity parameterized by α, and intolerance of false alarms parameterized by λ when the cost-loss ratio is 0.01 (b) The per cent reduction in the expected cost when the optimum threshold is used rather than the cost-loss ratio.

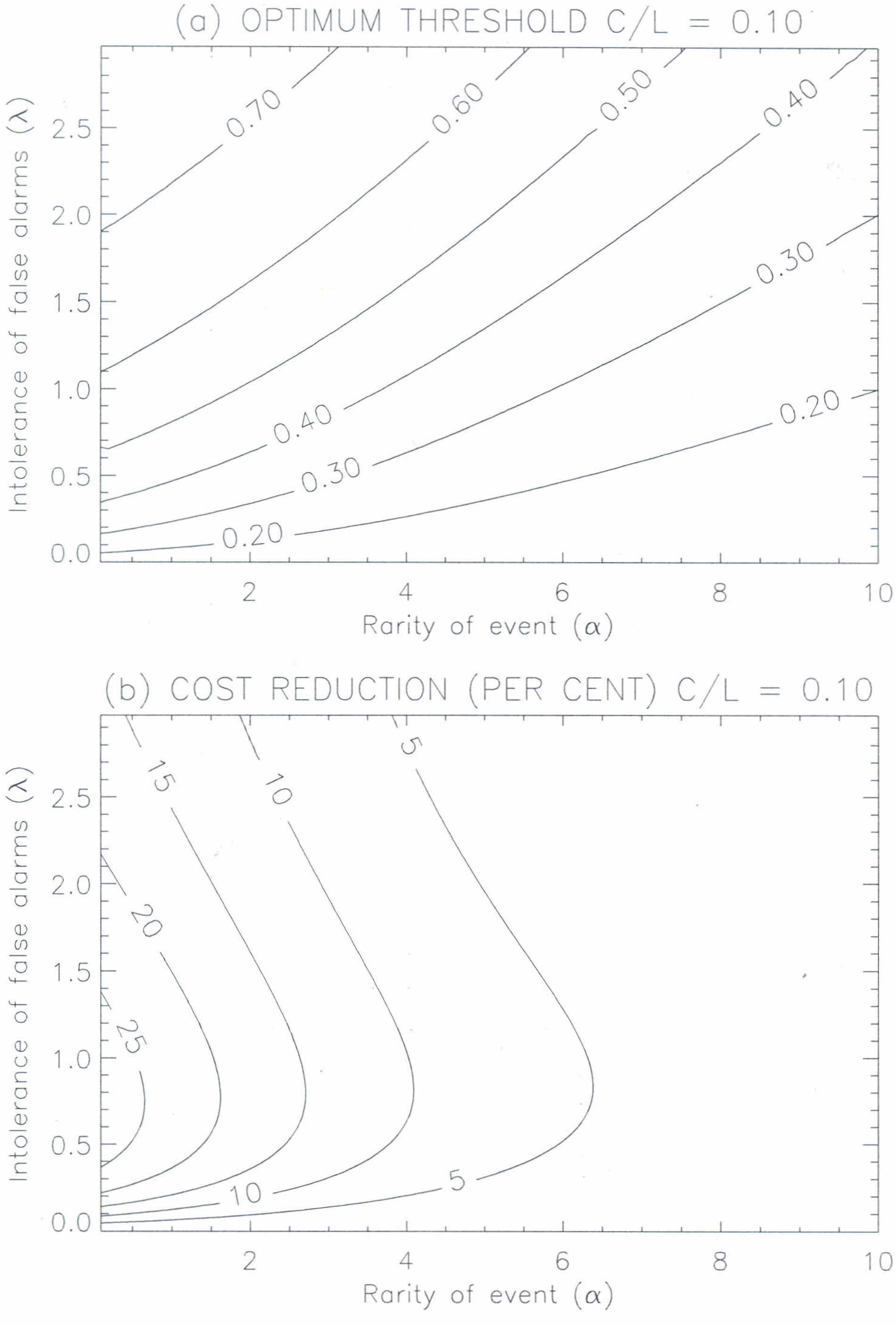

Figure 4: As Fig. 3 except for a cost-loss ratio of 0.1. Note that with this higher cost-loss ratio the maximum cost reduction is reduced to about 30%.

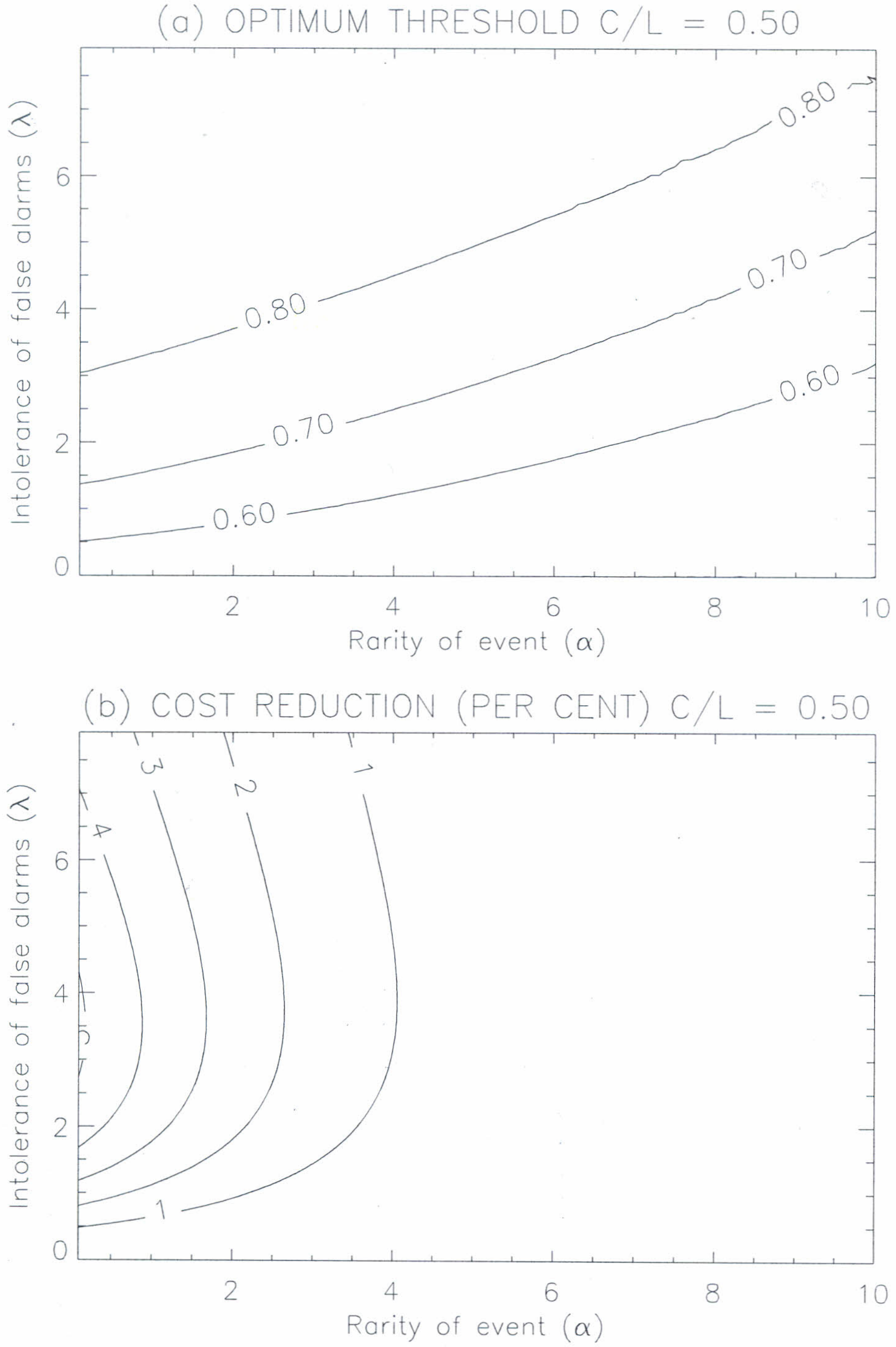

Figure 5: As Fig. 3 except for a cost-loss ratio of 0.5. Note that with this cost-loss ratio there is very little cost reduction and the optimum thresholds do not deviate from the cost-loss ratio as much as in Figs. 3 and 4.

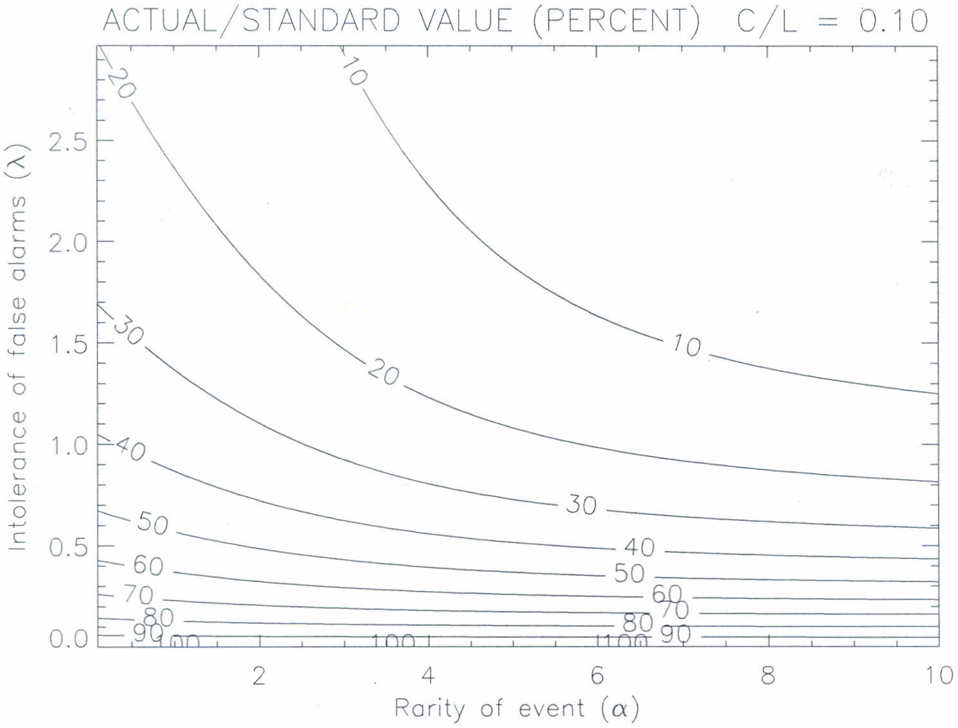

Figure 6: The ratio (in per cent) of the value (average cost reduction) of the forecasts when there is imperfect compliance to the value when there is perfect compliance. A value of 100% indicates that incompliance has no impact on forecast value, while a value of 10% indicates that the actual value of the forecast is only 10% of the value that would be calculated using the conventional cost-loss method. The optimum warning threshold for the parameter values was chosen for the imperfect compliance and a threshold of $p_w = c = 0.1$ was used for the standard cost-loss comparison.